Parametric instability in a collisional dusty plasma

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The present work investigates the parametric instability of parallel propagating circularly polarized Alfvén (pump) waves in a collisional dusty plasma. It is demonstrated that the relative drift between the charged dust and the electrons and ions gives rise to the Hall effect resulting in the modified pump wave characteristics. Although the linearized fluid equations with periodic coefficients are difficult to solve analytically, it is shown that a linear transformation can remove the periodic dependence. The resulting linearized equations with constant coefficients are used to derive an algebraic dispersion relation. The growth rate of the parametric instability is a sensitive function of the amplitude of the pump wave as well as to the ratio of the pump and the dust-cyclotron frequencies. The instability is insensitive to the plasma-beta parameter. The possible application of the result in the astrophysical context is discussed. © *2007 American Institute of Physics*.
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I. INTRODUCTION

The presence of charged grains plays an important role in the astrophysical and space environment. For example, the spoke formation in Saturn's ring, the Jovian ring formation, and the formation of the protoplanetary disks are but a few examples for which dust dynamics plays an important role. The planets are understood to have formed from a disk of gas and dust orbiting around the protostars. Observations lend credence to such a view since at least half of the protostellar objects are surrounded by the protoplanetary disks.[1] One of the principal difficulties in studying the dynamics of dusty plasma is related to the nonmonochromatic nature of the size, charge, and composition of the dust. For example, in the interstellar medium, one generally finds graphite, silicate, and metallic grains. The size of these grains can vary between a few angstroms to a few cm or more. The charge on the grain can vary between tens to hundreds of electrons.[2] The mass of the grain may vary as well between a few proton mass to a few billion proton mass. This complex situation is not typical only to the interstellar medium but also to the planetary environment.[3–5]

The presence of charged grains in the planetary and interstellar medium causes the excitation of very low frequency Alfvén waves.[6–13] These waves provide an important physical mechanism for the transport of angular momentum and energy in differentially rotating dusty disks.[14,15] Due to the large dispersion in the mass, charge, and size of the grain, Alfvén waves in dusty plasma have been studied in various limits. The excitation of the Alfvén wave in the presence of an immobile dusty background has been studied by several authors.[11,13,16]

The parametric instability of a finite-amplitude, circularly polarized Alfvén wave has been studied in space and astrophysical plasmas for the past four decades.[17–21] Such an investigation in a dusty medium is a relatively recent phenomenon.[5,22,23] The charged grain is assumed immobile and the pump wave is assumed magnetosonic or Alfvénic. A recent generalization of the parametric investigation into multicomponent plasma[24,25] removed the limitation of the immobile dust grains. It is known that the collisional processes in weakly ionized dusty plasma can strongly influence the nonlinear ambipolar diffusion and the ensuing current sheet formation.[26] It should be expected that the role of the plasma-dust collision may also be important for the parametric instability. It is known that the collision between the plasma particles and the grain is responsible for some of the novel features in dusty plasma. As an example, the new collective behavior is known to exist in such a plasma due to the charge fluctuations—an offshoot of collision.[5,8,9,27] Therefore, it is desirable to investigate the propagation of the large-amplitude Alfvén wave in a collisional dusty plasma.

The present work investigates the parametric instability of collisional dusty plasma. In a weakly ionized medium (for example, in a molecular cloud), the inertia of the plasma component is mainly due to the presence of the charged grains, i.e., $\rho_e \ll \rho_i \ll \rho_d$ (where $\rho_j = m_j n_j$ is the mass density with m_j as the mass and n_j as the number density of the jth particle). By a dusty plasma, a three-component plasma consisting of electrons, ions, and charged grains will be implied. Although neutral hydrogen gas is a major constituent of weakly ionized interstellar and planetary matter, inclusion of the neutrals is postponed in the present work and investigation is confined to the three-component plasma description in order to first understand the underlying physical principles before addressing the general problem. One of the difficulties associated with the investigation of the parametric instability of a circularly polarized Alfvén wave is related to the appearance of the periodic coefficients in the linearized magneto-

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hydrodynamic (MHD) equations. This restricts the normal-mode analysis of the spatial dependence, thus the resulting Floquet equation can only be solved in the long-wavelength limit. However, a recently used linear transformation by Ruderman and Simpson[20] reduces the periodic coefficients in the linearized MHD equations to constant coefficient. Although the transformation in Ref. 20 was applied to the non-dispersive Alfvén wave, it is shown in the present work that such a transformation is equally applicable to general, dispersive dusty Alfvén modes.

It is assumed in the present work that the charge on the dust particle is constant. Generally, grain charge is a function of the plasma parameter.[4,5,8,9] However, charge fluctuation will not be the concern of the present work. The basic set of equations is discussed in Sec. II. It is shown that, depending on the sign of the charge on the grain surface, ion-dust and electron-dust collision frequencies can become comparable in many astrophysical situations, suggesting that the bulk velocities of electrons and ions can become comparable. Assuming equality between the electron and ion bulk velocities, the momentum equations are reduced to a simple one-fluid description. The induction equation is shown to contain the Hall term, due to the relative drift between the plasma and the dust particles. It is known that this feature is a generic property of the dusty plasmas.[14–16] The Alfvén pump wave is discussed in Sec. III and the linearized equations are derived. Applying a linear transformation, a set of equations with constant coefficient is derived. The dispersion relation is solved numerically. The dependence of the parametric instability on the pump wave amplitude and on the ratio of the pump wave to the dust-cyclotron frequency is discussed. In Sec. IV, possible application of the result is discussed and a brief summary is presented.

II. BASIC MODEL

The simplest description of dusty plasma, consisting of the electrons, ions, and charged grains, is given in terms of continuity and momentum equations for respective species with a suitable closure model, viz., an equation of state. The continuity equation is

$$\frac{\partial \rho_j}{\partial t} + \nabla \cdot (\rho_j \mathbf{v}_j) = 0. \tag{1}$$

Here ρ_j is the mass density, $\mathbf{v}_j$ is the velocity, and j stands for electrons, ions, and grains. The momentum equations are

$$0 = -\nabla P_e - e n_e(\mathbf{E} + \mathbf{v}_e \times \mathbf{B}) - \rho_e \nu_{ed}(\mathbf{v}_e - \mathbf{v}_d), \tag{2}$$

$$0 = -\nabla P_i + e n_i(\mathbf{E} + \mathbf{v}_i \times \mathbf{B}) - \rho_i \nu_{id}(\mathbf{v}_i - \mathbf{v}_d), \tag{3}$$

$$\rho_d \frac{d\mathbf{v}_d}{dt} = -\nabla P_d + Z e n_d(\mathbf{E} + \mathbf{v}_d \times \mathbf{B}) + \sum_{j=e,i} \rho_j \nu_{jd}(\mathbf{v}_j - \mathbf{v}_d). \tag{4}$$

The electron and ion inertia has been neglected while writing Eqs. (2) and (3). This is motivated by the fact that the inertia in a dusty plasma is largely carried by the dust grains. The momentum Eqs. (2)–(4) are closed by assuming an isothermal equation of state $P_j = C_s^2 \rho_j$, where $C_s = \sqrt{T_j/m_j}$ is the sound speed. Equations (2)–(4) on the right-hand side have a Lorentz force term with $\mathbf{E}$ and $\mathbf{B}$ as the electric and magnetic fields, respectively, e is the electric charge, Z is the number of charge on the grain, n_j is the number density, and $\nu_{jd} = n_d \langle \sigma v \rangle_{jd}$ is the collision frequency of the dust with the jth species. The electron-dust and ion-dust collision rates for the negatively charged grains are given as[28]

$$\langle \sigma v \rangle_{ed} = \pi a^2 S_e \left(\frac{8T}{\pi m_e}\right)^{0.5} \left[1 + \left(\frac{1}{4\tau + 3Z}\right)^{0.5}\right]^2 \times \exp - \left[\frac{Z^{1.5}}{\tau(1 + Z^{0.5})}\right], \tag{5}$$

$$\langle \sigma v \rangle_{id} = \pi a^2 S_i \left(\frac{8T}{\pi m_e}\right)^{0.5} \left(1 + \frac{|Z|}{\tau}\right) \left[1 + \sqrt{\frac{2}{\tau + 2|Z|}}\right].$$

While writing Eq. (5), it has been assumed that the grains are spherical with radius a. Here S_j is the sticking coefficients and $\tau = aT/e^2$ with $T_e = T_i = T_d = T$. For larger, micrometer-sized grains, the grain-dust collision rate can vary between 10^{-10} and 10^{-5} for sizes ranging between a few angstroms to a few micrometers. This can be seen if we write the ion-dust collision rate as

$$\langle \sigma v \rangle_{id} = 2 \times 10^{-4} T_{30}^{1/2} a_{-5}^2, \tag{6}$$

where T_{30} is the gas temperature and a_{-5} is the grain radius in units of 30 K and 10^{-5} cm, respectively. The electron-grain collision rate for the micrometer-sized grain,

$$\langle \sigma v \rangle_{ed} = 2.4 \times 10^{-9} T_{30}^{1/2} a_{-5}^2, \tag{7}$$

is much smaller than the ion-dust collision rates. For positively charged dust grains, the electron-dust collision rate is similar to the above expression for the ion-dust collision rate. The collision between ions and positively charged dust grain is of the same order as the electron negatively charged dust grains. The dusty plasma in nature, viz., molecular clouds, protostellar disks, etc., is a mixture of electrons, ions, neutral and charged grains, and hydrogen gas. The probability of electrons colliding with the positively charged grain and ions colliding with the negatively charged grain is comparable and is much bigger than the electrons and ions colliding with the same sign grains. Therefore, it is realistic to assume that the bulk velocities of the electron and ion fluids in such a dusty medium can become comparable.

We shall define the mass density of the bulk fluid as $\rho = \rho_e + \rho_i + \rho_d \approx \rho_d$. Then the bulk velocity is $\mathbf{v} = (\rho_i \mathbf{v}_i + \rho_e \mathbf{v}_e + \rho_d \mathbf{v}_d)/\rho \approx \mathbf{v}_d$. The continuity equation [summing up Eq. (1)] for the bulk fluid becomes

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0. \tag{8}$$

The momentum equation can be derived by adding Eqs. (2) and (4),

$$\rho \frac{d\mathbf{v}}{dt} = -\nabla P + \frac{\mathbf{J} \times \mathbf{B}}{c}. \tag{9}$$

Here $P = P_e + P_i + P_d$ is the total plasma pressure.

Making use of the plasma quasineutrality condition, $n_e = n_i + Zn_d$, in the expression for the current density $\mathbf{J} = e(n_i \mathbf{v}_i - n_e \mathbf{v}_e)$, and assuming $\mathbf{v}_e \simeq \mathbf{v}_i$, the relative drift between the electrons and dust velocities can be written as

$$\mathbf{v}_e - \mathbf{v}_d \simeq \mathbf{J}/eZen_d. \tag{10}$$

Taking the curl of the electron momentum Eq. (2) and making use of Maxwell's equation, in the $\omega_{ce} \gg \nu_{ed}$ limit, the induction equation can be written as

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times \left[(\mathbf{v}_d \times \mathbf{B}) - \left(\frac{\mathbf{J} \times \mathbf{B}}{Zen_d} \right) \right]. \tag{11}$$

While writing Eq. (11), the electric field due to the pressure term in Eq. (2) has been neglected. In a weakly ionized dusty medium, barring shock regions, the contribution of the plasma pressure terms is generally small. We see from (11) that the ideal-MHD limit corresponds to $\nabla \times \mathbf{B}/Zen_d \to 0$.

One can investigate the dusty plasma dynamics with the help of Eqs. (8), (9), and (11) along with an isothermal equation of state.

III. THE PUMP WAVE

In what follows, a uniform background magnetic field $\mathbf{B} = (0, 0, B)$ is assumed along the z direction. It will be assumed that all physical quantities depend on z only. An exact solution of Eqs. (8), (9), and (11) is the finite-amplitude circularly polarized Alfvén wave,

$$B_x = A_0 \cos \phi, \quad B_y = A_0 \sin \phi,$$
$$V_x = V_0 \cos \phi, \quad V_y = V_0 \sin \phi. \tag{12}$$

Here $\phi = k_0 z - \omega_0 t$. The wave number k_0 and frequency ω_0 of the pump wave are related by the following dispersion relation:

$$\omega_0^2 = k_0^2 V_A^2 \left(1 \pm \frac{\omega_0}{\omega_{cd}} \right), \tag{13}$$

where $V_A^2 = B^2/\mu_0 \rho$ is the Alfvén speed and $\omega_{cd} = ZeB/m_d$ is the dust-cyclotron frequency. The amplitudes A_0 and V_0 of the pump waves are related as

$$V_0 = -\left(\frac{A_0 \omega_0}{k_0 B} \right) \left(1 \pm \frac{\omega_0}{\omega_{cd}} \right)^{-1}. \tag{14}$$

It will be assumed now that the steady-state background consists of the unperturbed plasma plus the circularly polarized pump waves, i.e., $\mathbf{B} = (B_x(z), B_y(z), B)$ and $\mathbf{v} = (V_x(z), V_y(z), 0)$ with constant density. The linearization of Eqs. (8), (9), and (11) gives

$$\frac{\partial \delta\rho}{\partial t} + \nabla \cdot (\rho \delta\mathbf{v} + \delta\rho \mathbf{v}) = 0,$$

$$\frac{\partial \delta\mathbf{v}}{\partial t} + (\mathbf{v} + \delta\mathbf{v} \cdot \nabla)(\mathbf{v} + \delta\mathbf{v})$$
$$= -\frac{\nabla \delta P}{\rho} + \frac{\delta\rho}{\rho^2} \nabla P$$
$$+ \frac{1}{\rho} \left(\delta\mathbf{J} \times \mathbf{B} + J \times \delta\mathbf{B} - \frac{\delta\rho}{\rho} \mathbf{J} \times \mathbf{B} \right), \tag{15}$$

$$\frac{\partial \delta\mathbf{B}}{\partial t} = \nabla \times \left[\delta\mathbf{v} \times \mathbf{B} + \mathbf{v} \times \delta\mathbf{B} \right.$$
$$\left. - \frac{1}{Zen_d} \left(\delta\mathbf{J} \times \mathbf{B} + \mathbf{J} \times \delta\mathbf{B} - \frac{\delta\rho}{\rho} \mathbf{J} \times \mathbf{B} \right) \right].$$

The above set of Eqs. (15) is supplemented with an equation of state $\delta P = C_s^2 \delta\rho$. In what follows, it will be assumed that all quantities depend on z only. Then writing the above set of equations in the component form, we have

$$\frac{\partial \delta\rho}{\partial t} + \rho \frac{\partial \delta v_z}{\partial z} = 0,$$

$$\frac{\partial \delta v_x}{\partial t} + \delta v_z \frac{\partial V_x}{\partial z} = \frac{B}{\mu_0 \rho} \left(\frac{\partial \delta B_x}{\partial z} - \frac{\delta\rho}{\rho} \frac{\partial B_x}{\partial z} \right),$$

$$\frac{\partial \delta v_y}{\partial t} + \delta v_z \frac{\partial V_y}{\partial z} = \frac{B}{\mu_0 \rho} \left(\frac{\partial \delta B_y}{\partial z} - \frac{\delta\rho}{\rho} \frac{\partial B_y}{\partial z} \right),$$

$$\frac{\partial \delta v_z}{\partial t} = -\frac{C_s^2}{\rho} \frac{\partial \delta\rho}{\partial z} - \frac{1}{\mu_0 \rho} \left(B_x \frac{\partial \delta B_x}{\partial z} + B_y \frac{\partial \delta B_y}{\partial z} \right)$$
$$+ \frac{\delta\rho}{2\mu_0 \rho} \frac{\partial (B_x^2 + B_y^2)}{\partial z}, \tag{16}$$

$$\frac{\partial \delta B_x}{\partial t} + \delta v_z \frac{\partial B_x}{\partial z} = B_z \frac{\partial \delta v_x}{\partial z} - B_x \frac{\partial \delta v_z}{\partial z}$$
$$+ \alpha \left(\frac{\partial^2 \delta B_y}{\partial z^2} - \frac{1}{\rho} \frac{\partial \delta\rho}{\partial z} \frac{\partial \delta B_y}{\partial z} \right),$$

$$\frac{\partial \delta B_y}{\partial t} + \delta v_z \frac{\partial B_y}{\partial z} = B_z \frac{\partial \delta v_y}{\partial z} - B_y \frac{\partial \delta v_z}{\partial z}$$
$$- \alpha \left(\frac{\partial^2 \delta B_x}{\partial z^2} - \frac{1}{\rho} \frac{\partial \delta\rho}{\partial z} \frac{\partial \delta B_x}{\partial z} \right),$$

where $\alpha = B/(\mu_0 Zen_d)$. While writing the above equations, $\delta B_z = 0$ has been assumed. Equations (16) have periodic coefficients and belong to a general class of equations known as the Floquet equation. The Floquet theorem prescribes the form of the solution for the differential equation with periodic coefficients. However, the Floquet theorem also states that there exists a linear transformation that can reduce the system of ordinary differential equations (ODEs) with periodic coefficient to a system of ODEs with constant coefficient. For a nondispersive MHD case, such a transformation has recently been proposed.[20] In the present case, applying the same transformation, it is shown that the system of Eqs. (16) can be reduced to ODEs with constant coefficient. Defining

$$\delta B_+ = \delta B_x \cos\phi + \delta B_y \sin\phi,$$
$$\delta B_- = \delta B_x \sin\phi - \delta B_y \cos\phi,$$
$$\delta v_+ = \delta v_x \cos\phi + \delta v_y \sin\phi, \tag{17}$$
$$\delta v_- = \delta v_x \sin\phi - \delta v_y \cos\phi,$$

Eqs. (16) can be reduced to the following set of equations with constant coefficient:

$$\frac{\partial \delta\rho}{\partial t} + \rho\frac{\partial \delta v_z}{\partial z} = 0,$$

$$\frac{\partial \delta v_+}{\partial t} - \omega_0 \delta v_- = \frac{B}{\mu_0 \rho}\left(\frac{\partial \delta B_+}{\partial z} + k_0 \delta B_-\right),$$

$$\frac{\partial \delta v_-}{\partial t} + \omega_0 \delta v_+ - k_0 V_0 \delta v_z = \frac{B}{\mu_0\rho}\left(\frac{\partial \delta B_-}{\partial z} - k_0 \delta B_+\right) + k_0 \frac{B A_0}{\mu_0 \rho^2}\delta\rho,$$

$$\frac{\partial \delta v_z}{\partial t} = -\frac{C_s^2}{\rho}\frac{\partial \delta\rho}{\partial z} - \frac{A_0}{\mu_0\rho}\frac{\partial \delta B_+}{\partial z}, \tag{18}$$

$$\frac{\partial \delta B_+}{\partial t} - \omega_0 \delta B_- = B\left(\frac{\partial \delta v_+}{\partial z} + k_0 \delta v_-\right) - \alpha\left(\frac{\partial^2 \delta B_-}{\partial z^2} - 2k_0 \frac{\partial \delta B_+}{\partial z} + k_0^2 \delta B_-\right) - A_0 \frac{\partial \delta v_z}{\partial z} - \frac{\alpha k_0 A_0}{\rho}\frac{\partial \delta\rho}{\partial z},$$

$$\frac{\partial \delta B_-}{\partial t} + \omega_0 \delta B_+ = B\left(\frac{\partial \delta v_-}{\partial z} - k_0 \delta v_+\right) + \alpha\left(\frac{\partial^2 \delta B_+}{\partial z^2} + 2k_0 \frac{\partial \delta B_-}{\partial z} - k_0^2 \delta B_+\right) + k_0 A_0 \delta v_z.$$

One notes that the x and y components of the induction equation [Eqs. (16)] are symmetric with respect to the Hall terms (with coefficient α) whereas in Eq. (18) such symmetry does not exist. The reason for this loss of symmetry is due to the form of δB_+ and δB_-. The terms proportional to the density fluctuation in the induction Eq. (16) cancel out for δB_-. This results in the asymmetric equations for δB_+ and δB_-.

The above set of equations has constant coefficient, thus one can Fourier analyze the spatial and temporal dependence of the fluctuations as $\sim\exp(i\omega t - ikz)$.

IV. DISPERSION RELATION

Denoting $\omega=\omega/\omega_0$, $k=k/k_0$, $F_\pm=1/(1\pm\omega/\omega_{cd})$, $\beta = C_s^2/V_A^2$, and $C_s^2=k^2\beta F$ following eighth-order dispersion relation is derived from Eq. (18),

$$\omega^8 + a_7\omega^7 + \cdots + a_1\omega + a_0 = 0. \tag{19}$$

The coefficients $a_7,\ldots,a_0$ are given in the Appendix. The transition from stability to instability will proceed through $\omega=0$. In the $\omega\to 0$ limit,

$$\omega = -\frac{a_0}{a_1}. \tag{20}$$

In the long-wavelength limit retaining only $\sim O(k), O(k^2)$ terms in the coefficients a_1 and a_0, one may write

$$\frac{a_1}{kF_\pm} = -2\frac{A_0^2}{B^2}\left[1 - F_\pm\left(1+\frac{\omega_0}{\omega_{cd}}\right)\right] + k\frac{A_0^2}{B^2}(1+F_\pm), \tag{21}$$

$$\frac{a_0}{kF_\pm} = \left[\beta\left(1+F_\pm\frac{\omega_0}{\omega_{cd}}\right) - F_\pm\frac{A_0^2}{B^2}\right]\left[1 - F_\pm\left(1+\frac{\omega_0}{\omega_{cd}}\right)\right]. \tag{22}$$

Recall that for the left circularly polarized pump waves, $F_+ = 1/(1+\omega_0/\omega_{cd})$ and hence, from Eq. (20), $a_0=0$. Thus in view of Eq. (22), one should anticipate that the instability will disappear in the vicinity of $k=0$. As will be shown later, the numerical solution of the full dispersion relation, Eq. (19), indeed supports this conclusion.

For the right circularly polarized pump waves, when $F_-=1/(1-\omega_0/\omega_{cd})$, the growth rate can be written as

$$\mathrm{Im}[\omega] = \frac{F_-}{2}\frac{A_0^2}{B^2}\left(\beta - \frac{A_0^2}{B^2}\right). \tag{23}$$

As is clear from above Eq. (23), the growth rate is inversely proportional to the factor $(1-\omega_0/\omega_{cd})$. This implies that near $\omega_0 \simeq \omega_{cd}$, when the pump is operating near the dust-cyclotron frequency, the instability can grow resonantly. Therefore, the growth of the parametric instability is quite different for the left and right circularly polarized pump. Whereas for the left circularly polarized waves the instability does not exist in the neighborhood of $k=0$, for the right circularly polarized pump, the instability may become large near $k=0$. Clearly, for the right circularly polarized pump, unbounded growth of the instability is not possible. The linear approximation in which the dispersion relation (19) has been derived breaks down for any dependent physical variable f once $\delta f \leqslant f$. Therefore, in the vicinity of resonance, nonlinear terms need to be considered. This task is beyond the scope of the present work.

Now the dispersion relation (19) is solved numerically. In Fig. 1(a), the normalized growth rate is plotted against the normalized wave number for different values of ω_0/ω_{cd}. The left circularly polarized pump is assumed. The instability does not exist in the vicinity of $k\to 0$, and $k=0.2$ is the threshold of the instability. This is in agreement with the above discussion, where in the vicinity of $k=0$ the wave does not grow. This is caused by the dissipation of the long-wavelength fluctuations by the plasma-dust collisions. With

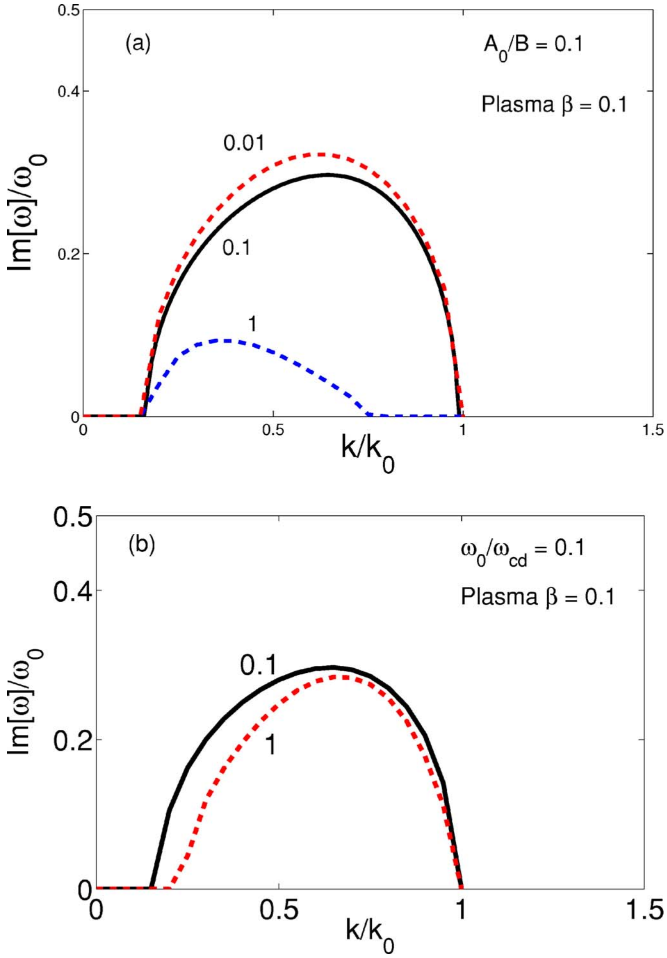

FIG. 1. The dependence of the growth rate on the ratio of the left circularly polarized pump wave to the dust-cyclotron frequency is shown in (a) for the corresponding physical parameters in the box. In (b), the dependence of the growth rate on the amplitude of the pump wave A_0/B is shown.

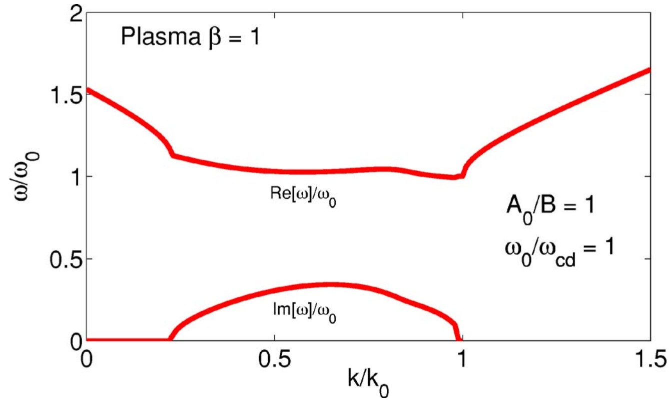

FIG. 2. Both $\mathrm{Re}[\omega_r/\omega_0]$ and $\mathrm{Im}[\omega_r/\omega_0]$ are plotted against k/k_0 for the plasma β, ω_0/ω_{cd}, and A_0/B shown in the box.

the increase in ω_0/ω_{cd}, the growth rate decreases. As is clear from the pump wave dispersion relation (13), the increase in ω_0/ω_{cd} implies the increasing importance of the Hall term $(\mathbf{J}\times\mathbf{B})$, which appears due to the relative drift between the plasma and the dust. This drift is caused entirely by the collisional momentum exchange. Therefore, the increase in ω_0/ω_{cd} implies the increased dissipation of the free energy. Hence with the increasing ω_0/ω_{cd}, one would expect a decrease in the growth rate. With decreasing ω_0/ω_{cd}, the growth rate is due to the nondispersive Alfvén pump and will correspond to a nondissipative, ideal regime. Hence one sees the saturation of the growth rate with decreasing ω_0/ω_{cd}.

In Fig. 1(b), the growth rate of the parametric instability is shown against the variation of the pump amplitude A_0/B. Recall that B is the amplitude of the constant magnetic field in the z direction, and A_0 is the amplitude of the transverse x-y fields. The growth rate is not very sensitive toward smaller wavelength, i.e., after $k/k_0 \geqslant 0.5$. However, the onset of instability is slightly more delayed when $A_0/B=1$ than when $A_0/B=0.1$, implying that the effect of collision is less pronounced for a stronger B_z field.

In Fig. 2, both real and imaginary frequencies are shown for $A_0/B=1$ and $\omega_0/\omega_{cd}=1$. One sees from 2 that from the onset until the demise of the instability, the real part of the frequency is constant. The part of free energy that is responsible for setting the normal oscillation in the system has been diverted toward the wave growth. If one superposes the imaginary part on the real frequency, then one gets an almost constant ω_r. This should be expected since the total energy available to the system is due to the Alfvén pump, and thus the growth of the fluctuation occurs at the expense of the oscillatory real part.

The growth rate is not sensitive to the plasma β. Unlike in the nondispersive, nondissipative case in which the increase of β leads to the increase in the growth rate,[18] in the present dissipative, dispersive case, plasma β appears to have no role. In the nondissipative case, the increase in β implies the unlikelihood of compressional wave excitation; the role of plasma β in the dissipative case is unclear. The dispersive nature of the pump wave is due to the collisional effects, thus β is not directly related to the growth of the waves.

In Figs. 3(a) and 3(b), respectively, the growth rate and the corresponding real part of the frequency are shown for the right circularly polarized pump. When $\omega_0/\omega_{cd}=0.9$, the growth rate becomes very large. The free energy is resonantly pumped into the fluctuations with increasing ω_0/ω_{cd}. The physical system behaves like a driven oscillator. The resonant driving is indirectly related to the dust-plasma collisions. The relative drift between the plasma and the dust causes a Hall field over the dust-cyclotron time. If the Alfvén-wave propagation time ω_0^{-1} becomes comparable to the dust-cyclotron time, the energy is freely fed to the fluctuation by the pump to the gyrating dust particles. The resulting free energy to the fluctuation causes the large growth rate. This behavior has been analytically predicted in the limiting case by Eq. (23). It should be noted from the corresponding curve in Fig. 3(b) that the real frequency shows a sharp decline for $\omega_0/\omega_{cd}=0.9$. This suggest that almost all the free pump energy is resonantly used in the fluctuation growth. Similar behavior is also noted for $\omega_0/\omega_{cd}=0.8$ and $\omega_0/\omega_{cd}=0.01$. Since the instability growth rate in this case is smaller than when $\omega_0/\omega_{cd}=0.9$, part of the available free energy remains in the real part of the frequency. This is in agreement with the well known behavior of the oscillators

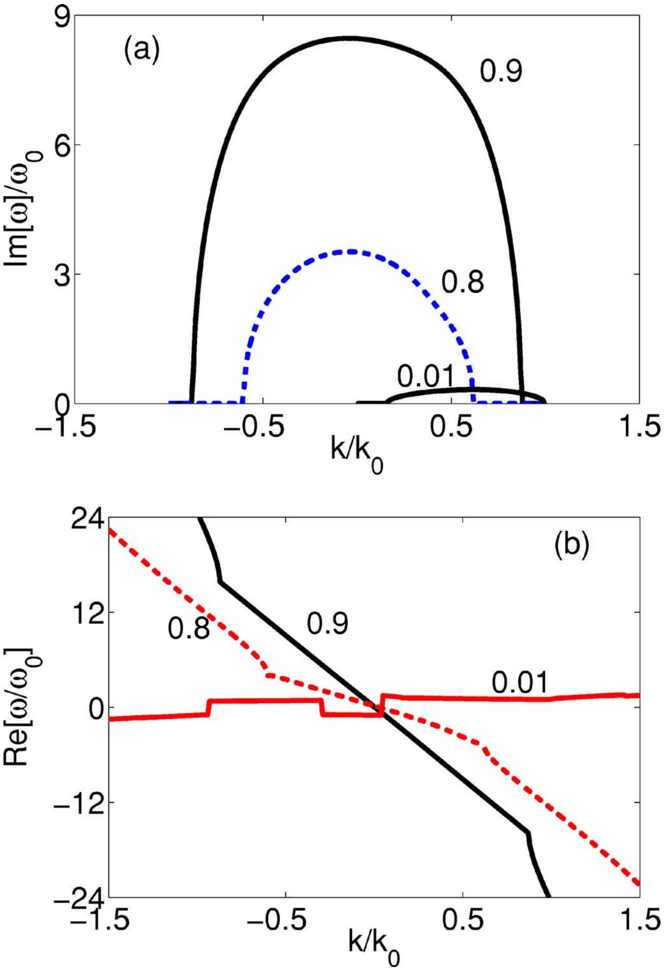

FIG. 3. The dependence of the growth rate on the amplitude of the right circularly polarized pump wave is shown in (a) and the corresponding real part of the frequency is shown in (b) for different values of ω/ω_{cd}. The value of plasma β and A_0/B is 0.1.

near resonance, although the present system is more complex.

V. APPLICATION AND SUMMARY

The grains are important charge carriers in the molecular clouds and their presence can significantly alter the dynamics of the protostellar disk. The ionization fraction of the plasma is strongly affected by the abundance and size distribution of the grains through the recombination process on the grain surface. The observations of the molecular cloud suggest a magnetically threaded supersonically turbulent environment,[29] and the generation of magnetohydrodynamic waves in such an environment is inevitable. Thus taking the upper value of the pump wave frequency,[29] $\omega_0 \simeq 10^{-4}$ yr^{-1}, the growth rate of the left circularly polarized wave $0.3\omega_0$ suggests that the parametric instability of the Alfvén wave could be relevant to the onset of turbulence. The Alfvén wavelength lies in the range of 0.07–0.35 pc (Ref. 29), and since the maximum growth rate occurs at $k/k_0=0.5$ [Fig. 1(a)], the parametric instability of the left circularly polarized mode will operate between 0.1 and 0.7 pc.

To check whether the resonance condition can prevail in the molecular clouds, assuming a typical magnetic field $B \sim 10\ \mu$G, and taking interstellar grain mass 10^{-15} g, one gets $\omega_{cd} \sim 10^{-4}$ yr^{-4}, which is comparable to the pump frequency. Therefore, the right circularly polarized wave can resonantly excite this instability. Furthermore, it can operate on very long wavelengths.

To summarize, parametric instability of a dusty medium may play an important role in exciting the turbulence in the interstellar medium. However, before a compelling argument is put forth in support of the present hypothesis, the dynamics of the neutrals need to be considered. This will be considered in future work. Following is an itemized summary of the present work.

(i) The parametric instability of collisional dusty plasma is studied in the present work. It is shown that the collision results in the dispersive nature of the pump waves.

(ii) Using a recently proposed linear transformation,[20] the linearized equations with periodic coefficients are reduced to a form with constant coefficient allowing the validity of the normal-mode analysis at all wavelengths.

(iii) The instability is sensitive to the change in the ratio of the pump and the cyclotron frequencies and is weakly sensitive to the ratio A_0/B. The growth rate is not sensitive to the plasma β.

(iv) The instability can become an order of magnitude larger for the right circularly polarized pump, particularly near the resonance, i.e., when $\omega_0 \simeq \omega_{cd}$.

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APPENDIX A: COEFFICIENT OF THE DISPERSION RELATION (19)

$$a_7 = 4kF_{\pm}, \tag{A1}$$

$$\frac{a_6}{F_{\pm}} = -\left[1 + 2\frac{\omega_0}{\omega_{cd}} + k^2\frac{\omega_0}{\omega_{cd}}\right]\left[1 + k^2\frac{\omega_0}{\omega_{cd}}\right] - k^2 - 2F_{\pm}^{-1} - \beta k^2 - k^2\left(\frac{A_0}{B}\right)^2 + 4k^2F_{\pm}\left(\frac{\omega_0}{\omega_{cd}}\right)^2 - (1+k^2), \tag{A2}$$

$$\frac{a_5}{kF_\pm} = \left[1+\left(\frac{A_0}{B}\right)^2\right]\left[1-\frac{\omega_0}{\omega_{cd}}(1-k^2)F_\pm\right] + 2\left[1+\frac{\omega_0}{\omega_{cd}}(1+k^2)F_\pm\right] - \left(\frac{A_0}{B}\right)^2 + 1 - 4\frac{\omega_0}{\omega_{cd}} - 2\frac{\omega_0}{\omega_{cd}}k^2\beta F_\pm - \left(\frac{A_0}{B}\right)^2 F_\pm k^2 \frac{\omega_0}{\omega_{cd}}$$
$$- 2\frac{\omega_0}{\omega_{cd}}\left[k^2F_\pm + 1 + k^2\beta F_\pm + \left(\frac{A_0}{B}\right)^2 F_\pm k^2\right] - 2\frac{\omega_0}{\omega_{cd}}[1+F_\pm(1+k^2)] + k^{-1}(1+k), \quad \text{(A3)}$$

$$\frac{a_4}{kF_\pm} = k\beta + k\left(\frac{A_0}{B}\right)^2 + k\left\{F_\pm\left(\frac{A_0}{B}\right)^2 + \left[k^2\beta F_\pm + \left(\frac{A_0}{B}\right)^2\right]\right\} + 2kF_\pm\frac{\omega_0}{\omega_{cd}}\left[-\left(\frac{A_0}{B}\right)^2 + 1 - 2\frac{\omega_0}{\omega_{cd}} - 2k^2\beta F_\pm\frac{\omega_0}{\omega_{cd}} - k^2F_\pm\frac{\omega_0}{\omega_{cd}}\left(\frac{A_0}{B}\right)^2\right]$$
$$+ k\left[1 + k^{-2}F_\pm^{-1} + \beta + \left(\frac{A_0}{B}\right)^2\right][1+F_\pm(1+k^2)] - 2\frac{\omega_0}{\omega_{cd}}F_\pm\left[2k\frac{\omega_0}{\omega_{cd}} - 1 - k\right] + k\left\{\left(\frac{A_0}{B}\right)^2 + k^{-2}\frac{\omega_0}{\omega_{cd}}(1+k^2)\right.$$
$$\left. + \beta\left[1 + F_\pm\frac{\omega_0}{\omega_{cd}}(1+k^2)\right] - 1 + k^{-2}F_\pm^{-1}\right\}\left[1 - \frac{\omega_0}{\omega_{cd}}(1-k^2)F_\pm\right] - 2kF_\pm\left[1+\left(\frac{A_0}{B}\right)^2\right]$$
$$+ \left[1 + \frac{\omega_0}{\omega_{cd}}(1+k^2)F_\pm\right]\left[k^{-1}F_\pm^{-1} - \frac{\omega_0}{\omega_{cd}}k^{-1}(1-k^2) - k^{-1}(1+k^2)\right], \quad \text{(A4)}$$

$$\frac{a_3}{kF_\pm} = -\left\{\left(\frac{A_0}{B}\right)^2 + k^2F_\pm\left[\beta + \left(\frac{A_0}{B}\right)^2\right] + \left(\frac{A_0}{B}\right)^2\right\}\left[1 - \frac{\omega_0}{\omega_{cd}}(1-k^2)F_\pm\right] - 2kF_\pm\left\{k\left[\left(\frac{A_0}{B}\right)^2 - 1\right] + \frac{1}{kF_\pm}\right.$$
$$\left. + k\beta\left[1 + \frac{\omega_0}{\omega_{cd}}(1+k^2)F_\pm\right] + \frac{\omega_0}{\omega_{cd}}\frac{(1+k^2)}{k}\right\} - \left[1 + \left(\frac{A_0}{B}\right)^2\right]\left[1 - \frac{\omega_0}{\omega_{cd}}F_\pm(1-k^2) - F_\pm(1+k^2)\right]$$
$$+ 2k^2F_\pm\frac{\omega_0}{\omega_{cd}}\left[\beta + \left(\frac{A_0}{B}\right)^2\right] - \left[k^2\beta F_\pm + \left(\frac{A_0}{B}\right)^2 F_\pm k^2\right] + 2kF_\pm\frac{\omega_0}{\omega_{cd}}\left(k\left[\beta + \left(\frac{A_0}{B}\right)^2\right] + k\left\{F_\pm\left(\frac{A_0}{B}\right)^2\right.\right.$$
$$\left.\left. + k\left[k^2\beta F_\pm + \left(\frac{A_0}{B}\right)^2\right]\right\}\right) + \left[\left(\frac{A_0}{B}\right)^2 - 1 + 2\frac{\omega_0}{\omega_{cd}} + 2\frac{\omega_0}{\omega_{cd}}k^2\beta F_\pm + \left(\frac{A_0}{B}\right)^2 F_\pm k\frac{\omega_0}{\omega_{cd}}\right][1+F_\pm(1+k^2)]$$
$$+ \left[k^2F_\pm + 1 + k^2\beta F_\pm + \left(\frac{A_0}{B}\right)^2 F_\pm k^2\right]\left[2kF_\pm\frac{\omega_0}{\omega_{cd}} - F_\pm(1+k)\right], \quad \text{(A5)}$$

$$\frac{a_2}{kF_\pm} = -\left\{k\beta\left[1 + \frac{\omega_0}{\omega_{cd}}(1+k^2)F_\pm\right] - kF_\pm\left[k^2\beta + \left(\frac{A_0}{B}\right)^2\right]\right\}\left[1 - \frac{\omega_0}{\omega_{cd}}(1-k^2)F_\pm\right] + 2kF_\pm\left\{2\left(\frac{A_0}{B}\right)^2 + k^2F_\pm\left[\beta + \left(\frac{A_0}{B}\right)^2\right]\right\}$$
$$+ \left\{k\left[\left(\frac{A_0}{B}\right)^2 - 1\right] + \frac{1}{kF_\pm} + \frac{\omega_0}{\omega_{cd}}F_\pm\frac{1+k^2}{k} + k\beta\left[1 + \frac{\omega_0}{\omega_{cd}}(1+k^2)F_\pm\right]\right\}\left[-1 + \frac{\omega_0}{\omega_{cd}}F_\pm(1-k^2) + F_\pm(1+k^2)\right]$$
$$+ 2kF_\pm\left\{2k\beta\frac{\omega_0}{\omega_{cd}} + k\frac{\omega_0}{\omega_{cd}}\left(\frac{A_0}{B}\right)^2 - k^2F_\pm\frac{\omega_0}{\omega_{cd}}\left[\beta + \left(\frac{A_0}{B}\right)^2\right]\right\} - \left\{k\beta + k\left(\frac{A_0}{B}\right)^2 + kF_\pm\left(\frac{A_0}{B}\right)^2\right.$$
$$\left. + k\left[k^2\beta F_\pm + \left(\frac{A_0}{B}\right)^2\right]\right\}[1+F_\pm(1+k^2)] + F_\pm\left\{\left(\frac{A_0}{B}\right)^2 - 1 + \frac{\omega_0}{\omega_{cd}} + 2k^2F_\pm\frac{\omega_0}{\omega_{cd}}\left[2\beta + \left(\frac{A_0}{B}\right)^2\right]\right\}\left(2k\frac{\omega_0}{\omega_{cd}} - 1 - k\right), \quad \text{(A6)}$$

$$\frac{a_1}{kF_\pm} = 2k^2\beta F_\pm\left\{1 + F_\pm\left[\frac{\omega_0}{\omega_{cd}} + k^2\left(\frac{\omega_0}{\omega_{cd}} - 1\right)\right] - \frac{F_\pm A_0^2}{\beta B^2}\right\} - \left(\frac{A_0}{B}\right)^2\left\{2 + F_\pm k^2\left[1 + \beta\left(\frac{B}{A_0}\right)^2\right]\right\}\left\{1 - F_\pm\left[\frac{\omega_0}{\omega_{cd}}(1-k^2) + 1 + k^2\right]\right\}$$
$$- k^2F_\pm\left[\beta\left(2\frac{\omega_0}{\omega_{cd}} - 1\right) + \left(\frac{A_0}{B}\right)^2\left(\frac{\omega_0}{\omega_{cd}} - 1\right)\right][1+F_\pm(1+k^2)] - k^2F_\pm\left[\beta(1+k^2F_\pm) + \left(\frac{A_0}{B}\right)^2(2+F_\pm)\right]\left(2\frac{\omega_0}{\omega_{cd}} - \frac{1+k^2}{k}\right),$$
$$\text{(A7)}$$

$$\frac{a_0}{k^2F_\pm} = kF_\pm^2\left[\left(2\beta + \frac{A_0^2}{B^2}\right)\frac{\omega_0}{\omega_{cd}} - \left(\beta + \frac{A_0^2}{B^2}\right)\right]\left(-2F_\pm\frac{\omega_0}{\omega_{cd}} + 1 + k\right) + \left[\beta F_\pm\frac{\omega_0}{\omega_{cd}}(1+k^2) + \beta - F_\pm\left(\beta k^2 + \frac{A_0^2}{B^2}\right)\right]$$
$$\times\left\{\left[1 - \frac{\omega_0}{\omega_{cd}}(1-k^2)F_\pm\right] - F_\pm(1+k^2)\right\}. \quad \text{(A8)}$$

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