

## DUALITY IN LINEAR PROGRAMMING PROBLEMS RELATED TO DETERMINISTIC LONG RUN AVERAGE PROBLEMS OF OPTIMAL CONTROL\*

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**Abstract.** It has been established recently that, under mild conditions, deterministic long run average problems of optimal control are “asymptotically equivalent” to infinite-dimensional linear programming problems (LPPs) and that these LPPs can be approximated by finite-dimensional LPPs. In this paper we introduce the corresponding infinite- and finite-dimensional dual problems and study duality relationships. We also investigate the possibility of using solutions of finite-dimensional LPPs and their duals for numerical construction of the optimal controls in periodic optimization problems. The construction is illustrated with a numerical example.

**Key words.** long run average optimal control, occupational measures, averaging, linear programming, duality

**AMS subject classifications.** 34E15, 34C29, 34A60, 93C70

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**1. Introduction.** Infinite horizon problems of optimal control have been studied intensively in both deterministic and stochastic settings (see [1], [2], [3], [6], [7], [8], [9], [12], [14], [18], [19], [20], [21], [23], [26], [31], [37], [42], [43], [47], [56], and references therein for a sample of the literature on the subject) with linear programming formulations being one of the main tools of treating stochastic problems (see, e.g., [13], [16], [5], [38], [39], [41], [46], [51], [52], [60]).

A linear programming approach to deterministic long run average problems of optimal control was considered in [32] and [33], where it was established that these problems are “asymptotically equivalent” to infinite-dimensional (I-D) linear programming problems (LPPs) similar to those arising in stochastic control (see [13], [16], [41], [52]), and it has been shown that these I-D LPPs can be approximated by finite-dimensional (F-D) LPPs (F-D approximations of I-D LPPs arising in stochastic control problems and in deterministic problems on finite intervals of time have been studied in [38], [46], and in [48], respectively; F-D approximations of I-D LPPs arising in certain problems of calculus of variations have been considered in [24]).

In this paper we introduce problems dual to the LPPs considered in [32] and [33] (both F-D and I-D), and we study the corresponding duality relationships. We also investigate the possibility of using solutions of F-D LPPs and their duals for numerical construction of optimal controls in periodic optimization problems (thus refining the corresponding results of [32], where the option of using dual solutions was not considered).

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Periodic optimization problems (POPs) constitute a special family of long run average problems of optimal control, in which solutions are sought over the class of periodic regimes. POPs arise in various applications, including vibration damping, production planning, flight control, chemical engineering, ecological modeling (see [40], [45], [50], [53], [59], and references therein). A linear programming approach that we continue to develop in the present paper provides a new analytical and numerical tool for dealing with this important family of control problems.

The paper is organized as follows. Sections 2 and 3 contain some preliminaries, the purpose of which is to help the reader put the developments of the subsequent sections into perspective. In section 2, long run average problems of optimal control and their reformulations in terms of occupational measures are considered. In section 3, some results of [32] and [33] that establish connections between the long run average problems of optimal control and the I-D LPPs are restated.

Sections 4, 5, and 6 contain the main duality results. In section 4, the problem dual to the I-D LPP is introduced and duality relationships are established (Theorem 4.1). The dual problem proves to be closely related to the Hamilton–Jacobi–Bellman equation. Its solution is used to state some necessary and sufficient optimality conditions (Proposition 4.3 and Corollaries 4.4 and 4.5). In section 5, duality relationships for an I-D LPP with a finite number of constraints are discussed (Theorem 5.2) and convergence properties (as the number of constraints goes to infinity) are established (Proposition 5.1 and Theorem 5.6). In section 6, an F-D LPPs (defined on grid points) and its dual are considered, with their convergence properties (as the grid size goes to zero) being established (Theorems 6.1 and 6.2). Also, in this section, a construction of an approximate solution of the problem dual to the I-D LPPs introduced in section 4 is discussed (Proposition 6.3).

Sections 7 and 8 are devoted to construction of solutions of periodic optimization problems. In section 7, it is established that, under certain conditions, the control found with the help of an approximate solution of the problem dual to the I-D LPP (constructed in section 6) converges to the optimal control (Theorem 7.1 and Corollary 7.2). In section 8, a numerical example is considered to illustrate the construction.

In sections 9 and 10, the proofs for sections 4, 5, 6, and 7, respectively, are collected.

**2. Preliminaries I: Occupational measures formulations.** Consider the control system

$$(2.1) \quad \dot{y}(\tau) = f(u(\tau), y(\tau)), \quad \tau \in [0, S],$$

where the function  $f(u, y) : U \times \mathbb{R}^m \rightarrow \mathbb{R}^m$  is continuous in  $(u, y)$  and satisfies Lipschitz conditions in  $y$ ; the controls are Lebesgue measurable functions  $u(\tau) : [0, S] \rightarrow U$  and  $U$  is a compact metric space.

A pair  $(u(\tau), y(\tau))$  will be called *admissible* on the interval  $[0, S]$  if (2.1) is satisfied for almost all  $\tau \in [0, S]$  and  $y(\tau) \in Y \quad \forall \tau \in [0, S]$ , where  $Y$  is a given compact subset of  $\mathbb{R}^m$ . The pair will be called admissible on  $[0, \infty)$  if it is admissible on any interval  $[0, S]$ ,  $S > 0$ .

Let  $\mathcal{P}(U)$  be the space of probability measures defined on the Borel subsets of  $U$  and  $\bar{f}(\nu, y) \stackrel{\text{def}}{=} \int_U f(u, y)\nu(du)$ . Along with (2.1), let us consider a relaxed control system

$$(2.2) \quad \dot{y}(\tau) = \bar{f}(\nu(\tau), y(\tau)), \quad \tau \in [0, S],$$

in which controls are measurable functions  $\nu(\tau) \in \mathcal{P}(U)$  (see [58]).

A pair  $(\nu(\tau), y(\tau))$  will be called *relaxed admissible* on the interval  $[0, S]$  if (2.2) is satisfied for almost all  $\tau \in [0, S]$  and  $y(\tau) \in Y \forall \tau \in [0, S]$ . The pair will be called relaxed admissible on  $[0, \infty)$  if it is relaxed admissible on any interval  $[0, S]$ ,  $S > 0$ .

Let  $g(u, y) : U \times \mathbb{R}^m \rightarrow \mathbb{R}^1$  be a continuous function. We will be considering the optimal control problem

$$(2.3) \quad \frac{1}{S} \inf_{(u(\cdot), y(\cdot))} \int_0^S g(u(\tau), y(\tau)) d\tau \stackrel{\text{def}}{=} G(S)$$

and the corresponding relaxed optimal control problem

$$(2.4) \quad \frac{1}{S} \inf_{(\nu(\cdot), y(\cdot))} \int_0^S \bar{g}(\nu(\tau), y(\tau)) d\tau \stackrel{\text{def}}{=} \bar{G}(S),$$

where  $\bar{g}(\nu, u) \stackrel{\text{def}}{=} \int_U g(u, y)\nu(du)$ , and the first (second) inf is over all admissible (respectively, relaxed admissible) pairs.

Let  $\mathcal{P}(U \times Y)$  stand for the space of probability measures defined on the Borel subsets of  $U \times Y$ . Given an admissible pair  $(u(\cdot), y(\cdot))$ , a probability measure  $\gamma \in \mathcal{P}(U \times Y)$  will be said to be generated by this pair on the interval  $[0, S]$  if

$$(2.5) \quad \int_{U \times Y} q(u, y)\gamma(du, dy) = \frac{1}{S} \int_0^S q(u(\tau), y(\tau)) d\tau$$

for any  $q(\cdot) \in C(U \times Y)$  (the space of continuous functions on  $U \times Y$ ). A probability measure  $\gamma \in \mathcal{P}(U \times Y)$  will be said to be generated by an admissible pair  $(u(\cdot), y(\cdot))$  on  $[0, \infty)$  if

$$(2.6) \quad \int_{U \times Y} q(u, y)\gamma(du, dy) = \lim_{S \rightarrow \infty} \frac{1}{S} \int_0^S q(u(\tau), y(\tau)) d\tau$$

for any  $q(\cdot) \in C(U \times Y)$  (the limit in the right-hand side of (2.6) is assumed to exist). Given a relaxed admissible pair  $(\nu(\tau), y(\tau))$ , a probability measure  $\gamma \in \mathcal{P}(U \times Y)$  will be said to be generated by this pair on the interval  $[0, S]$  if

$$(2.7) \quad \int_{U \times Y} q(u, y)\gamma(du, dy) = \frac{1}{S} \int_0^S \bar{q}(\nu(\tau), y(\tau)) d\tau$$

for any  $q(\cdot) \in C(U \times Y)$ , with

$$(2.8) \quad \bar{q}(\nu, u) \stackrel{\text{def}}{=} \int_U q(u, y)\nu(du).$$

A probability measure  $\gamma \in \mathcal{P}(U \times Y)$  will be said to be generated by an admissible pair  $(\nu(\cdot), y(\cdot))$  on  $[0, \infty)$  if

$$(2.9) \quad \int_{U \times Y} q(u, y)\gamma(du, dy) = \lim_{S \rightarrow \infty} \frac{1}{S} \int_0^S \bar{q}(\nu(\tau), y(\tau)) d\tau$$

for any  $q(\cdot) \in C(U \times Y)$  and  $\bar{q}(\cdot)$  is as in (2.8) (the limit in the right-hand side of (2.9) is assumed to exist).

Thus defined probability measures are called occupational measures (see, e.g., [7] and [30]). In what follows  $\gamma^{(S, u(\cdot), y(\cdot))}$ ,  $\gamma^{(u(\cdot), y(\cdot))}$  will stand for the occupational measures generated by an admissible pair  $(u(\cdot), y(\cdot))$  on  $[0, S]$  and  $[0, \infty)$ , respectively, and  $\gamma^{(S, \nu(\cdot), y(\cdot))}$ ,  $\gamma^{(\nu(\cdot), y(\cdot))}$  will stand for the occupational measures generated by a relaxed admissible pair  $(\nu(\cdot), y(\cdot))$  on  $[0, S]$  and  $[0, \infty)$ , respectively.

Denote by  $\Gamma(S) \subset \mathcal{P}(U \times Y)$  and  $\bar{\Gamma}(S) \subset \mathcal{P}(U \times Y)$  the set of all occupational measures generated by the admissible (respectively, relaxed admissible) pairs on the interval  $[0, S]$ . That is,

$$(2.10) \quad \Gamma(S) \stackrel{\text{def}}{=} \bigcup_{(u(\cdot), y(\cdot))} \left\{ \gamma^{(S, u(\cdot), y(\cdot))} \right\}, \quad \bar{\Gamma}(S) \stackrel{\text{def}}{=} \bigcup_{(\nu(\cdot), y(\cdot))} \left\{ \gamma^{(S, \nu(\cdot), y(\cdot))} \right\},$$

where the unions are over all admissible and, respectively, relaxed admissible pairs on  $[0, S]$ . Using these notations, one can rewrite the problem (2.3) as

$$(2.11) \quad \inf_{\gamma \in \Gamma(S)} \int_{U \times Y} g(u, y) \gamma(du, dy) = G(S)$$

and the problem (2.4) as

$$(2.12) \quad \inf_{\gamma \in \bar{\Gamma}(S)} \int_{U \times Y} g(u, y) \gamma(du, dy) = \bar{G}(S).$$

Note that  $\Gamma(S) \subset \bar{\Gamma}(S)$  and  $G(S) \geq \bar{G}(S)$ . In a special case when  $Y$  is forward invariant with respect to the solutions of (2.1), from Filippov–Wazewski’s theorem (see, e.g., Theorem 10.4.4 in [11]) it follows that  $\bar{\Gamma}(S)$  is equal to the closure of  $\Gamma(S)$  and  $G(S) = \bar{G}(S)$  for any  $S > 0$ .

Let us introduce a metric  $\rho(\cdot, \cdot)$  on  $\mathcal{P}(U \times Y)$  by the equation

$$(2.13) \quad \rho(\gamma', \gamma'') \stackrel{\text{def}}{=} \sum_{j=1}^{\infty} \frac{1}{2^j} \left| \int_{U \times Y} q_j(u, y) \gamma'(du, dy) - \int_{U \times Y} q_j(u, y) \gamma''(du, dy) \right|,$$

where  $\gamma', \gamma'' \in \mathcal{P}(U \times Y)$ , and  $q_j(\cdot)$ ,  $j = 1, 2, \dots$ , is a sequence of Lipschitz continuous functions that is dense in the unit ball of  $C(U \times Y)$ . Note that this metric is consistent with the topology of weak convergence in  $\mathcal{P}(U \times Y)$ . Namely, a sequence  $\gamma^k \in \mathcal{P}(U \times Y)$  converges to  $\gamma \in \mathcal{P}(U \times Y)$  in this metric if and only if

$$(2.14) \quad \lim_{k \rightarrow \infty} \int_{U \times Y} q(u, y) \gamma^k(du, dy) = \int_{U \times Y} q(u, y) \gamma(du, dy)$$

for any  $q(\cdot) \in C(U \times Y)$ . Note also that the space  $\mathcal{P}(U \times Y)$  is known to be compact in the topology of weak convergence (see, e.g., [15]), and, hence, being equipped with the metric (2.13), it becomes a compact metric space.

Using  $\rho(\cdot, \cdot)$ , one can define the Hausdorff metric on the subsets of  $\mathcal{P}(U \times Y)$  as follows: for any  $\Gamma_1 \subset \mathcal{P}(U \times Y)$  and  $\Gamma_2 \subset \mathcal{P}(U \times Y)$ ,

$$(2.15) \quad \rho_H(\Gamma_1, \Gamma_2) \stackrel{\text{def}}{=} \max \left\{ \sup_{\gamma \in \Gamma_1} \rho(\gamma, \Gamma_2), \sup_{\gamma \in \Gamma_2} \rho(\gamma, \Gamma_1) \right\},$$

$$\rho(\gamma, \Gamma_i) \stackrel{\text{def}}{=} \inf_{\gamma' \in \Gamma_i} \rho(\gamma, \gamma'), \quad i = 1, 2.$$

Note that, although, by some abuse of terminology, we call  $\rho_H(\cdot, \cdot)$  a metric on the set of subsets of  $\mathcal{P}(U \times Y)$ , it is, in fact, a semimetric on this set (since  $\rho_H(\Gamma_1, \Gamma_2) = 0$  implies that  $\Gamma_1 = \Gamma_2$  if and only if  $\Gamma_1$  and  $\Gamma_2$  are closed).

**3. Preliminaries II: Limit linear programming problem.** Define the set  $W \subset \mathcal{P}(U \times Y)$  by the equation

$$(3.1) \quad W \stackrel{\text{def}}{=} \left\{ \gamma \in \mathcal{P}(U \times Y) : \int_{U \times Y} (\phi'(y))^T f(u, y) \gamma(du, dy) = 0 \quad \forall \phi(\cdot) \in C^1 \right\},$$

where  $C^1$  is the space of continuously differentiable functions  $\phi(y) : \mathbb{R}^m \rightarrow \mathbb{R}^1$  and  $\phi'(y)$  is a vector column of partial derivatives (the gradient) of  $\phi(y)$ .

It is easy to see that  $W$  is closed and compact in weak convergence topology of  $\mathcal{P}(U \times Y)$ . Also, it is easy to see that  $W$  is convex.

Assuming that  $W$  is not empty (see Remark 4.2 below about a necessary and sufficient condition for this to be the case), let us consider the problem

$$(3.2) \quad \min_{\gamma \in W} \int_{U \times Y} g(u, y) \gamma(du, dy) \stackrel{\text{def}}{=} G_*,$$

where  $g(\cdot)$  is the same as in (2.3) and (2.11). Since both the objective function and the constraints defining  $W$  are linear in  $\gamma$ , the problem (3.2) is that of I-D linear programming (see, e.g., [4]). Note that a problem similar to (3.2) was introduced in [41] and [52] in a much more general stochastic setting. The problem (3.2) will in what follows be referred to as the limit linear programming problem (LLPP). In the rest of this section we restate some results of [32] and [33] that are important for our further consideration.

**THEOREM 3.1.** (i) *If the set  $W$  is empty, then there exists  $S_0 > 0$  such that  $\bar{\Gamma}(S)$  (and, hence,  $\Gamma(S)$ ) are empty for  $S \geq S_0$ . If  $W$  is not empty, then  $\bar{\Gamma}(S)$  is not empty for  $S > 0$ . If  $W$  is not empty and, in addition, the set  $f(U, y) \stackrel{\text{def}}{=} \{\eta : \eta = f(u, y), u \in U\}$  is convex for all  $y \in Y$ , then  $\Gamma(S)$  is not empty for  $S > 0$ .*

(ii) *Let  $W$  be not empty. Then*

$$(3.3) \quad \lim_{S \rightarrow \infty} \rho_H(\text{co}\bar{\Gamma}(S), W) = 0$$

and

$$(3.4) \quad \lim_{S \rightarrow \infty} \bar{G}(S) = G_*.$$

(iii) *If  $Y$  is forward invariant, then*

$$(3.5) \quad \lim_{S \rightarrow \infty} \rho_H(\text{co}\Gamma(S), W) = 0$$

and

$$(3.6) \quad \lim_{S \rightarrow \infty} G(S) = G_*.$$

*Proof.* The statements (i) and (ii) have been established in [33], their proofs being similar to the proofs of Theorem 2.1 in [29] and Proposition 5 in [32]. The statement (iii) is implied by Proposition 5 in [32].  $\square$

**Remark 3.2.** In the terminology introduced in [10], the fact that the set  $\Gamma(s)$  (or  $\bar{\Gamma}(s)$ ) is not empty for all  $S > 0$  means that the *viability kernel* of the control system (2.1) (respectively, relaxed control system (2.2)) is not empty in  $Y$ . Hence, from Theorem 3.1(i) it follows that the viability kernel of the relaxed control system (2.2) is not empty in  $Y$  if and only if  $W$  is not empty, and, under the condition that

the set  $f(U, y)$  is convex for all  $y \in Y$ , the viability kernel of the control system (2.1) is not empty in  $Y$  if and only if  $W$  is not empty. Note that an alternative way of establishing this statement was proposed by Quincampoix, and it will be included in a separate publication.

**THEOREM 3.3.** *Assume that  $W$  is not empty. Then, corresponding to any extreme point  $\gamma$  of  $W$ , there exists a relaxed admissible pair  $(\nu(\cdot), y(\cdot))$  such that  $\gamma$  is generated by this pair on  $[0, \infty)$  (that is,  $\gamma = \gamma^{(\nu(\cdot), y(\cdot))}$ ).*

*Proof.* The theorem has been established in [33]. The proof is based on Proposition 4.4 in [17] and on Lemma 5.1 in [29].  $\square$

Consider the problem

$$(3.7) \quad \inf_{(u(\cdot), y(\cdot))} \lim_{S \rightarrow \infty} \frac{1}{S} \int_0^S g(u(\tau), y(\tau)) d\tau \stackrel{\text{def}}{=} G_\infty$$

and the problem

$$(3.8) \quad \inf_{(\nu(\cdot), y(\cdot))} \lim_{S \rightarrow \infty} \frac{1}{S} \int_0^S \bar{g}(\nu(\tau), y(\tau)) d\tau \stackrel{\text{def}}{=} \bar{G}_\infty,$$

where the infs in (3.7) and (3.8) are over all admissible and, respectively, over all relaxed admissible pairs on the interval  $[0, \infty)$  such that the corresponding limits exist.

**COROLLARY 3.4.** *If  $W$  is not empty, then  $\bar{G}_\infty = G_*$  and the problem (3.8) has a solution. That is, there exists  $(\nu(\cdot), y(\cdot))$ , a relaxed admissible pair on  $[0, \infty)$ , such that*

$$(3.9) \quad \lim_{S \rightarrow \infty} \frac{1}{S} \int_0^S \bar{g}(\nu(\tau), y(\tau)) d\tau = G_*.$$

*Proof.* The statement is implied by Theorem 3.3 and by the fact that LLPP (3.2) always has an extreme point solution.  $\square$

For the problem (3.7) to have a solution, one needs to introduce additional assumptions. In particular, one can use Corollary 3.4 and a standard measurable selection argument to show that, if  $(g, f)(U, y) \stackrel{\text{def}}{=} \{(\zeta, \eta) : \zeta = g(u, y), \eta = f(u, y), u \in U\}$  is convex for all  $y \in Y$ , then  $G_\infty = G_*$  and there exists  $(u(\cdot), y(\cdot))$ , an admissible pair on  $[0, \infty)$ , such that

$$(3.10) \quad \lim_{S \rightarrow \infty} \frac{1}{S} \int_0^S g(u(\tau), y(\tau)) d\tau = G_*.$$

Along with the problems considered above, let us also consider the problem

$$(3.11) \quad \inf_{(u(\cdot), y(\cdot))_{\text{per}}} \lim_{S \rightarrow \infty} \frac{1}{S} \int_0^S g(u(\tau), y(\tau)) d\tau \stackrel{\text{def}}{=} G_{\text{per}},$$

where inf is over all periodic admissible pairs, that is, over the admissible pairs such that

$$(3.12) \quad (u(\tau), y(\tau)) = (u(\tau + T), y(\tau + T)) \quad \forall \tau \geq 0,$$

for some  $T > 0$ . Note that the problem (3.11) is equivalent to a so-called periodic optimization problem

$$(3.13) \quad \inf_{(u(\cdot), y(\cdot))} \frac{1}{T} \int_0^T g(u(\tau), y(\tau)) d\tau = G_{\text{per}},$$

where inf is over the length of the time interval  $T$  and over the admissible pairs defined on  $[0, T]$  which satisfy the periodicity condition  $y(0) = y(T)$ .

In the general case the optimal values of the problems (2.3), (3.7), and (3.11) satisfy the inequalities

$$(3.14) \quad \overline{\lim}_{S \rightarrow \infty} G(S) \leq G_\infty \leq G_{per} \Rightarrow G_* \leq G_{per},$$

and, under some additional assumptions,

$$(3.15) \quad \lim_{S \rightarrow \infty} G(S) = G_\infty = G_{per} \Rightarrow G_* = G_{per}.$$

Sufficient conditions for the equalities (3.15) to be valid have been considered in [27], [28], [35], and [36].

LEMMA 3.5. *If there exists a solution of the LLPP (3.2) that is generated by a periodic admissible pair, then this pair is a solution of the periodic optimization problem (3.13), and the inequalities (3.14) turn into equalities (3.15). Conversely, if equalities (3.15) are valid and if a solution of the periodic optimization problem (3.13) exists, then the occupational measure generated by this pair is a solution of the LLPP (3.2).*

*Proof.* The proof follows from (3.14) and from the definition of the occupational measures.  $\square$

**4. Dual problem and duality relationships.** Define the problem dual to LLPP (D-LLPP) by the equation

$$(4.1) \quad \sup_{(\mu, \psi(\cdot)) \in \mathcal{D}} \mu \stackrel{\text{def}}{=} \mu_*,$$

with the feasible set  $\mathcal{D} \subset \mathbb{R}^1 \times C^1$  defined as

$$(4.2) \quad \mathcal{D} \stackrel{\text{def}}{=} \{(\mu, \psi(\cdot)) : \mu = \min_{(u, y) \in U \times Y} \{g(u, y) + (\psi'(y))^T f(u, y)\}, \psi(\cdot) \in C^1\}.$$

It can be readily seen that, if  $W \neq \emptyset$  and  $\gamma \in W$ , then for any  $(\mu, \psi(\cdot)) \in \mathcal{D}$  (note that  $\mathcal{D}$  is never empty),

$$(4.3) \quad \mu \leq \int_{U \times Y} (g(u, y) + (\psi'(y))^T f(u, y)) \gamma(du, dy) = \int_{U \times Y} g(u, y) \gamma(du, dy)$$

$$(4.4) \quad \Rightarrow \mu_* \leq G_*.$$

The following statements establish more elaborate connections between LLPP (3.2) and D-LLPP (4.1).

THEOREM 4.1.

(i) *The optimal value of D-LLPP (4.1) is bounded (that is,  $\mu_* < \infty$ ) if and only if the set  $W$  is not empty.*

(ii) *If the optimal value of D-LLPP (4.1) is bounded, then*

$$(4.5) \quad \mu_* = G_*.$$

(iii) *The optimal value of D-LLPP (4.1) is unbounded (that is,  $\mu_* = \infty$ ) if and only if there exists a function  $\psi(\cdot) \in C^1$  such that*

$$(4.6) \quad \max_{(u, y) \in U \times Y} (\psi'(y))^T f(u, y) < 0.$$

*Proof.* The proof of Theorem 4.1 is given in section 9.  $\square$

Note that, in a stochastic setting, a duality result similar to Theorem 4.1(ii) (with  $Y = \mathbb{R}^m$ ) has been obtained in [16].

*Remark 4.2.* From the statements (i) and (ii) it follows that the set  $W$  and, hence, the set  $\bar{\Gamma}(S)$  (and also the set  $\Gamma(S)$  provided that  $f(U, y)$  is convex for all  $y \in Y$ ) are not empty (see Theorem 3.1(i)) if and only if a function  $\psi(\cdot) \in C^1$  satisfying (4.6) does not exist. Note that, if such a function  $\psi(\cdot)$  exists, then the fact that  $\bar{\Gamma}(S)$  (and, hence,  $\Gamma(S)$ ) are empty for  $S \geq S_0$  (for some  $S_0 > 0$ ) follow from the fact that this  $\psi(\cdot)$  can be used as a Liapunov function decreasing along the trajectories of the system (2.2) (respectively, (2.1)) and “forcing” them to leave  $Y$  in a finite time.

Assume that  $\mu_* < \infty$ . A function  $\psi_*(\cdot) \in C^1$  will be called a solution of D-LLPP (4.1) if

$$(4.7) \quad \mu_* = \min_{(u,y) \in U \times Y} \{g(u, y) + (\psi'_*(y))^T f(u, y)\}.$$

Defining  $H(p, y)$  by the equation

$$(4.8) \quad H(p, y) \stackrel{\text{def}}{=} \min_{u \in U} \{g(u, y) + p^T f(u, y)\}$$

and rewriting (4.7) in the form

$$(4.9) \quad \mu_* = \min_{y \in Y} H(\psi'_*(y), y) \quad \Rightarrow \quad \mu_* \leq H(\psi'_*(y), y) \quad \forall y \in Y,$$

one can come to a conclusion that a solution  $\psi_*(\cdot)$  of D-LLPP (4.1) is a smooth viscosity subsolution of the corresponding Hamilton–Jacobi–Bellman equation (see [12] and [26] for relevant definitions and developments).

Given a solution  $\psi_*(\cdot)$  of D-LLPP (4.1), define the set  $\Omega_* \subset U \times Y$  by the equation

$$(4.10) \quad \Omega_* \stackrel{\text{def}}{=} \{(u, y) \in U \times Y : g(u, y) + (\psi'_*(y))^T f(u, y) = \mu_*\}.$$

**PROPOSITION 4.3.** *Let  $\mu_* < \infty$  and  $\psi_*(\cdot)$  be a solution of D-LLPP (4.1). Then  $\gamma \in W$  will be a solution of the LLPP (3.2) if and only if*

$$(4.11) \quad \gamma(\Omega_*) = 1.$$

*Proof.* By Theorem 4.1(ii), a probability measure  $\gamma$  belonging to  $W$  will be a solution of the LLPP (3.2) if and only if

$$(4.12) \quad \mu_* = \int_{U \times Y} g(u, y) \gamma(du, dy) = \int_{U \times Y} (g(u, y) + (\psi'_*(y))^T f(u, y)) \gamma(du, dy).$$

Since (see (4.7))

$$(4.13) \quad \mu_* \leq g(u, y) + (\psi'_*(y))^T f(u, y) \quad \forall (u, y) \in U \times Y,$$

the equality (4.12) can be valid if and only if (4.11) is true. This proves the proposition.  $\square$

**COROLLARY 4.4.** *For a relaxed admissible pair  $(\nu(\cdot), y(\cdot))$  generating the occupational measure  $\gamma^{\nu(\cdot), y(\cdot)}$  on  $[0, \infty)$  to be optimal in the problem (3.8) it is necessary and sufficient that  $\gamma^{\nu(\cdot), y(\cdot)}(\Omega_*) = 1$ .*

*Proof.* By Corollary 3.4, the pair  $(\nu(\cdot), y(\cdot))$  will be optimal in (3.8) (that is, it satisfies (3.9)) if and only if  $\gamma^{\nu(\cdot), y(\cdot)}$  is a solution of the LLPP (3.2). Hence, the statement is implied by Proposition 4.3.  $\square$

Let

$$(4.14) \quad \mathcal{Y}_* \stackrel{\text{def}}{=} \{y \in Y : H(\psi_*(y), y) = \mu_*\}$$

and

$$(4.15) \quad \mathcal{U}_*(y) \stackrel{\text{def}}{=} \{u \in U : g(u, y) + (\psi'_*(y))^T f(u, y) = H(\psi'_*(y), y)\}.$$

**COROLLARY 4.5.** *Let  $\psi_*(\cdot)$  be a solution of D-LLPP (4.1). A  $T$ -periodic admissible pair  $(u(\tau), y(\tau))$  will be a solution of the periodic optimization problem (3.13) if and, under the condition that the equalities (3.15) are valid, only if*

$$(4.16) \quad y(\tau) \in \mathcal{Y}_* \quad \forall \tau \in [0, T], \quad u(\tau) \in \mathcal{U}_*(y(\tau)) \quad \text{for almost all } \tau \in [0, T].$$

*Proof.* By Lemma 3.5, a  $T$ -periodic admissible pair  $(u(\tau), y(\tau))$  will be a solution of the periodic optimization problem (3.13) if and, under the condition that the equalities (3.15) are valid, only if the occupational measure  $\gamma^{u(\cdot), y(\cdot)}$  generated by this pair is a solution of LLPP (3.2). According to Proposition 4.3, this is the case if and only if  $\gamma^{u(\cdot), y(\cdot)}(\Omega_*) = 1$ , which is equivalent to  $(u(\tau), y(\tau)) \in \Omega_*$  for almost all  $\tau \in [0, T]$ . The latter, in turn, is equivalent to the inclusions (4.16) being satisfied for almost all  $\tau \in [0, T]$  since

$$(4.17) \quad \Omega_* = \{(u, y) : u \in \mathcal{U}_*(y), y \in \mathcal{Y}_*\}.$$

The fact that the first inclusion in (4.16) is valid for all  $\tau \in [0, T]$  follows from the fact that the function  $y(\cdot)$  is continuous and that the set  $\mathcal{Y}_*$  is compact.  $\square$

Note that a solution of D-LLPP (4.1) defined as a  $C^1$  function satisfying (4.9) may not exist, and one can consider the possibility of defining the solution as a nondifferentiable function, which satisfies (4.9) in the viscosity sense (see, e.g., [12, p. 399]). We, however, do not follow this path. Instead, in the next sections we discuss a way of constructing  $C^1$  functions that solve D-LLPP (4.1) approximately. First, in section 5, we approximate the LLPP (3.2) with I-D LLPs having a finite number of constraints  $N$  ( $N = 1, 2, \dots$ ), and we study the problem dual to the latter (section 5). Then, in section 6, we approximate the problem with  $N$  constraints and its dual by a family of F-D LPPs defined on grid points of  $U \times Y$  and the corresponding dual, and we show that, by solving these F-D LPPs, one can construct a family of functions  $\psi_{N,\Delta}(\cdot) \in C^1$  ( $\Delta$  being the parameter of the grid) such that, for any  $\delta > 0$ ,

$$(4.18) \quad \mu_* - \delta \leq \min_{(u,y) \in U \times Y} \{g(u, y) + (\psi'_{N,\Delta}(y))^T f(u, y)\} = \min_{y \in Y} H(\psi'_{N,\Delta}(y), y)$$

if  $N$  is large and  $\Delta$  is small enough (see (4.7) and (4.9)). We also show (see section 7) that, under some additional conditions, a feedback control function  $u_{N,\Delta}(\cdot) : Y \rightarrow U$  satisfying the inclusion

$$(4.19) \quad u_{N,\Delta}(y) \in \mathcal{U}_{N,\Delta}(y) \stackrel{\text{def}}{=} \{u \in U : g(u, y) + (\psi'_{N,\Delta}(y))^T f(u, y) = H(\psi'_{N,\Delta}(y), y)\}$$

can serve as an approximation to the optimal feedback control defined on the optimal periodic orbit (provided that the latter exists).

Note also that D-LLPP (4.1) can be rewritten in the form

$$(4.20) \quad \sup_{\psi(\cdot) \in C^1} \min_{y \in Y} H(\psi'(y), y) = \mu_*.$$

Problems similar to (4.20) (but in a different setting and under a different set of conditions) have been considered in the literature, particularly, in relation to finding so-called effective Hamiltonians arising in the homogenization theory (see [34] and references therein). It seems plausible that results that we discuss below could be applicable in this area, but we do not investigate this matter in the present paper.

Finally, to conclude this section, let us also mention that problems dual to I-D LPPs that arise in dealing with deterministic optimal control problems on finite time intervals and the way of how these dual problems can be used for formulating necessary and sufficient optimality conditions have been studied in [57] (see also references therein).

**5.  $N$ -approximating problem and its dual.** Let  $\{\phi_i(\cdot) \in C^1, i = 1, 2, \dots\}$  be a sequence of functions having continuous partial derivatives of second order such that any function  $\psi(\cdot) \in C^1$  and its gradient  $\psi'(\cdot)$  can be simultaneously approximated on  $Y$  by linear combinations of functions from  $\{\phi_i(\cdot), i = 1, 2, \dots\}$  and their corresponding gradients. That is, for any  $\psi(\cdot) \in C^1$  and any  $\delta > 0$ , there exist  $\beta_1, \dots, \beta_k$  (real numbers) such that

$$(5.1) \quad \max_{y \in Y} \left\{ \left| \psi(y) - \sum_1^k \beta_i \phi_i(y) \right| + \left\| \psi'(y) - \sum_1^k \beta_i \phi'_i(y) \right\| \right\} \leq \delta,$$

with  $\|\cdot\|$  being a norm in  $\mathbb{R}^m$ . An example of such an approximating sequence is the sequence of monomials  $y_1^{i_1} \dots y_m^{i_m}$ ,  $i_1, \dots, i_m = 0, 1, \dots$ , where  $y_j$  ( $j = 1, \dots, m$ ) stands for the  $j$ th component of  $y$  (see, e.g., [44]). In what follows it will be assumed that the gradients  $\phi'_i(y)$ ,  $i = 1, 2, \dots, N$ , are linearly independent on  $Y$ . That is,

$$(5.2) \quad \sum_{i=1}^N v_i \phi'_i(y) = 0 \quad \forall y \in Y \quad \Rightarrow \quad v_i = 0, \quad i = 1, 2, \dots, N.$$

Note that this is satisfied automatically if  $\phi_i(y)$  are chosen to be monomials and  $\text{int}Y$  (interior of  $Y$ ) is not empty.

Let us define the set  $W_N$  by the equation

$$(5.3) \quad W_N \stackrel{\text{def}}{=} \left\{ \gamma \mid \gamma \in \mathcal{P}(U \times Y); \int_{U \times Y} (\phi'_i(y))^T f(u, y) \gamma(du, dy) = 0, \quad i = 1, 2, \dots, N \right\}$$

and consider the linear programming problem

$$(5.4) \quad \min_{\gamma \in W_N} \int_{U \times Y} q(u, y) \gamma(du, dy) \stackrel{\text{def}}{=} G_N.$$

This problem will be referred to as  $N$ -approximating LLPP (or just  $N$ -LLPP). Note that  $W_N$  is a convex and compact subset of  $\mathcal{P}(U \times Y)$  and  $W \subset W_N$ , which implies

$$(5.5) \quad G_* \geq G_N.$$

The connection between LLPP (3.2) and  $N$ -LLPP (5.4) is established by the following proposition (see [32]).

PROPOSITION 5.1. *The set  $W$  is not empty if and only if there exists  $N_0 \geq 1$  such that  $W_N$  is not empty for  $N \geq N_0$ . If  $W$  is not empty, then*

$$(5.6) \quad \lim_{N \rightarrow \infty} \rho_H(W_N, W) = 0$$

and

$$(5.7) \quad \lim_{N \rightarrow \infty} G_N = G_*.$$

Also, if  $\gamma^N$  is a solution of the problem (5.4) and  $\lim_{N' \rightarrow \infty} \rho(\gamma^{N'}, \gamma) = 0$  for some subsequence of integers  $N'$  tending to infinity, then  $\gamma$  is a solution of (3.2). If the solution  $\gamma_*$  of the problem (3.2) is unique, then, for any solution  $\gamma^N$  of (5.4),  $\lim_{N \rightarrow \infty} \rho(\gamma^N, \gamma_*) = 0$ .

*Proof.* The proof follows from Proposition 7 in [32].  $\square$

Let us define the problem dual to  $N$ -LLPP (5.4) (referred in what follows to as  $D$ - $N$ -LLPP) by the equation

$$(5.8) \quad \sup_{(\mu, v) \in \mathcal{D}_N} \mu \stackrel{\text{def}}{=} \mu_N,$$

with the feasible set  $\mathcal{D}_N \subset \mathbb{R}^1 \times \mathbb{R}^N$  defined as

$$(5.9) \quad \mathcal{D}_N \stackrel{\text{def}}{=} \left\{ (\mu, v) : \mu = \min_{(u, y) \in U \times Y} \left\{ g(u, y) + \sum_{i=1}^N v_i (\phi'_i(y))^T f(u, y) \right\}, v = (v_i) \in \mathbb{R}^N \right\}.$$

For a fixed  $N$ , the relationships between  $N$ -LLPP (5.4) and  $D$ - $N$ -LLPP (5.8) are similar to those between (3.2) and (4.1). For example, similarly to (4.3), one can obtain that, if  $W_N \neq \emptyset$  and  $\gamma \in W_N$ , then for any  $(\mu, v) \in \mathcal{D}_N$  ( $\mathcal{D}_N$  is never empty),

$$(5.10) \quad \mu \leq \int_{U \times Y} \left( g(u, y) + \sum_{i=1}^N v_i (\phi'_i(y))^T f(u, y) \right) \gamma(du, dy) = \int_{U \times Y} g(u, y) \gamma(du, dy)$$

$$(5.11) \quad \Rightarrow \mu_N \leq G_N.$$

Also, the following result similar to Theorem 4.1 is valid.

THEOREM 5.2. (i) *The optimal value of  $D$ - $N$ -LLPP (5.8) is bounded (that is,  $\mu_N < \infty$ ) if and only if the set  $W_N$  is not empty.*

(ii) *If the optimal value of  $D$ - $N$ -LLPP (5.8) is bounded, then*

$$(5.12) \quad \mu_N = G_N.$$

(iii) *The optimal value of  $D$ - $N$ -LLPP (5.8) is unbounded (that is,  $\mu_N = \infty$ ) if and only if there exists  $v = (v_1, \dots, v_N)$  such that*

$$(5.13) \quad \max_{(u, y) \in U \times Y} (\psi'_v(y))^T f(u, y) < 0, \quad \psi_v(y) \stackrel{\text{def}}{=} \sum_{i=1}^N v_i \phi_i(y).$$

*Proof.* The proof of the theorem follows exactly the same steps as those in the proof of Theorem 4.1. For completeness we briefly outline these steps in section 9.  $\square$

A vector  $v = (v_i), i = 1, \dots, N$ , will be called a solution of D- $N$ -LLPP (5.8) if

$$(5.14) \quad \mu_N = \min_{(u,y) \in U \times Y} \{g(u,y) + (\psi'_v(y))^T f(u,y)\}, \quad \psi_v(y) \stackrel{\text{def}}{=} \sum_{i=1}^N v_i \phi_i(y).$$

Let us introduce the following two assumptions used in the consideration below.

*Assumption 1.* The inequality

$$(5.15) \quad (\psi'_v(y))^T f(u,y) \leq 0 \quad \forall (u,y) \in U \times Y, \quad \psi_v(y) \stackrel{\text{def}}{=} \sum_{i=1}^N v_i \phi_i(y),$$

can be valid only with  $v_i = 0, i = 1, \dots, N$ .

*Assumption 2.* The inequality

$$(5.16) \quad (\psi'(y))^T f(u,y) \leq 0 \quad \forall (u,y) \in U \times Y, \quad \psi(\cdot) \in C^1,$$

can be valid only with  $\psi'(y) = 0 \forall y \in Y$ .

PROPOSITION 5.3. *Assumption 1 is equivalent to the assumption that*

$$(5.17) \quad 0 \in \text{int}(coK_N),$$

where  $\text{int}(coK_N)$  stands for the interior of the convex hull of  $K_N$ ,

$$(5.18) \quad K_N \stackrel{\text{def}}{=} \{z \in \mathbb{R}^N : z = (z_i), z_i = (\phi'_i(y))^T f(u,y), i = 1, \dots, N, (u,y) \in U \times Y\}.$$

*Proof.* The proof is in section 9.  $\square$

*Remark 5.4.* (i) It is easy to see that Assumption 2 implies the validity of Assumption 1 with any  $N = 1, 2, \dots$ . In fact, if the former is satisfied, then the inequality  $\sum_{i=1}^N v_i (\phi'_i(y))^T f(u,y) \leq 0 \forall (u,y) \in U \times Y$  implies that  $\sum_{i=1}^N v_i \phi'_i(y) = 0 \forall y \in Y$  and, hence, by (5.2),  $v_i = 0, i = 1, \dots, N$ . Sufficient conditions for Assumption 2 to be satisfied and special cases in which it is satisfied have been considered in [32] (see Proposition 8 and Remark 2 of [32]).

(ii) Assumption 1 implies that the function  $\psi_v(\cdot)$  satisfying (5.13) does not exist, and Assumption 2 implies that the function  $\psi(\cdot)$  satisfying (4.6) does not exist. Thus, from the former it follows that  $\mu_N$  is bounded (by Theorem 5.2(iii)) and from the latter that  $\mu_*$  is bounded (by Theorem 4.1(iii)).

THEOREM 5.5. (i) *If Assumption 1 is satisfied, then the set of solutions of D- $N$ -LLPP (5.8) is nonempty and bounded. That is (see (5.14)),*

$$(5.19) \quad \emptyset \neq V_N \stackrel{\text{def}}{=} \left\{ v = (v_i) : \mu_N = \min_{(u,y) \in U \times Y} \left\{ g(u,y) + \left( \sum_{i=1}^N v_i \phi'_i(y) \right)^T f(u,y) \right\} \right\},$$

$$(5.20) \quad \max_{v \in V_N} \|v\| \leq c_N = \text{const.}$$

(ii) *Conversely, if (5.19) and (5.20) are true, then Assumption 1 is satisfied.*

*Proof.* Proof is in section 9.  $\square$

Let us now establish that the optimal values of the D- $N$ -LLPP (5.8) converge to the optimal value of D-LLPP (4.1) as  $N \rightarrow \infty$ . Note that, as follows directly from the definitions of D- $N$ -LLPP and D-LLPP (see (5.8) and (4.1)),

$$(5.21) \quad \mu_1 \leq \mu_2 \leq \dots \leq \mu_N \leq \dots \leq \mu_*.$$

THEOREM 5.6. *The optimal value of D-N-LLPP (5.8) converges to the optimal value of D-LLPP (4.1). That is,*

$$(5.22) \quad \lim_{N \rightarrow \infty} \mu_N = \mu_*,$$

the statement being valid for both  $\mu_* < \infty$  and  $\mu_* = \infty$ .

*Proof.* If Assumption 2 is satisfied, then  $\mu_* = G_*$  and  $\mu_N = G_N$  for  $N = 1, 2, \dots$ . Hence, (5.22) follows from (5.7). The proof of the general case is in section 9.  $\square$

In conclusion of this section, let us introduce a regularity condition, which will be used in section 7 below. Let Assumption 2 be satisfied and let  $v = (v_i) \in V_N$ ,  $\psi_v(y) \stackrel{\text{def}}{=} \sum_{i=1}^N v_i \phi_i(y)$ . Define the sets  $\Omega_N^v, \mathcal{Y}_N^v$  by the equations

$$(5.23) \quad \Omega_N^v \stackrel{\text{def}}{=} \{(u, y) \in U \times Y : g(u, y) + (\psi'_v(y))^T f(u, y) = \mu_N\},$$

$$(5.24) \quad \mathcal{Y}_N^v \stackrel{\text{def}}{=} \{y : (u, y) \in \Omega_N^v\} = \{y \in Y : H(\psi'_v(y), y) = \mu_N\}$$

and the set  $\mathcal{U}_N^v(y)$  by the equation

$$(5.25) \quad \mathcal{U}_N^v(y) \stackrel{\text{def}}{=} \{u \in U : g(u, y) + (\psi'_v(y))^T f(u, y) = H(\psi'_v(y), y)\},$$

where  $H(p, y)$  is as in (4.8).

*N-regularity condition (N-RC) on Q.* Given  $Q \subset Y$ , we shall say that N-RC is satisfied on  $Q$  if, for any  $v \in V_N$  and any  $y \in \mathcal{Y}_N^v \cap Q$ , a solution of the problem

$$(5.26) \quad \max_{u \in U} \{g(u, y) + (\psi'_v(y))^T f(u, y)\}$$

is unique. That is,  $\mathcal{U}_N^v(y)$  is a singleton for  $v \in V_N$  and  $y \in \mathcal{Y}_N^v \cap Q$ .

Note that N-RC is satisfied on any  $Q \subset Y$  and with any  $N = 1, 2, \dots$  if  $g(u, y)$  is strictly convex and  $f(u, y)$  is linear in  $u$  and if  $U$  is a convex set.

**6. NΔ-approximating problem and its dual.** Assume first that  $N$  is fixed and that, for any  $\Delta > 0$ , Borel sets  $Q_{l,k}^\Delta \subset U \times Y$  ( $l = 1, \dots, L^\Delta, k = 1, \dots, K^\Delta$ ) (called cells) are defined in such a way that two different cells do not intersect, the union of all cells is equal to  $U \times Y$ , and

$$(6.1) \quad \sup_{(u,y) \in Q_{l,k}^\Delta} \|(u, y) - (u_l, y_k)\| \leq c\Delta, \quad c = \text{const},$$

for some point  $(u_l, y_k) \in Q_{l,k}^\Delta$ . For simplicity of notation, it is assumed (from now on) that  $U$  is a compact subset of  $\mathbb{R}^n$  and  $\|\cdot\|$  stands for a norm in  $\mathbb{R}^{n+m}$ . Let us fix the points  $(u_l, y_k)$  ( $l = 1, \dots, L^\Delta, k = 1, \dots, K^\Delta$ ) and define a polyhedral set  $W_{N,\Delta} \subset \mathbb{R}^{L^\Delta + K^\Delta}$  by the equation

$$(6.2) \quad W_{N,\Delta} \stackrel{\text{def}}{=} \left\{ \gamma = \{\gamma_{l,k}\} \geq 0 : \sum_{l,k} \gamma_{l,k} = 1, \sum_{l,k} (\phi'_i(y_k))^T f(u_l, y_k) \gamma_{l,k} = 0, i = 1, \dots, N \right\},$$

where  $\sum_{l,k} \stackrel{\text{def}}{=} \sum_{l=1}^{L^\Delta} \sum_{k=1}^{K^\Delta}$ . Consider the problem

$$(6.3) \quad \min_{\gamma \in W_{N,\Delta}} \sum_{l,k} \gamma_{l,k} g(u_l, y_k) \stackrel{\text{def}}{=} G_{N,\Delta}.$$

This is an F-D LLP which will be referred to as  $N\Delta$ -approximating LLPP (or  $N\Delta$ -LLPP).

Note that the polyhedral set  $W_{N,\Delta}$  is the set of probability measures on  $U \times Y$  which assign nonzero probabilities only to the points  $(u_l, y_k)$ , and, as such,

$$(6.4) \quad W_{N,\Delta} \subset W_N \quad \Rightarrow \quad G_{N,\Delta} \geq G_N.$$

**THEOREM 6.1.** *Let Assumption 1 be satisfied. Then the set  $W_N$  is not empty and there exists  $\Delta_0 > 0$  such that  $W_{N,\Delta}$  is not empty for  $\Delta \leq \Delta_0$ . Also*

$$(6.5) \quad \lim_{\Delta \rightarrow 0} \rho_H(W_{N,\Delta}, W_N) = 0$$

and

$$(6.6) \quad \lim_{\Delta \rightarrow 0} G_{N,\Delta} = G_N.$$

If  $\gamma^{N,\Delta}$  is a solution of the problem (6.3) and  $\lim_{\Delta' \rightarrow 0} \rho(\gamma^{N,\Delta'}, \gamma^N) = 0$  for some sequence of  $\Delta'$  tending to zero, then  $\gamma^N$  is a solution of (5.4). If the solution  $\gamma^N$  of the problem (5.4) is unique, then, for any solution  $\gamma^{N,\Delta}$  of (6.3),  $\lim_{\Delta \rightarrow 0} \rho(\gamma^{N,\Delta}, \gamma^N) = 0$ .

*Proof.* The fact that  $W_N$  is not empty if Assumption 1 is satisfied follows from Theorem 5.2(i) and from the fact that  $\mu_N$  is bounded (see Remark 5.4(ii)). The proofs of all other statements of the theorem follow from Proposition 9 in [32].  $\square$

Consider the F-D LPP

$$(6.7) \quad \max_{(\mu, \lambda) \in \mathbb{R}^1 \times \mathbb{R}^N} \left\{ \mu : \mu \leq g(u_l, y_k) + \sum_{i=1}^N \lambda_i (\phi'_i(y_k))^T f(u_l, y_k) \quad \forall (u_l, y_k) \right\} \stackrel{\text{def}}{=} \mu_{N,\Delta},$$

which is dual to  $N\Delta$ -LLPP (6.3) and which will be referred to as D- $N\Delta$ -LLPP. From the duality theory for F-D LPPs (see, e.g., [22]) it follows, in particular, that, if  $W_{N,\Delta}$  is not empty, then the optimal value of  $N\Delta$ -LLPP (6.3) is equal to the optimal value of D- $N\Delta$ -LLPP (6.7)

$$(6.8) \quad G_{N,\Delta} = \mu_{N,\Delta}$$

and the solutions set of D- $N\Delta$ -LLPP (6.7) is not empty:

$$(6.9) \quad \emptyset \neq \Lambda_{N,\Delta} \stackrel{\text{def}}{=} \left\{ \lambda = (\lambda_i) : \mu_{N,\Delta} = \min_{(u_l, y_k)} \left\{ g(u_l, y_k) + \sum_{i=1}^N \lambda_i (\phi'_i(y_k))^T f(u_l, y_k) \right\} \right\}.$$

**THEOREM 6.2.** *Let Assumption 1 be satisfied. Then*

$$(6.10) \quad \lim_{\Delta \rightarrow 0} \max_{\lambda \in \Lambda_{N,\Delta}} \text{dist}(\lambda, V_N) = 0, \quad \text{dist}(\lambda, V_N) \stackrel{\text{def}}{=} \min_{v \in V_N} \|\lambda - v\|,$$

where  $V_N$  is the solutions set of D- $N$ -LLPP (5.8) (explicitly defined in (5.19)).

*Proof.* The proof is in section 10. Let us only note here that, by Theorem 6.1, the set  $W_{N,\Delta}$  is not empty under Assumption 1, and hence, (6.8) and (6.9) are valid for  $\Delta$  small enough.  $\square$

Let us now address the issue of constructing  $\psi_{N,\Delta}(\cdot)$  that approximately solves D-LLPP (4.1) (see the end of section 4).

PROPOSITION 6.3. *Let Assumption 2 be satisfied. Then, for any  $\delta > 0$ , there exist  $N_\delta > 0$  and  $\Delta_N > 0$  such that the function  $\psi_{N,\Delta}(\cdot)$  defined by the equation*

$$(6.11) \quad \psi_{N,\Delta}(y) \stackrel{\text{def}}{=} \sum_{i=1}^N \lambda_i^{N,\Delta} \phi_i(y), \quad \lambda^{N,\Delta} = (\lambda_i^{N,\Delta}) \in \Lambda_{N,\Delta},$$

satisfies (4.18) with  $N \geq N_\delta$  and  $\Delta \leq \Delta_N$ .

*Proof.* As was mentioned above (see Remark 5.4(i)), the validity of Assumption 2 implies the validity of Assumption 1 with all  $N$ . Having this in mind, let us choose  $N_\delta$  in such a way that

$$(6.12) \quad \mu_* - \frac{\delta}{2} \leq \mu_N$$

for any  $N \geq N_\delta$  (this is possible due to Theorem 5.6). By Theorem 5.5(i), the set  $V_N$  is not empty and

$$(6.13) \quad \mu_* - \frac{\delta}{2} \leq \min_{(u,y) \in U \times Y} \left\{ g(u,y) + \left( \sum_{i=1}^N v_i \phi'_i(y) \right)^T f(u,y) \right\} \quad \forall v = (v_i) \in V_N.$$

From (6.10) it follows that, for any  $\Delta \leq \Delta_N$  ( $\Delta_N$  being positive small enough) and any  $\lambda \in \Lambda_{N,\Delta}$ , there exists  $v^{N,\Delta} = (v_i^{N,\Delta}) \in V_N$  such that

$$\begin{aligned} & \min_{(u,y) \in U \times Y} \left\{ g(u,y) + \left( \sum_{i=1}^N v_i^{N,\Delta} \phi'_i(y) \right)^T f(u,y) \right\} - \frac{\delta}{2} \\ & \leq \min_{(u,y) \in U \times Y} \left\{ g(u,y) + \left( \sum_{i=1}^N \lambda_i^{N,\Delta} \phi'_i(y) \right)^T f(u,y) \right\} \\ \Rightarrow \quad & \mu_* - \delta \leq \min_{(u,y) \in U \times Y} \left\{ g(u,y) + \left( \sum_{i=1}^N \lambda_i^{N,\Delta} \phi'_i(y) \right)^T f(u,y) \right\}. \end{aligned}$$

The latter is (4.18) with  $\psi_{N,\Delta}(\cdot)$  as in (6.11).  $\square$

**7. Convergence to the optimal periodic solution.** In this section we will show that, under certain additional conditions, the control satisfying (4.19), that is, defined as a solution of the problem

$$(7.1) \quad \min_{u \in U} \{ g(u,y) + (\psi'_{N,\Delta}(y))^T f(u,y) \},$$

where  $\psi_{N,\Delta}(\cdot)$  is as in (6.11), converges (as  $N \rightarrow \infty$  and  $\Delta \rightarrow 0$ ) to the optimal feedback control defined on the optimal periodic orbit. To ensure the existence of the latter, throughout this section it is assumed that there exists a solution  $\gamma_*$  of LLPP (3.2) that is generated by a  $T$ -periodic admissible pair  $(u_*(\cdot), y_*(\cdot))$  (note that this assumption implies that the pair is a solution of the periodic optimization problem (3.13); see Lemma 3.5).

Define sets  $\Theta_*$  and  $\mathcal{Y}_*$  by the equations

$$(7.2) \quad \Theta_* \stackrel{\text{def}}{=} \{ (u,y) : (u,y) = (u_*(\tau), y_*(\tau)) \text{ for some } \tau \in [0, T] \},$$

$$(7.3) \quad \mathcal{Y}_* \stackrel{\text{def}}{=} \{y : y = y_*(\tau) \text{ for some } \tau \in [0, T]\} = \{y : (u, y) \in \Theta_*\}.$$

The set  $\Theta_*$  can be considered as the graph of the optimal feedback control function, which is defined on the optimal orbit  $\mathcal{Y}_*$  by the equation  $u_*(y) \stackrel{\text{def}}{=} u \forall (u, y) \in \Theta_*$ . For this definition to make sense, it is assumed that from the fact that  $(u', y) \in \Theta_*$  and  $(u'', y) \in \Theta_*$  it follows that  $u' = u''$  (this assumption is satisfied if the closed curve defined by  $y_*(\tau), \tau \in [0, T]$ , does not intersect itself).

Note that the set  $\mathcal{Y}_*$  defined in (7.3) is different from the set defined in (4.14) (despite the fact that the same notation is used in both cases). Underline that in this section it is not assumed that a solution  $\psi_*(\cdot)$  of D-LLPP (4.1) (which is used in (4.14)) exists. If, however, it does exist, then by Corollary 4.5, the set defined in (7.3) is contained in the set defined in (4.14).

Let us introduce the following assumption about the set  $\Theta_*$ .

*Assumption 3.* For any  $(u, y) \in \bar{\Theta}_*$  (the closure of  $\Theta_*$ ) and any  $r > 0$ , the set  $B_r(u, y) \stackrel{\text{def}}{=} ((u, y) + rB) \cap (U \times Y)$  ( $B$  being the open unit ball in  $\mathbb{R}^{n+m}$ ) has a nonzero  $\gamma_*$ -measure:  $\gamma_*(B_r(u, y)) > 0$ .

Note that this assumption is satisfied if the optimal control function  $u_*(\cdot) : [0, T] \rightarrow U$  is piecewise continuous and at every discontinuity point  $\tau$  the value of  $u_*(\tau)$  is equal either to the limit from the left ( $u_*(\tau) = \lim_{\tau' \rightarrow \tau^-} u_*(\tau')$ ) or to the limit from the right ( $u_*(\tau) = \lim_{\tau' \rightarrow \tau^+} u_*(\tau')$ ).

Under the validity of Assumption 2, from Proposition 5.1 and Theorem 6.1 it follows that there exist solutions  $\gamma^{N,\Delta} = \{\gamma_{l,k}^{N,\Delta}\}$  of the  $N\Delta$ -LLPP (6.3) (considered with  $N \rightarrow \infty$  and  $\Delta \rightarrow 0$ ) such that

$$(7.4) \quad \lim_{N \rightarrow \infty} \lim_{\Delta \rightarrow 0} \rho(\gamma^{N,\Delta}, \gamma_*) = 0,$$

with the latter being valid for any solution of the  $N\Delta$ -LLPP (6.3) if  $\gamma_*$  is a unique solution of LLPP (3.2).

For any Borel subset  $\Theta$  of  $U \times Y$ , let  $\gamma^{N,\Delta}(\Theta)$  stand for the  $\gamma^{N,\Delta}$  measure of  $\Theta$ . That is,  $\gamma^{N,\Delta}(\Theta) \stackrel{\text{def}}{=} \sum_{(u_l, y_k) \in \Theta} \gamma_{l,k}^{N,\Delta}$ .

**THEOREM 7.1.** *Let Assumptions 2 and 3 be satisfied.*

(i) *Let  $Q$  be an open subset of  $Y$  such that the  $N$ -regularity condition (see the end of section 5) is satisfied on  $Q$  for all  $N$  large enough and such that the solution  $u_{N,\Delta}(\cdot)$  of the problem (7.1) is unique on  $Q$ , and it is uniformly equicontinuous on any closed subset  $\bar{Q}'$  of  $Q$  for  $N$  large and  $\Delta$  small enough. That is,*

$$(7.5) \quad \|u_{N,\Delta}(y') - u_{N,\Delta}(y'')\| \leq \omega(\|y' - y''\|) \quad \forall y', y'' \in \bar{Q}',$$

where  $\omega(\theta)$  tends to zero as  $\theta$  tends to zero and  $N \geq N_0, \Delta \leq \Delta_N$  ( $\omega(\cdot)$  and  $N_0$  may be different for different  $\bar{Q}' \subset Q$ ). Then

$$(7.6) \quad \lim_{N \rightarrow \infty} \lim_{\Delta \rightarrow 0} \|u_{N,\Delta}(y) - u_*(y)\| = 0 \quad \forall y \in \mathcal{Y}_* \cap Q,$$

with the convergence being uniform on any closed subset of  $\mathcal{Y}_* \cap Q$ .

(ii) *Assume, in addition, that*

$$(7.7) \quad \overline{\lim}_{N \rightarrow \infty} \overline{\lim}_{\Delta \rightarrow 0} \gamma^{N,\Delta}(\bar{Q}) = 1,$$

where  $\bar{Q}$  is the closure of  $Q$  and that  $\gamma_*(\bar{Q}/Q) = 0$ . Then  $\gamma_*(\mathcal{Y}_* \cap Q) = 1$ . That is,  $u_{N,\Delta}(y)$  converges to  $u_*(y)$  for  $\gamma_*$ -almost all  $y \in Y$ .

The second statement of the theorem allows an extension, which we state in the form of the following corollary.

**COROLLARY 7.2.** *Let Assumptions 2 and 3 be satisfied, and let  $Q_j, j = 1, \dots, J$ , be nonintersecting open subsets of  $Y$ , such that, for every  $j$ , the  $N$ -regularity condition is satisfied on  $Q_j$ , and (7.5) is valid on any closed  $\bar{Q}' \subset Q_j$ . Assume also that*

$$(7.8) \quad \overline{\lim}_{N \rightarrow \infty} \overline{\lim}_{\Delta \rightarrow 0} \gamma^{N,\Delta}(\cup_j Q_j) = 1$$

and that

$$(7.9) \quad \gamma_*(\overline{\cup_j Q_j} / \cup_j Q_j) = 0,$$

where  $\overline{\cup_j Q_j}$  is the closure of  $\cup_j Q_j$ . Then

$$(7.10) \quad \lim_{N \rightarrow \infty} \overline{\lim}_{\Delta \rightarrow 0} \|u_{N,\Delta}(y) - u_*(y)\| = 0$$

for  $\gamma_*$ -almost all  $y \in Y$ .

*Proof.* The proofs of Theorem 7.1 and Corollary 7.2 are given in section 10. They are based on a result of [32] establishing that the sets

$$(7.11) \quad \Theta_{N,\Delta} \stackrel{\text{def}}{=} \{(u_l, y_k) : \gamma_{l,k}^{N,\Delta} > 0\},$$

$$(7.12) \quad \mathcal{Y}_{N,\Delta} \stackrel{\text{def}}{=} \{y : (u, y) \in \Theta_{N,\Delta}\} = \left\{ y_k : \sum_l \gamma_{l,k}^{N,\Delta} > 0 \right\}$$

are “approaching” the sets  $\Theta_*$  and  $\mathcal{Y}_*$  as  $N \rightarrow \infty$  and  $\Delta \rightarrow 0$  (see Theorem 7.3 below).  $\square$

**THEOREM 7.3.** *Let Assumption 2 be satisfied, (7.4) be valid, and  $\gamma_*, \gamma^{N,\Delta}$  be as above. Then the following hold.*

(i) *Corresponding to an arbitrary small  $r > 0$  and arbitrary small  $\delta > 0$ , there exists  $N_0$  such that, for  $N \geq N_0$  and  $\Delta \leq \Delta_N$ ,*

$$(7.13) \quad \gamma^{N,\Delta}(\Theta_{N,\Delta}/(\Theta_* + rB)) < \delta, \quad \gamma^{N,\Delta}(\mathcal{Y}_{N,\Delta}/(\mathcal{Y}_* + rD)) < \delta$$

and

$$(7.14) \quad \Theta_{N,\Delta,\delta} \subset \Theta_* + rB, \quad \mathcal{Y}_{N,\Delta,\delta} \subset \mathcal{Y}_* + rD,$$

where  $B$  and  $D$  are open unit balls in  $\mathbb{R}^{n+m}$  and  $\mathbb{R}^m$ , respectively, and

$$\Theta_{N,\Delta,\delta} \stackrel{\text{def}}{=} \{(u_l, y_k) : \gamma_{l,k}^{N,\Delta} \geq \delta\}, \quad \mathcal{Y}_{N,\Delta,\delta} \stackrel{\text{def}}{=} \left\{ y_k : \sum_l \gamma_{l,k}^{N,\Delta} \geq \delta \right\}.$$

(ii) *If, in addition, Assumption 3 is satisfied, then, corresponding to an arbitrary small  $r > 0$ , there exists  $N_0$  such that, for  $N \geq N_0$  and  $\Delta \leq \Delta_N$ ,*

$$(7.15) \quad \Theta_* \subset \Theta_{N,\Delta} + rB, \quad \mathcal{Y}_* \subset \mathcal{Y}_{N,\Delta} + rB.$$

*Proof.* The validity of (i) and (ii) is implied by Propositions 10 and 11 from [32].  $\square$

Note that from (7.4) it follows that  $\gamma^{N,\Delta} \stackrel{\text{def}}{=} \{\gamma_{l,k}^{N,\Delta}\}$  can be considered as an “approximation” of  $\gamma_*$  for  $N$  large and  $\Delta$  small enough. Due to the fact that  $\gamma_*$  is the occupational measure generated by the optimal periodic pair  $(u^*(\cdot), y^*(\cdot))$ , an element  $\gamma_{l,k}^{N,\Delta}$  of  $\gamma^{N,\Delta}$  can be interpreted as an estimate of “proportion” of time spent by the optimal pair in a “small” vicinity of the point  $(u_l, y_k)$ , while the fact that  $\gamma_{l,k}^{\Delta}$  is positive or zero can be interpreted as an indication of whether or not the optimal pair attends this vicinity. Theorem 7.3 can be viewed as a justification of the latter interpretation.

Based on the consideration above, one can propose the following steps to construct an approximate solution to the periodic optimization problem.

(1) Find the optimal value  $G_{N,\Delta}$  and a solution  $\gamma^{N,\Delta}$  of  $N\Delta$ -LLPP (6.3), and also find a solution  $\mu_{N,\Delta}, \lambda^{N,\Delta}$  of problem (6.7) dual to  $N\Delta$ -LLPP (6.3).

(2) Define the sets  $\Theta_{N,\Delta}$  and  $\mathcal{Y}_{N,\Delta}$  according to (7.11) and (7.12).

(3) Construct the function  $\psi_{N,\Delta}(y)$  according to (6.11) and find the control  $u_{N,\Delta}(y)$  by solving the problem (7.1) for every  $y$  in a neighborhood of  $\mathcal{Y}_{N,\Delta}$ .

(4) Integrate the system (2.1) starting from an initial point  $y(0) \in \mathcal{Y}_{N,\Delta}$  and using  $u_{N,\Delta}(y)$  as the feedback control. One can expect that the obtained solution of the system will return to a small vicinity of the starting point  $y(0)$  and it will be possible to identify the end point of the integration period,  $T_{N,\Delta}$ , as the moment of entering this vicinity.

(5) Adjust the initial condition and/or control to obtain a periodic admissible pair  $(u_{N,\Delta}(\tau), y_{N,\Delta}(\tau))$  defined on the interval  $[0, T_{N,\Delta}]$ . Calculate the integral

$$\frac{1}{T_{N,\Delta}} \int_0^{T_{N,\Delta}} g(u_{N,\Delta}(\tau), y_{N,\Delta}(\tau)) d\tau$$

and compare it with  $G_{N,\Delta}$ . If the value of the integral proves to be close to  $G_{N,\Delta}$ , then the constructed admissible pair is a “good” approximation to the solution of the periodic optimization problem.

Note that these steps are similar to the procedure described in [32]. The difference is that the approximation  $u_{N,\Delta}(\cdot)$  to the optimal control is determined as a solution of the problem (7.1). The objective function of the latter is constructed with the help of a solution of the problem (6.7) dual to  $N\Delta$ -LLPP (6.3) (the opportunity of using dual solutions for finding an approximation to the optimal control was not discussed in [32]). In the next section we consider a numerical example to illustrate the above steps.

Let us conclude this section by noting that most of the existing computational methods for periodic optimization problems are either aimed at solving the system of necessary optimality conditions (so-called dynamic optimization schemes; see, e.g., [45], [53], and references therein) or based on approximating the POP with F-D nonlinear mathematical programming problems. The latter are defined either via a discretization of the time interval (as in direct optimization schemes; see, e.g., [45]) or with the help of special parametrization techniques (as in the flatness-based algorithm of [54]). In all instances the time interval (period) is assumed to be finite, with its length being one of the optimization parameters.

In contrast to these methods, the approach of this paper is based on the fact that, if the optimal periodic regime exists, then, under mild conditions, it is a solution of the long run average optimal control problem, and, also, it generates the occupational measure that is a solution of the LLPP (3.2); the latter does not involve the time parameter at all. As has been noticed in section 4, finding a solution of the problem

dual to LLPP (3.2) is equivalent to finding a subsolution of the Hamilton–Jacobi–Bellman equation for the corresponding long run average problem of optimal control. Thus, the approach under consideration can be characterized as one belonging to the class of algorithms that solve Hamilton–Jacobi–Bellman equations for infinite time horizon optimal control problems via a discretization of the state and control spaces (see [25], [26], [43], and references therein). The obvious advantage of the approach is in its simplicity (no software except standard linear programming and ordinary differential equation solvers is needed). Its disadvantage is in computer memory requirements (especially in problems of higher dimensions). A comparison of the linear programming approach with other algorithms of the given class will be a matter of further research.

**8. A numerical example.** Let  $m = 2$ ,  $n = 1$ , and let

$$y(t) = (y_1(t), y_2(t)) \in Y \stackrel{\text{def}}{=} [-0.25, 0.25] \times [1.3, 1.9], \quad u(t) \in U \stackrel{\text{def}}{=} [-0.4, 0.4].$$

Also, let  $f(\cdot)$  and  $g(\cdot)$  be defined by the equations

$$(8.1) \quad f^T(u, y_1, y_2) \stackrel{\text{def}}{=} (y_2 - 1.6, u),$$

$$(8.2) \quad g(u, y_1, y_2) \stackrel{\text{def}}{=} ((y_2 - 3)(y_2 - 1)^2 + 2.5)(y_2 - 1) + 3 + 1.126y_1^2 + 0.4u^2.$$

The periodic optimization problem (3.13) with this data was considered and numerically solved using the “flatness-based algorithm” in [54]. Below we present a numerical solution of the problem obtained by following the steps outlined in section 7. Define the grid of  $U \times Y$  by the equations

$$u_i^\Delta \stackrel{\text{def}}{=} i\Delta - 0.4, \quad y_{1,j}^\Delta \stackrel{\text{def}}{=} j\Delta - 0.25, \quad y_{2,k}^\Delta \stackrel{\text{def}}{=} k\Delta + 1.3,$$

where  $i = 0, 1, \dots, \frac{0.8}{\Delta}$ ,  $j = 0, 1, \dots, \frac{0.5}{\Delta}$ , and  $k = 0, 1, \dots, \frac{0.6}{\Delta}$  ( $\Delta$  is chosen in such a way that  $\frac{0.8}{\Delta}$ ,  $\frac{0.5}{\Delta}$ , and  $\frac{0.6}{\Delta}$  are integers). In this case, the  $N\Delta$ -approximating LLPP (6.3) can be written in the form

$$(8.3) \quad G_{N,\Delta} \stackrel{\text{def}}{=} \min_{\gamma \in W_{N,\Delta}} \sum_{i,j,k} g(u_i^\Delta, y_{1,j}^\Delta, y_{2,k}^\Delta) \gamma_{i,j,k},$$

with  $W_{N,\Delta}$  being the polyhedral set defined by the equation

$$(8.4) \quad W_{N,\Delta} \stackrel{\text{def}}{=} \left\{ \gamma = \{\gamma_{i,j,k}\} \geq 0 : \sum_{i,j,k} \gamma_{i,j,k} = 1, \right. \\ \left. \sum_{i,j,k} (\phi'_{i_1,i_2}(y_{1,j}^\Delta, y_{2,k}^\Delta))^T f(u_i^\Delta, y_{1,j}^\Delta, y_{2,k}^\Delta) \gamma_{i,j,k} = 0, (i_1, i_2) \in I_N \right\}.$$

Here,  $\phi_{i_1,i_2}(y_1, y_2) \stackrel{\text{def}}{=} y_1^{i_1} y_2^{i_2}$  and  $I_N$  is the set of multi-indices

$$I_N \stackrel{\text{def}}{=} \{i : i = (i_1, i_2), \quad i_1, i_2 = 0, 1, \dots, N, \quad i_1 + i_2 \geq 1\}.$$

Note that  $g(\cdot)$  in (8.3) and  $f(\cdot)$  in (8.4) are as in (8.2) and (8.1), respectively.

The problem (8.3) was solved using the CPLEX linear programming solver [61] for different values of  $N$  and  $\Delta$ . The obtained optimal values  $G_{N,\Delta}$  are summarized in the following table:

$\Delta$	$N$		
	4	5	6
0.05	4.1959526	4.1959709	4.1959726
0.025	4.1958774	4.1958808	4.1958807
0.0125	4.1958750	4.1958751	4.1958751

On the basis of this data, one may conclude that  $G_6 = \lim_{\Delta \rightarrow 0} G_{6,\Delta} \approx 4.1958751 \approx 4.196$ , the latter coinciding with the optimal value of the given periodic optimization problem obtained in [54]. Thus, if for some  $T$ -periodic admissible pair  $(u(\tau), y(\tau))$ ,

$$(8.5) \quad \frac{1}{T} \int_0^T \left[ ((y_2(\tau) - 3)(y_2(\tau) - 1)^2 + 2.5)(y_2(\tau) - 1) + 3 + 1.126y_1(\tau)^2 + 0.4u(\tau)^2 \right] d\tau \approx 4.196,$$

then this pair is an approximate solution of the periodic optimization problem under consideration.

Let  $\gamma^{N,\Delta} = \{\gamma_{i,j,k}^{N,\Delta}\}$  stand for the solution of (8.3), and let  $\lambda^{N,\Delta} = \{\lambda_{i_1,i_2}^{N,\Delta}\}$  stand for the solution of the problem dual to (8.3)

$$(8.6) \quad \max_{(\mu,\lambda)} \left\{ \mu : \mu \leq g(u_i^\Delta, y_{1,j}^\Delta, y_{2,k}^\Delta) - \sum_{(i_1,i_2) \in I_N} \lambda_{i_1,i_2} (\phi'_{i_1,i_2}(y_{1,j}^\Delta, y_{2,k}^\Delta))^T f(u_i^\Delta, y_{1,j}^\Delta, y_{2,k}^\Delta) \quad \forall (u_i^\Delta, y_{1,j}^\Delta, y_{2,k}^\Delta) \right\}$$

(see (6.7) for comparison). Define the sets  $\Theta_{N,\Delta}$  and  $\mathcal{Y}_{N,\Delta}$  by the equations

$$(8.7) \quad \Theta_{N,\Delta} \stackrel{\text{def}}{=} \left\{ (u_i^\Delta, y_{1,j}^\Delta, y_{2,k}^\Delta) : \gamma_{i,j,k}^{N,\Delta} \neq 0 \right\},$$

$$(8.8) \quad \mathcal{Y}_{N,\Delta} \stackrel{\text{def}}{=} \left\{ (y_{1,j}^\Delta, y_{2,k}^\Delta) : \sum_i \gamma_{i,j,k}^{N,\Delta} \neq 0 \right\}$$

(see (7.11) and (7.12) for comparison). For  $N = 6$  and  $\Delta = 0.0125$ , the graphs of  $\Theta_{N,\Delta}$  and  $\mathcal{Y}_{N,\Delta}$  are depicted in Figures 1 and 2 below (the points belonging to these sets are marked). As follows from Theorem 7.3, these graphs can serve as approximations, respectively, to the graph of the optimal periodic control  $\Theta_*$  and to the optimal periodic orbit  $\mathcal{Y}_*$ .

Define the function  $\psi_{N,\Delta}(y_1, y_2)$  by the equation (see (6.11) for comparison)

$$(8.9) \quad \psi_{N,\Delta}(y_1, y_2) \stackrel{\text{def}}{=} \sum_{(i_1,i_2) \in I_N} \lambda_{i_1,i_2}^{N,\Delta} \phi_{i_1,i_2}(y_1, y_2).$$

The problem (7.1) is in this case of the form

$$(8.10) \quad \min_{u \in [-0.4, 0.4]} \Phi_{N,\Delta}(u, y_1, y_2),$$

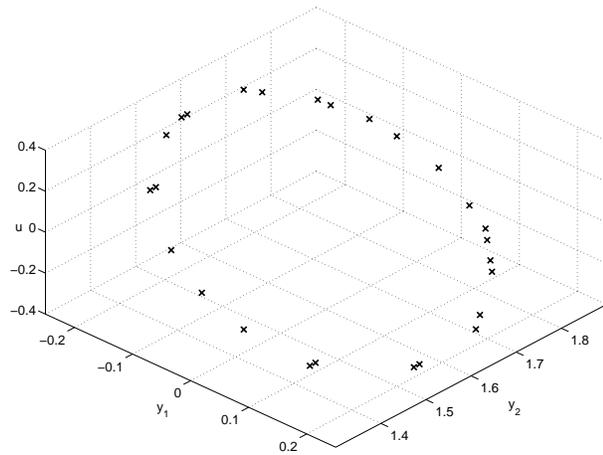


FIG. 1.  $\Theta_{N,\Delta}$ . Approximation to the graph of the optimal periodic control  $\Theta_*$ .

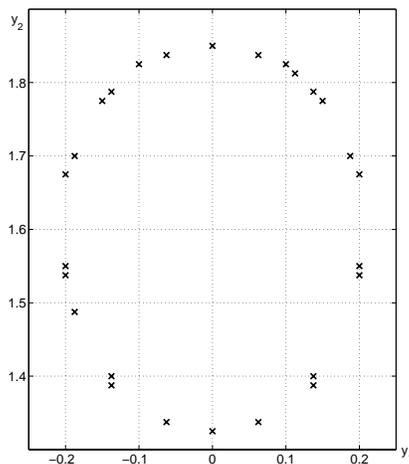


FIG. 2.  $\mathcal{Y}_{N,\Delta}$ . Approximation to the optimal periodic orbit  $\mathcal{Y}_*$ .

where

$$\begin{aligned}
 \Phi_{N,\Delta}(u, y_1, y_2) = & \left[ ((y_2 - 3)(y_2 - 1)^2 + 2.5)(y_2 - 1) + 3 + 1.126y_1^2 + 0.4u^2 \right. \\
 (8.11) \quad & \left. + \left( \frac{\partial \psi_{N,\Delta}(y_1, y_2)}{\partial y_1} (y_2 - 1.6) + \frac{\partial \psi_{N,\Delta}(y_1, y_2)}{\partial y_2} u \right) \right].
 \end{aligned}$$

Note that the function  $\Phi_{N,\Delta}(u, y_1, y_2)$  is strictly convex in  $u$  and, hence, the  $N$ -regularity condition is satisfied on every subset  $Q$  of  $Y$ . Also, the solution  $u_{N,\Delta}(y_1, y_2)$

of the problem (8.10) is unique and is defined by the equation (8.12)

$$u_{N,\Delta}(y_1, y_2) = \begin{cases} -1.25 \frac{\partial \psi_{N,\Delta}(y_1, y_2)}{\partial y_2}, & -0.4 \leq -1.25 \frac{\partial \psi_{N,\Delta}(y_1, y_2)}{\partial y_2} \leq 0.4, \\ -0.4, & -1.25 \frac{\partial \psi_{N,\Delta}(y_1, y_2)}{\partial y_2} < -0.4, \\ 0.4, & -1.25 \frac{\partial \psi_{N,\Delta}(y_1, y_2)}{\partial y_2} > 0.4. \end{cases}$$

The graph of this function (with  $N$  and  $\Delta$  as in Figures 1 and 2 is depicted in Figure 3, where the points belonging to the set  $\Theta_{N,\Delta}$  are marked, for convenience, as well (the latter being visibly close to the surface defined by  $u_{N,\Delta}(\cdot)$ ).

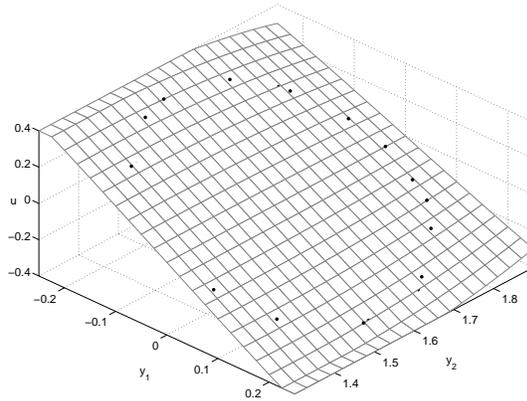


FIG. 3.  $u_{N,\Delta}(y_1, y_2)$ . Obtained feedback control function.

As can be seen from Figures 2 and 3, the function  $u_{N,\Delta}(y_1, y_2)$ , being defined by the right-hand side of (8.12), is smooth in a neighborhood  $Q$  of  $\mathcal{Y}_{N,\Delta}$ , the latter being close to the optimal periodic orbit  $\mathcal{Y}_*$ . Hence, one can assume (based on Theorem 7.1) that  $u_{N,\Delta}(y_1, y_2)$  can serve as an approximation for the optimal control  $u_*(y_1, y_2)$  on  $\mathcal{Y}_*$ . To verify that this is the case, we used MATLAB to integrate our system with the feedback control  $u_{N,\Delta}(y_1, y_2)$  and with the initial conditions  $y_1(0) = 0$ ,  $y_2(0) = 1.848$  (a point in  $\mathcal{Y}_{N,\Delta}$ ). The result of such integration is the  $T$ -periodic solution  $y(\tau) = (y_1(\tau), y_2(\tau))$  and the corresponding  $T$ -periodic control  $u(\tau) \stackrel{\text{def}}{=} u_{N,\Delta}(y_1(\tau), y_2(\tau))$ , with  $T = 4.8597$ . The graphs of these functions are shown in Figures 4 and 5 below (which look very similar to those obtained in [54]). The value of the objective function obtained on this periodic solution is evaluated to be equal to  $4.195877 \approx 4.196$ . That is, (8.5) is satisfied and, hence, an approximate solution to the periodic optimization problem is found. Note that the closed curve  $\mathcal{Y}$  defined by  $(y_1(\tau), y_2(\tau))$ ,

$$(8.13) \quad \mathcal{Y} \stackrel{\text{def}}{=} \{(y_1, y_2) : (y_1, y_2) = (y_1(\tau), y_2(\tau)) \text{ for some } \tau \in [0, T]\},$$

passes very close to the points of  $\mathcal{Y}_{N,\Delta}$  as illustrated by Figure 6.

**9. Proofs for sections 4 and 5.** Due to the approximating property of the sequence of the functions  $\phi_i(\cdot)$ ,  $i = 1, 2, \dots$ , (see (5.1)), the set  $W$  can be presented in the form

$$(9.1) \quad W = \left\{ \gamma \in \mathcal{P}(U \times Y) : \int_{U \times Y} (\phi'_i(y))^T f(u, y) \gamma(du, dy) = 0, \quad i = 1, 2, \dots \right\},$$

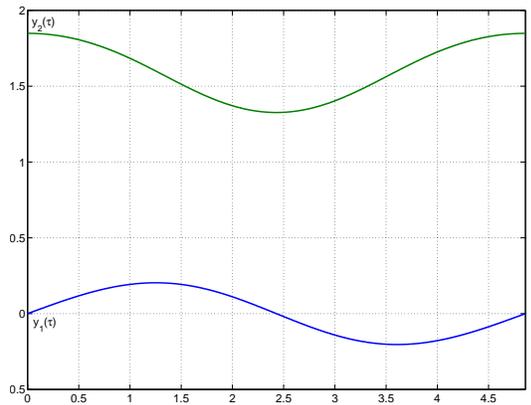


FIG. 4.  $(y_1(\tau), y_2(\tau))$ . Obtained periodic solution.

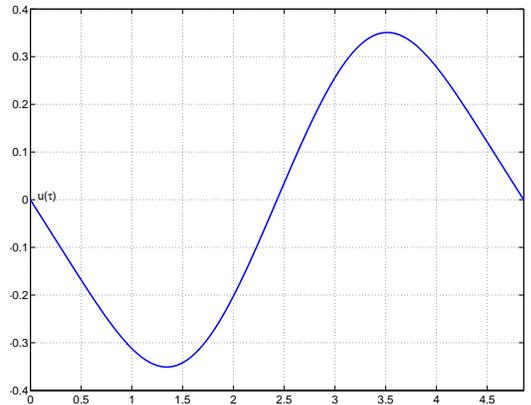


FIG. 5.  $u(\tau)$ . Obtained periodic control.

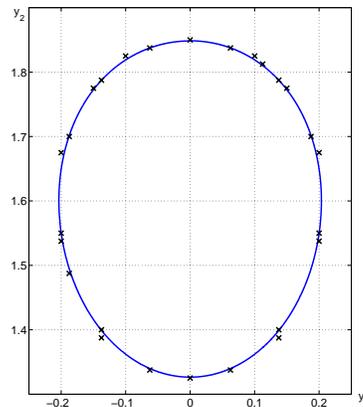


FIG. 6.  $\mathcal{Y}$ . Obtained periodic orbit.

where, without loss of generality, one may assume that the functions  $\phi_i(\cdot)$  satisfy the following normalization conditions:

$$(9.2) \quad \max_{y \in D} \{|\phi_i(y)|, \|\phi'_i(y)\|, \|\phi''_i(y)\|\} \leq \frac{1}{2^i}, \quad i = 1, 2, \dots$$

In the above expression,  $\|\phi''_i(y)\|$  is a norm of the Hessian (the matrix of second derivatives of  $\phi_i(y)$ ) in  $\mathbb{R}^m \times \mathbb{R}^m$ ,  $\|\phi'_i(y)\|$  is a norm of  $\phi'_i(y)$  in  $\mathbb{R}^m$ , and  $\hat{D}$  is a closed ball in  $\mathbb{R}^m$  that contains  $Y$  in its interior.

Let  $l_1$  and  $l_\infty$  stand for the Banach spaces of infinite sequences such that, for any  $x = (x_1, x_2, \dots) \in l_1$ ,  $\|x\|_{l_1} \stackrel{\text{def}}{=} \sum_i |x_i| < \infty$  and, for any  $\lambda = (\lambda_1, \lambda_2, \dots) \in l_\infty$ ,  $\|\lambda\|_{l_\infty} \stackrel{\text{def}}{=} \sup_i |\lambda_i| < \infty$ . It is easy to see that, given an element  $\lambda \in l_\infty$ , one can define a linear continuous functional  $\lambda(\cdot) : l_1 \rightarrow \mathbb{R}^1$  by the equation

$$(9.3) \quad \lambda(x) = \sum_i \lambda_i x_i \quad \forall x \in l_1, \quad \|\lambda(\cdot)\| = \|\lambda\|_{l_\infty}.$$

It is also known (see, e.g., [49, p. 86]) that any continuous linear functional  $\lambda(\cdot) : l_1 \rightarrow \mathbb{R}^1$  can be presented in the form (9.3) with some  $\lambda \in l_\infty$ .

By (9.2),  $(\phi_1(y), \phi_2(y), \dots) \in l_1$  and  $(\frac{\partial \phi_1}{\partial y_j}, \frac{\partial \phi_2}{\partial y_j}, \dots) \in l_1$  for any  $y \in Y$ . Hence, the function  $\psi_\lambda(y)$ ,

$$(9.4) \quad \psi_\lambda(y) \stackrel{\text{def}}{=} \sum_i \lambda_i \phi_i(y), \quad \lambda = (\lambda_1, \lambda_2, \dots) \in l_\infty,$$

is continuously differentiable, with  $\psi'_\lambda(y) = \sum_i \lambda_i \phi'_i(y)$ .

*Proof of Theorem 4.1(iii).* If the function  $\psi(\cdot)$  satisfying (4.6) exists, then  $\min_{(u,y) \in U \times Y} (-\psi'(y))^T f(u, y) > 0$  and, hence,

$$(9.5) \quad \lim_{\alpha \rightarrow \infty} \min_{(u,y) \in U \times Y} \{g(u, y) + \alpha (-\psi'(y))^T f(u, y)\} = \infty.$$

This implies that the optimal value of the dual problem is unbounded ( $\mu_* = \infty$ ).

Assume now that the optimal value of the dual problem is unbounded. That is, there exists a sequence  $(\mu_k, \psi_k(\cdot))$  such that

$$(9.6) \quad \mu_k \leq g(u, y) + (\psi'_k(y))^T f(u, y) \quad \forall (u, y) \in U \times Y, \quad \lim_{k \rightarrow \infty} \mu_k = \infty$$

$$(9.7) \quad \Rightarrow 1 \leq \frac{1}{\mu_k} g(u, y) + \frac{1}{\mu_k} (\psi'_k(y))^T f(u, y) \quad \forall (u, y) \in U \times Y.$$

For  $k$  large enough,  $\frac{1}{\mu_k} |g(u, y)| \leq \frac{1}{2} \quad \forall (u, y) \in U \times Y$ . Hence

$$\frac{1}{2} \leq \frac{1}{\mu_k} (\psi'_k(y))^T f(u, y) \quad \forall (u, y) \in U \times Y.$$

That is, the function  $\psi(y) \stackrel{\text{def}}{=} -\frac{1}{\mu_k} \psi_k(y)$  satisfies (4.6).  $\square$

*Proof of Theorem 4.1(i).* From (4.4) it follows that, if  $W$  is not empty, then the optimal value of the dual problem is bounded.

Conversely, let us assume that the optimal value  $\mu_*$  of the dual problem is bounded and let us establish that  $W$  is not empty. Assume that this is not true and  $W$  is empty. Define the set  $\mathcal{Q}$  by the equation

$$(9.8) \quad \mathcal{Q} \stackrel{\text{def}}{=} \left\{ x = (x_1, x_2, \dots) : x_i = \int_{u,y} (\phi'_i(y))^T f(u, y) \gamma(du, dy), \quad \gamma \in \mathcal{P}(U \times Y) \right\}.$$

It is easy to see that the set  $\mathcal{Q}$  is a convex and compact subset of  $l_1$  (the fact that  $\mathcal{Q}$  is relatively compact in  $l_1$  is implied by (9.2); the fact that it is closed follows from that  $\mathcal{P}(U \times Y)$  is compact in weak convergence topology).

By (9.1), the assumption that  $W$  is empty is equivalent to the assumption that the set  $\mathcal{Q}$  does not contain the “zero element” ( $0 \notin \mathcal{Q}$ ). Hence, by a separation theorem (see, e.g., [49, p. 59]), there exists  $\bar{\lambda} = (\bar{\lambda}_1, \bar{\lambda}_2, \dots) \in l_\infty$  such that

$$\begin{aligned} 0 = \bar{\lambda}(0) &> \max_{x \in \mathcal{Q}} \sum_i \bar{\lambda}_i x_i = \max_{\gamma \in \mathcal{P}(U \times Y)} \int_{U \times Y} (\psi'_{\bar{\lambda}}(y))^T f(u, y) \gamma(du, dy) \\ &= \max_{(u, y) \in U \times Y} (\psi'_{\bar{\lambda}}(y))^T f(u, y), \end{aligned}$$

where  $\psi_{\bar{\lambda}}(y) = \sum_i \bar{\lambda}_i \phi_i(y)$  (see (9.4)). This implies that the function  $\psi(y) \stackrel{\text{def}}{=} \psi_{\bar{\lambda}}(y)$  satisfies (4.6), and, by Theorem 4.1(iii),  $\mu_*$  is unbounded. Thus, we have obtained a contradiction that proves that  $W$  is not empty.  $\square$

*Proof of Theorem 4.1(ii).* By Theorem 4.1(i), if the optimal value of the dual problem (4.1) is bounded, then  $W$  is not empty and, hence, a solution of the problem (3.2) exists.

Define the set  $\hat{\mathcal{Q}} \subset \mathbb{R}^1 \times l_1$  by the equation

$$\begin{aligned} \hat{\mathcal{Q}} &\stackrel{\text{def}}{=} \left\{ (\theta, x) : \theta \geq \int_{U \times Y} g(u, y) \gamma(du, dy), \right. \\ (9.9) \quad x &= (x_1, x_2, \dots), \quad x_i = \int_{U \times Y} (\phi'_i(y))^T f(u, y) \gamma(du, dy), \quad \gamma \in \mathcal{P}(U \times Y) \left. \right\}. \end{aligned}$$

The set  $\hat{\mathcal{Q}}$  is convex and closed. Also, for any  $j = 1, 2, \dots$ , the point  $(\theta_j, 0) \notin \hat{\mathcal{Q}}$ , where  $\theta_j \stackrel{\text{def}}{=} G_* - \frac{1}{j}$  and 0 is the zero element of  $l_1$ . On the basis of a separation theorem (see [49, p. 59]), one may conclude that there exists a sequence  $(\kappa^j, \lambda^j) \in \mathbb{R}^1 \times l_\infty$ ,  $j = 1, 2, \dots$  (with  $\lambda^j \stackrel{\text{def}}{=} (\lambda_1^j, \lambda_2^j, \dots)$ ) such that

$$\begin{aligned} \kappa^j \left( G_* - \frac{1}{j} \right) + \delta^j &\leq \inf_{(\theta, x) \in \hat{\mathcal{Q}}} \left\{ \kappa^j \theta + \sum_i \lambda_i^j x_i \right\} = \inf_{\gamma \in \mathcal{P}(U \times Y)} \left\{ \kappa^j \theta \right. \\ (9.10) \quad &+ \left. \int_{U \times Y} (\psi'_{\lambda^j}(y))^T f(u, y) \gamma(du, dy) \text{ s.t. } \theta \geq \int_{U \times Y} g(u, y) \gamma(du, dy) \right\}, \end{aligned}$$

where  $\delta^j > 0$  for all  $j$  and  $\psi_{\lambda^j}(y) = \sum_i \lambda_i^j \phi_i(y)$ . From (9.10) it immediately follows that  $\kappa^j \geq 0$ . Let us show that  $\kappa^j > 0$ . In fact, if this were not the case, one would obtain that

$$\begin{aligned} 0 < \delta^j &\leq \min_{\gamma \in \mathcal{P}(U \times Y)} \int_{U \times Y} (\psi'_{\lambda^j}(y))^T f(u, y) \gamma(du, dy) = \min_{(u, y) \in U \times Y} \{ (\psi'_{\lambda^j}(y))^T f(u, y) \} \\ &\Rightarrow \max_{(u, y) \in U \times Y} \{ (-\psi'_{\lambda^j}(y))^T f(u, y) \} \leq -\delta^j < 0. \end{aligned}$$

The latter would lead to the validity of the inequality (4.6) with  $\psi(y) = -\psi_{\lambda^j}(y)$ , which, by Theorem 4.1(iii), would imply that the optimal value of the dual problem is unbounded. Thus,  $\kappa^j > 0$ .

Dividing (9.10) by  $\kappa^j$  one can obtain that

$$\begin{aligned} G_* - \frac{1}{j} &< \left(G_* - \frac{1}{j}\right) + \frac{\delta^j}{\kappa_j} \\ &\leq \min_{\gamma \in \mathcal{P}(U \times Y)} \left\{ \int_{U \times Y} \left(g(u, y) + \frac{1}{\delta^j} (\psi'_{\lambda^j}(y))^T f(u, y)\right) \gamma(du, dy) \right\} \\ &= \min_{(u, y) \in U \times Y} \left\{ g(u, y) + \frac{1}{\delta^j} (\psi'_{\lambda^j}(y))^T f(u, y) \right\} \leq \mu^* \Rightarrow G_* \leq \mu^*. \end{aligned}$$

The latter and (4.4) prove (4.5).  $\square$

*Proof of Theorem 5.2.* The proof statement (iii) of the theorem follows the argument used in the proof of Theorem 4.1(iii), with the replacement of  $\psi(y)$  in (9.5) by  $\psi_v(y) = \sum_1^N v_i \phi_i(y)$ ,  $v = (v_i) \in \mathbb{R}^N$ , and with the replacement of  $\psi_k(y)$  in (9.6), (9.7) by  $\psi_{v^k}(y) = \sum_1^N v_i^k \phi_i(y)$ ,  $v^k = (v_i^k) \in \mathbb{R}^N$ .

The proofs of statements (i) and (ii) of the theorem are based on a separation theorem in F-D spaces and follow the argument used in the proofs of Theorem 4.1(i) and Theorem 4.1(ii), with the replacement of the set  $\mathcal{Q}$  defined in (9.8) by the set  $\mathcal{Q}' \subset \mathbb{R}^N$ ,

$$(9.11) \quad \mathcal{Q}' \stackrel{\text{def}}{=} \left\{ x = (x_1, \dots, x_N) : x_i = \int_{u, y} (\phi'_i(y))^T f(u, y) \gamma(du, dy), \gamma \in \mathcal{P}(U \times Y) \right\},$$

and with the replacement of the set  $\hat{\mathcal{Q}}$  defined in (9.9) by the set  $\hat{\mathcal{Q}}' \subset \mathbb{R}^1 \times \mathbb{R}^N$ ,

$$(9.12) \quad \hat{\mathcal{Q}}' \stackrel{\text{def}}{=} \left\{ (\theta, x) : \theta \geq \int_{U \times Y} g(u, y) \gamma(du, dy), \right. \\ \left. x = (x_1, \dots, x_N), x_i = \int_{U \times Y} (\phi'_i(y))^T f(u, y) \gamma(du, dy), \gamma \in \mathcal{P}(U \times Y) \right\}. \quad \square$$

*Proof of Proposition 5.3.* Let (5.17) be valid. Then, for some positive  $r$ ,

$$r\bar{B} \subset \text{co}K_N,$$

where  $\bar{B}$  is a closed unit ball in  $\mathbb{R}^N$ . By (5.18), the inequality (5.15) is equivalent to

$$(9.13) \quad \max_{z=(z_i) \in K_N} \sum_{i=1}^N v_i z_i \leq 0 \Rightarrow \max_{z=(z_i) \in r\bar{B}} \sum_{i=1}^N v_i z_i \leq 0.$$

Since  $\max_{z=(z_i) \in r\bar{B}} \sum_{i=1}^N v_i z_i = r\|v\|$ , the second inequality in (9.13) implies that  $v = 0$ . That is, Assumption 1 is satisfied.

Assume now that (5.17) is not valid, that is, either 0 is a boundary point of  $\text{co}K_N$  or it does not belong to the closure of  $\text{co}K_N$ . Then, by separation theorem, there exists a vector  $v = (v_i) \neq 0$  such that

$$(9.14) \quad \max_{z=(z_i) \in K_N} \sum_{i=1}^N v_i z_i = \max_{z=(z_i) \in \text{co}K_N} \sum_{i=1}^N v_i z_i \leq 0.$$

This implies that Assumption 1 is not satisfied.  $\square$

*Proof of Theorem 5.5.* The fact that Assumption 1 is satisfied implies, in particular, that the function  $\psi_v(\cdot)$  satisfying (5.13) does not exist. Hence, by Theorem 5.2(iii), the optimal value  $\mu_N$  of the D-N-LLPP (5.8) is bounded. Let  $v^k = (v_i^k) \in \mathbb{R}^N$  be such that

$$(9.15) \quad \mu_N - \frac{1}{k} \leq g(u, y) + \left( \sum_{i=1}^N v_i^k \phi'_i(y) \right)^T f(u, y) \quad \forall (u, y) \in U \times Y, \quad k = 1, 2, \dots$$

Show that the sequence  $v^k$  is bounded. That is,

$$(9.16) \quad \|v^k\| \leq c_N, \quad k = 1, 2, \dots$$

In fact, if  $v^k, k = 1, 2, \dots$ , were not bounded, then there would exist a subsequence  $v^{k'}$  such that

$$(9.17) \quad \lim_{k' \rightarrow \infty} \|v^{k'}\| = \infty, \quad \lim_{k' \rightarrow \infty} \frac{v^{k'}}{\|v^{k'}\|} \stackrel{\text{def}}{=} \tilde{v}, \quad \|\tilde{v}\| = 1.$$

Dividing (9.15) by  $\|v^k\|$  and passing to the limit over the subsequence  $\{k'\}$ , one would obtain the following inequality:

$$(9.18) \quad 0 \leq \left( \sum_{i=1}^N \tilde{v}_i \phi'_i(y) \right)^T f(u, y) \quad \forall (u, y) \in U \times Y.$$

By Assumption 1, the fact that (9.18) is valid implies that  $\tilde{v} = (\tilde{v}_i) = 0$ , which is in contradiction to (9.17). Thus, the validity of (9.16) is established.

Due to (9.16), there exists a subsequence  $\{k'\}$  such that there exists a limit

$$(9.19) \quad \lim_{k' \rightarrow \infty} v^{k'} \stackrel{\text{def}}{=} v.$$

Passing over this subsequence to the limit in (9.15), one obtains

$$(9.20) \quad \mu_N \leq g(u, y) + \left( \sum_{i=1}^N v_i \phi'_i(y) \right)^T f(u, y) \quad \forall (u, y) \in U \times Y \quad \Rightarrow \quad v = (v_i) \in V_N.$$

The latter proves that  $V_N$  is not empty. The proof of that  $V_N$  is bounded follows the same argument as that used above to establish (9.16). This completes the proof of statement (i) of the theorem.

To prove statement (ii), let us assume that (5.19) and (5.20) are valid but Assumption 1 is not satisfied, and, hence, there exists a vector  $v = (v_i) \neq 0$  such that

$$(9.21) \quad \left( \sum_{i=1}^N v_i \phi'_i(y) \right)^T f(u, y) \leq 0 \quad \forall (u, y) \in U \times Y.$$

By (9.21), for an arbitrary  $v^0 = (v_i^0) \in V_N$  and for an arbitrary  $\alpha \geq 0$ ,

$$(9.22) \quad \mu_N \leq g(u, y) + \left( \sum_{i=1}^N (v_i^0 - \alpha v_i) \phi'_i(y) \right)^T f(u, y) \quad \forall (u, y) \in U \times Y.$$

The latter implies that  $(v^0 - \alpha v) \in V_N \forall \alpha \geq 0$ , which contradicts the fact that  $V_N$  is bounded (postulated by (5.20)). This proves statement (ii) of the theorem.  $\square$

*Proof of Theorem 5.6.* Consider first the case when  $\mu_* < \infty$ . For arbitrary small  $\delta > 0$ , there exists a function  $\psi(\cdot) \in C^1$  such that

$$\min_{(u,y) \in U \times Y} \{g(u, y) + \psi'(y)f(u, y)\} \geq \mu_* - \delta.$$

By the approximating property (5.1), there exist numbers  $v_1, v_2, \dots, v_{\bar{N}}$  (with  $\bar{N}$  being large enough) such that

$$(9.23) \quad \max_{y \in Y} \left\| \psi'(y) - \sum_{i=1}^{\bar{N}} v_i \phi'_i(y) \right\| \leq \delta$$

$$(9.24) \quad \begin{aligned} &\Rightarrow \left| \min_{(u,y) \in U \times Y} \{g(u, y) + \psi'(y)f(u, y)\} \right. \\ &\quad \left. - \min_{(u,y) \in U \times Y} \left\{ g(u, y) + \sum_{i=1}^{\bar{N}} v_i (\phi'_i(y))^T f(u, y) \right\} \right| \leq c\delta, \end{aligned}$$

where  $c \stackrel{\text{def}}{=} \max_{(u,y) \in U \times Y} \|f(u, y)\|$ . From the above expressions and from the fact that

$$(9.25) \quad \min_{(u,y) \in U \times Y} \left\{ g(u, y) + \sum_{i=1}^N v_i (\phi'_i(y))^T f(u, y) \right\} \leq \mu_N,$$

it follows that  $\mu_* - \delta \leq \mu_{\bar{N}} + c\delta$ . This and (5.21) imply that  $0 \leq \mu_* - \mu_N \leq c(1 + \delta)$  for  $N \geq \bar{N}$ . Since  $\delta$  is arbitrarily small, the latter proves (5.22) for bounded  $\mu_*$ .

Assume now that  $\mu_* = \infty$ . Then, for an arbitrary large  $A > 0$ , there exists a function  $\psi(\cdot) \in C^1$  such that

$$\min_{(u,y) \in U \times Y} \{g(u, y) + \psi'(y)f(u, y)\} \geq A.$$

By the approximating property (5.1), there exist numbers  $v_1, v_2, \dots, v_{\bar{N}}$  (with  $\bar{N}$  being large enough), such that (9.23) and (9.24) are valid. Also, the inequality (9.25) remains valid due to the definition of  $\mu_N$  (see (5.8)). Consequently,  $\mu_{\bar{N}} \geq A - c\delta$ . Since  $\delta > 0$  can be chosen arbitrarily small, the latter implies that  $\mu_{\bar{N}} \geq A$ . Hence,  $\lim_{N \rightarrow \infty} \mu_N = \infty$ .  $\square$

## 10. Proofs for sections 6 and 7.

*Proof of Theorem 6.2.* First, let us show that the set  $\Lambda_{N,\Delta}$  is bounded for  $\Delta$  small enough. That is, show that

$$(10.1) \quad \sup_{\lambda \in \Lambda_{N,\Delta}} \|\lambda\| \leq c_N = \text{const}$$

for  $\Delta \leq \Delta_N$  ( $\Delta_N > 0$ ). Assume that this is not true and, hence, there exist sequences  $\Delta_s$  and  $\lambda^{N,\Delta_s} \in \Lambda_{N,\Delta_s}$ ,  $s = 1, 2, \dots$ , such that

$$\lim_{s \rightarrow \infty} \Delta_s = 0, \quad \lim_{s \rightarrow \infty} \|\lambda^{N,\Delta_s}\| = \infty.$$

Without loss of generality one may assume that there exists a limit

$$(10.2) \quad \lim_{s \rightarrow \infty} \frac{\lambda^{N, \Delta_s}}{\|\lambda^{N, \Delta_s}\|} \stackrel{\text{def}}{=} v, \quad \|v\| = 1.$$

From the definition of  $\Lambda_{N, \Delta}$  (see (6.9)) it follows that the inequality

$$(10.3) \quad \mu_{N, \Delta} \leq g(u_l, y_k) + \sum_{i=1}^N \lambda_i^{N, \Delta} (\phi'_i(y_k))^T f(u_l, y_k)$$

is valid for any grid point  $(u_l, y_k) \in U \times Y$ . Substituting  $\Delta_s$  for  $\Delta$  in (10.3) and then dividing the latter by  $\|\lambda^{N, \Delta_s}\|$  and passing to the limit as  $s \rightarrow \infty$ , one can prove that

$$(10.4) \quad 0 \leq \sum_{i=1}^N v_i (\phi'_i(y))^T f(u, y) \quad \forall (u, y) \in U \times Y.$$

Note that the proof of the above inequality is based on the fact that (see (5.12), (6.6), and (6.8))

$$(10.5) \quad \lim_{\Delta \rightarrow 0} \mu_{N, \Delta} = \mu_N,$$

which, in particular, implies that  $\mu_{N, \Delta_s}$  remains bounded as  $s \rightarrow \infty$ , and also on the fact that any point  $(u, y)$  in  $U \times Y$  can be presented as a limit of a sequence of grid points. From Assumption 1 it now follows that  $v = (v_i) = 0$ , which contradicts (10.2). This proves (10.1).

Let us now prove (6.10). Assuming that it is not true, one can come to a conclusion that there exist a positive number  $\alpha$  and sequences  $\Delta_s$  and  $\lambda^{N, \Delta_s} \in \Lambda_{N, \Delta_s}$ ,  $s = 1, 2, \dots$ , such that

$$(10.6) \quad \lim_{s \rightarrow \infty} \Delta_s = 0, \quad \text{dist}(\lambda^{N, \Delta_s}, V_N) \geq \alpha, \quad s = 1, 2, \dots$$

Due to (10.1), one may assume without loss of generality that there exists a limit

$$(10.7) \quad \lim_{s \rightarrow \infty} \lambda^{N, \Delta_s} \stackrel{\text{def}}{=} v^N \quad \Rightarrow \quad \text{dist}(v^N, V_N) \geq \alpha.$$

Substituting  $\Delta_s$  for  $\Delta$  in (10.3), taking into account (10.5), and passing to the limit as  $s \rightarrow \infty$ , one can obtain that

$$(10.8) \quad \mu_N \leq g(u, y) + \sum_{i=1}^N v_i^N (\phi'_i(y))^T f(u, y) \quad \forall (u, y) \in U \times Y \quad \Rightarrow \quad v^N = (v_i^N) \in V_N.$$

The latter contradicts (10.7) and, thus, proves (6.10).  $\square$

*Proofs of Theorem 7.1 and Corollary 7.2.* Fix an arbitrary  $\bar{y} \in \mathcal{Y}_* \cap Q$ . From Theorem 7.3(ii) it follows that there exists  $(u_{l_{N, \Delta}}, y_{k_{N, \Delta}}) \in \Theta_{N, \Delta}$  such that

$$(10.9) \quad \lim_{N \rightarrow \infty} \overline{\lim}_{\Delta \rightarrow 0} \|(u_{l_{N, \Delta}}, y_{k_{N, \Delta}}) - (u_*(\bar{y}), \bar{y})\| = 0.$$

Note that, since  $Q$  is open, from (10.9) it follows that there exists  $r > 0$  such that, for  $N$  large and  $\Delta$  small enough,

$$(10.10) \quad y_{k_{N, \Delta}} \in \bar{y} + r\bar{D} \subset Q, \quad \bar{D} \stackrel{\text{def}}{=} \{y : \|y\| \leq 1\}.$$

Show that the validity of (10.9) also implies the validity of

$$(10.11) \quad \lim_{N \rightarrow \infty} \overline{\lim}_{\Delta \rightarrow 0} \|u_{N,\Delta}(y_{k_{N,\Delta}}) - u_*(\bar{y})\| = 0$$

if  $N$ -RC is satisfied.

Assume that  $N$ -RC is satisfied on  $Q$  but (10.11) is not true. Then, due to (10.9), there exist a number  $\alpha > 0$  and sequences  $N_s$  ( $\lim_{s \rightarrow \infty} N_s = \infty$ ),  $\Delta_{s,j}$  ( $\lim_{j \rightarrow \infty} \Delta_{s,j} = 0$ ) such that

$$(10.12) \quad \|u_{N_s, \Delta_{s,j}}(y_{k_{N_s, \Delta_{s,j}}}) - u_{l_{N_s, \Delta_{s,j}}}\| \geq \alpha.$$

Without loss of generality, one may assume that there exist limits

$$(10.13) \quad \lim_{j \rightarrow \infty} y_{k_{N_s, \Delta_{s,j}}} \stackrel{\text{def}}{=} y_{N_s}, \quad \lim_{j \rightarrow \infty} u_{N_s, \Delta_{s,j}}(y_{k_{N_s, \Delta_{s,j}}}) \stackrel{\text{def}}{=} \tilde{u}_{N_s}, \quad \lim_{j \rightarrow \infty} u_{l_{N_s, \Delta_{s,j}}} \stackrel{\text{def}}{=} \tilde{u}_{N_s},$$

and also that there exists a limit

$$(10.14) \quad \lim_{j \rightarrow \infty} \lambda^{N_s, \Delta_{s,j}} \stackrel{\text{def}}{=} v^{N_s}, \quad v^{N_s} = (v_i^{N_s}) \in V_{N_s},$$

where the validity of the last inclusion is implied by Theorem 6.2 (remember that  $\lambda^{N_s, \Delta_{s,j}} \in \Lambda_{N_s, \Delta_{s,j}}$ ). From the duality relationships between the solutions of  $N\Delta$ -LLPP (6.3) and the solutions of  $D$ - $N\Delta$ -LLPP (6.7) it follows that, for any  $(u_l, y_k) \in \Theta_{N,\Delta}$ ,

$$(10.15) \quad \mu_{N,\Delta} = g(u_l, y_k) + \sum_{i=1}^N \lambda_i^{N,\Delta} (\phi'_i(y_k))^T f(u_l, y_k).$$

From (10.15) and from the fact that  $u_{N,\Delta}(\cdot)$  is defined as a solution of (7.1) it also follows that, for any  $y_k \in \mathcal{Y}_{N,\Delta}$ ,

$$(10.16) \quad \mu_{N,\Delta} \geq g(u_{N,\Delta}(y_k), y_k) + \sum_{i=1}^N \lambda_i^{N,\Delta} (\phi'_i(y_k))^T f(u_{N,\Delta}(y_k), y_k).$$

Via substituting  $N_s, \Delta_{s,j}, u_{l_{N_s, \Delta_{s,j}}}, y_{k_{N_s, \Delta_{s,j}}}$  for, respectively,  $N, \Delta, u_l, y_k$  in (10.15) and then passing to the limit as  $j \rightarrow \infty$ , one can obtain (see (10.13), (10.14) and (5.12), (6.6), (6.8)) that

$$(10.17) \quad \mu_{N_s} = g(\tilde{u}_{N_s}, y_{N_s}) + \sum_{i=1}^N v_i^{N_s} (\phi'_i(y_{N_s}))^T f(\tilde{u}_{N_s}, y_{N_s}).$$

Similarly, via substituting  $N_s, \Delta_{s,j}, u_{N_s, \Delta_{s,j}}(y_{k_{N_s, \Delta_{s,j}}}), y_{k_{N_s, \Delta_{s,j}}}$  for, respectively,  $N, \Delta, u_{N,\Delta}(y_k), y_k$  in (10.16) and passing to the limit as  $j \rightarrow \infty$ , one obtains that

$$(10.18) \quad \mu_{N_s} \geq g(\tilde{u}_{N_s}, y_{N_s}) + \sum_{i=1}^N v_i^{N_s} (\phi'_i(y_{N_s}))^T f(\tilde{u}_{N_s}, y_{N_s})$$

$$(10.19) \quad \Rightarrow \quad \mu_{N_s} = g(\tilde{u}_{N_s}, y_{N_s}) + \sum_{i=1}^N v_i^{N_s} (\phi'_i(y_{N_s}))^T f(\tilde{u}_{N_s}, y_{N_s}).$$

The equality (10.19) is implied by the inequality (10.18) due to the fact that

$$(10.20) \quad \mu_{N_s} = \min_{(u,y) \in U \times Y} \left\{ g(u,y) + \sum_{i=1}^N v_i^{N_s} (\phi'_i(y))^T f(u,y) \right\},$$

which is a consequence of  $v^{N_s} \in V_{N_s}$  (see (5.19)).

From (10.17) and (10.19) it follows (see the notation introduced in (5.23), (5.24), and (5.25)) that

$$(10.21) \quad y_{N_s} \in \mathcal{Y}_{N_s}^{v^{N_s}}, \quad \tilde{u}_{N_s} \in \mathcal{U}_{N_s}^{v^{N_s}}(y_{N_s}), \quad \tilde{u}_{N_s} \in \mathcal{U}_{N_s}^{v^{N_s}}(y_{N_s}).$$

By (10.10),  $y_{N_s} \in Q \cap \mathcal{Y}_{N_s}^{v^{N_s}}$  for  $s$  large enough. Hence, due to the validity of  $N$ -RC on  $Q$  (with  $N = N_s$ ),  $\mathcal{U}_{N_s}^{v^{N_s}}(y_{N_s})$  is a singleton. Consequently, by (10.21),

$$(10.22) \quad \tilde{u}_{N_s} = \tilde{u}_{N_s}.$$

Since, on the other hand, by (10.12) and (10.13),  $\|\tilde{u}_{N_s} - \tilde{u}_{N_s}\| \geq \alpha$ , one obtains a contradiction that establishes the validity of (10.11).

To prove (7.6), let us assume that it is not valid for the given  $\bar{y}$  and, hence, there exist sequences  $N' \rightarrow \infty$  and  $\Delta' \rightarrow 0$  such that

$$(10.23) \quad \lim_{N' \rightarrow \infty} \lim_{\Delta' \rightarrow 0} u_{N', \Delta'}(\bar{y}) = \tilde{u} \neq u_*(\bar{y}).$$

The functions  $u_{N', \Delta'}(\cdot)$  are uniformly equicontinuous on  $\bar{Q}' = \bar{y} + r\bar{D}$ . Consequently, by the Arzela–Ascoli theorem, there exist subsequences  $N'' \rightarrow \infty$  and  $\Delta'' \rightarrow 0$  of  $N'$  and  $\Delta'$ , and there exist a uniformly continuous function  $u_0(\cdot) : \bar{y} + r\bar{D} \rightarrow U$  such that

$$(10.24) \quad \lim_{N'' \rightarrow \infty} \lim_{\Delta'' \rightarrow 0} \max_{x \in \bar{y} + r\bar{D}} \|u_{N'', \Delta''}(x) - u_0(x)\| = 0,$$

where, by (10.23),

$$(10.25) \quad u_0(\bar{y}) = \tilde{u} \neq u_*(\bar{y}).$$

On the other hand, passing to the limit over the subsequences  $N''$  and  $\Delta''$  in the inequality

$$\|u_0(\bar{y}) - u_*(\bar{y})\| \leq \|u_0(\bar{y}) - u_{N, \Delta}(\bar{y})\|$$

$$+ \|u_{N, \Delta}(\bar{y}) - u_{N, \Delta}(y_{k_{N, \Delta}})\| + \|u_{N, \Delta}(y_{k_{N, \Delta}}) - u_*(\bar{y})\|$$

and using (7.5), (10.11), and (10.24), one can obtain that  $\|u_0(\bar{y}) - u_*(\bar{y})\| = 0$ , which contradicts (10.25) and, thus, proves (7.6). The uniform convergence of  $u_{N, \Delta}(\cdot)$  to  $u_*(\cdot)$  on any closed subset of  $\mathcal{Y}_* \cap Q$  follows from (7.6) and the Arzela–Ascoli theorem. Thus, statement (i) of the theorem is proved.

Statement (ii) of theorem is a special case ( $J = 1$ ) of Corollary 7.2, which we now will prove. By Theorem 7.1(i) established above, (7.10) is valid for any

$$y \in \cup_j (\mathcal{Y}_* \cap Q_j) = \mathcal{Y}_* \cap (\cup_j Q_j).$$

Thus, to establish the validity of the required result, one needs to verify that

$$(10.26) \quad \gamma_*(\mathcal{Y}_* \cap (\cup_j Q_j)) = 1.$$

From (7.4) and (7.8) it follows that  $\gamma_*(\overline{\cup_j Q_j}) = 1$ , which implies that

$$y_*(\tau) \in \overline{\cup_j Q_j} \quad \forall \tau \in [0, T].$$

The latter is equivalent to  $\mathcal{Y}_* \subset \overline{\cup_j Q_j}$  and, hence,

$$(10.27) \quad \gamma_*(\mathcal{Y}_* \cap (\cup_j Q_j)) = \gamma_*(\mathcal{Y}_* \cap (\overline{\cup_j Q_j})) - \gamma_*(\mathcal{Y}_* \cap (\overline{\cup_j Q_j} / \cup_j Q_j)).$$

This and (7.9) imply (10.26).  $\square$

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