Physics of the dusty Hall plasmas

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The presence of the immobile charged dust in the plasma modifies the scale over which the Hall effect becomes important. For a positively charged dusty background this scale can become arbitrarily large. It is shown that the emergence of the Hall effect in an immobile charged background is related to the presence of an electric field that operates over the plasma gyration period. The generalized flux, which is a combination of the magnetic and fluid vortex flux, can decay due to the presence of the charge or the density inhomogeneities. The normal mode behavior of such a dusty plasma could be very different for positively and negatively charged grains. Whereas for negatively charged grains the usual magnetohydrodynamic (MHD) modes are present in the system, for positively charged grains, the Alfvén mode may not exist if $Zn_d \sim n_e$, where Z is the charge of the dust and $n_d(n_e)$ are the dust (electron) number densities. In the presence of the inhomogeneities, inertialess dusty plasma is subject to the Hall instability. It is shown that the growth rate of the Hall instability is proportional to the whistler frequency. Since Hall drift is nondissipative in nature, this instability can play important role in redistributing the magnetic energy from the large to small scales. © *2006 American Institute of Physics*. [DOI: 10.1063/1.2402148]

I. INTRODUCTION

Dusty plasma dynamics plays an important role in the space and astrophysical environment. For example, spoke formation in the Saturn's ring, Jovian ring formation and formation of the protoplanetary disk are but a few example where dust plays an important role. Owing to the ambient radiative environment, the dust particles are generally charged. Therefore, charged dust is a major constituent of the planetary and interstellar environment.

By dusty plasmas one generally implies a three component plasma consisting of electrons, ions and charged grains. In the interstellar medium and in the planetary environment neutrals also play an important role in the dynamics. However, owing to the complexities introduced by grains in a plasma, dusty plasma research has largely been confined to the three component plasma description in order to first uncover the underlying physical principles before addressing the general problem. Due to novel spatial and temporal scales introduced by dust in an electron-ion plasma, new waves and instabilities have been discovered in such a plasma.[1,2]

The dust surface acts as a physical, thermal, and, momentum sink for the plasma particles. The ambient plasma conditions determine the suitable charging model of the grain. For example, in a dark molecular cloud where grains carry ±2, ±1, 0 charge,[3] a discrete or stochastic model of charge dynamics is suitable.[4] However, in the planetary rings, where grains may carry more than thousands of electronic charges,[5] a continuum model of the grain charging is appropriate. The grain charge dynamics not only alters the collective behavior of the plasma but also excites new modes.[6–9] The charge wave can also be generated due to the density variation of the dust.[10,11] In turn, this may affect the low frequency inertial waves. Clearly then, the variation of various physical parameters introduces a somewhat wide array of physical process in such a plasma.

Due to the large difference in masses (grain mass $\sim 10^{-15} - 10^{-5}$ g, $m_e \sim 10^{-27}$ g, $m_i \sim 10^{-24}$ g), the dusty plasma dynamics can be studied in either of the two limits: (i) the dust particles are so heavy that they provide a stationary background for the perturbations propagating in a much lighter electron-ion plasma, and, (ii) the perturbations are of the order of or less than the typical plasma frequencies of the dusty fluid (of the order of Hz) and wavelengths are visible to the bare eyes. Although large dust grains undergo temperature fluctuations due to collision, the mean square fluctuation in their temperature is much less than the equilibrium temperature.

In the present work, it is assumed that the dust particles provide a stationary background. Generally grain charge is a function of plasma parameters. However, charge fluctuation will not be the concern of the present work. The basic set of equations is discussed in Sec. II and it is shown that the presence of the stationary dust background modifies the induction equation in a significant manner. This modification is contrasted with the two-component plasma behavior. In the presence of a positively charged dust background, the ion-inertial scale can become comparable to the size of the system. This indicates that in many dusty plasma situations, Hall MHD is the only proper description of the dynamics. The gyration of the plasma particles against the fixed dust background will cause an electric field over plasma gyroradius. It is shown that such an electric field can appear in no

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time if the dust is positively charged. Further, the presence of charged grains destroys the magnetic flux freezing in the plasma fluid. This has important ramifications for the dark molecular clouds, galactic disks, protoplanetary disks, etc., where at least 1% of the matter is dust. The magnetic field diffusion is a difficult problem in most of the astrophysical environment since owing to the large scale length the Ohmic diffusion is negligible. The magnetic flux is frozen and this has lead to the search for a viable diffusion mechanism that could provide a meaningful evolution of the magnetic field during star formation or for the dynamo theories.[12] The presence of charged dust removes flux-freezing quite naturally and permits the magnetic field to evolve. Therefore, charged dust in the interstellar medium should hold clue to the evolution of the magnetic field.

By employing the basic set of equations in Sec. III, MHD waves in various limits are investigated. The wave characteristic is modified considerably in a dusty medium. For a positively charged grain, Alfvén waves may altogether disappear if the grain number density, is comparable to the electron number density.

In Sec. IV, a local Hall instability of an inertialess dusty plasma is discussed in the presence of an inhomogeneous magnetic field. It is shown that the instability grows at the whistler frequency. Furthermore, for positively charged grains, the growth rate is smaller than when the grains are negatively charged. Section V of this paper presents a brief summary of the results.

II. BASIC MODEL

One starts with the three component description of a dusty plasma and assume that the dust grains provide a stationary background. The continuity equation is given as

$$\frac{\partial \rho_j}{\partial t} + \nabla \cdot (\rho_j \mathbf{v}_j) = 0. \tag{1}$$

Here ρ_j is the mass density and $\mathbf{v}_j$ is the velocity and j stands for electrons and ions. The momentum equations for plasma particles are

$$0 = \mathbf{E} + \frac{\mathbf{v}_e \times \mathbf{B}}{c}, \tag{2}$$

$$\rho_i \frac{d\mathbf{v}_i}{dt} = en_i\left(\mathbf{E} + \frac{\mathbf{v}_i \times \mathbf{B}}{c}\right). \tag{3}$$

The electron inertia has been neglected while writing Eq. (2). The momentum Eqs. (2) and (3) on the right-hand side have Lorentz force term with $\mathbf{E}$ and $\mathbf{B}$ as the electric and magnetic fields, respectively, e is the electric charge, n_j is the number density, and c is the speed of light.

The mass density of the bulk fluid is defined as $\rho = \rho_e + \rho_i \approx \rho_i$. Then the bulk velocity $\mathbf{v} = (\rho_i \mathbf{v}_i + \rho_e \mathbf{v}_e)/\rho \approx \mathbf{v}_i$. The continuity equation [summing up Eqs. (1)] for the bulk fluid is given as

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0. \tag{4}$$

The momentum equation can be derived by adding Eqs. (2) and (3)

$$\rho \frac{d\mathbf{v}}{dt} = -Zen_d\mathbf{E} + \frac{\mathbf{J} \times \mathbf{B}}{c}. \tag{5}$$

Here the plasma quasineutrality condition, $n_e = n_i + Zn_d$, have been used while writing Eq. (5), Z is the number of charge on the grain that can be either positive or negative and n is the number density. The current density $\mathbf{J} = e(n_i\mathbf{v}_i - n_e\mathbf{v}_e)$ and thus

$$\mathbf{v}_e = \left(1 - \frac{Zn_d}{n_e}\right)\mathbf{v}_i - \mathbf{J}/en_e. \tag{6}$$

In an immobile dusty background, the motion of the lighter plasma component will cause a charge separation between plasma particles and dust grains. This will invariably lead to the generation of an electric field. Often such an electric field is inferred from force balance equation. However, the equation of motion (5) does not provide a mechanism for the electric field generation but merely states that ions are accelerated if acted upon by the Lorentz force. It does not gives us the cause of the force. The Ampère's and Faraday's laws should be used to deduce the field. The qualitative picture of electric field generation in a dusty plasma can be formalized by writing an equation for the electric field generation. This can be done by retaining the displacement current in the Ampère's law. Note that although the magnitude of the displacement current in the MHD framework is very small, neglecting this term will obscure the mechanism and generation of the field.[13] Thus in order to give a dynamical equation for field generation that will enable us to estimate the typical time scales, displacement term in the Ampère's law is retained

$$\nabla \times \mathbf{B} = \frac{4\pi}{c}\mathbf{J} + \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}. \tag{7}$$

To make the formulation transparent, physical quantities are expressed in the eigenbasis of the rotation operator $\hat{\mathbf{e}}_\pm = (\mathbf{e}_x \pm i\mathbf{e}_y)/\sqrt{2}$. Since $\hat{\mathbf{B}} \times \hat{\mathbf{e}}_\pm = \mp i\hat{\mathbf{e}}_\pm$, from Eqs. (2) and (6), $\mathbf{E}$ may be expressed in the orthogonal basis $(\hat{\mathbf{e}}_+, \hat{\mathbf{e}}_-, \hat{\mathbf{B}})$ as

$$\mathbf{E}_\pm = \frac{iB}{en_ec}\mathbf{J}_\pm - \frac{iB}{c}\left(1 - \frac{Zn_d}{n_e}\right)\mathbf{v}_\pm. \tag{8}$$

Here $\hat{\mathbf{B}} = \mathbf{B}/B_0$. Combining Eqs. (7) and (8), a dynamical equation for the electric field can be written as

$$\left[\frac{\partial}{\partial t} - \frac{i\omega_{pi}^2/\omega_{ci}}{(1 - Zn_d/n_e)}\right]\mathbf{E}_\pm = c(\nabla \times \hat{\mathbf{B}})_\pm - i\omega_{pi}^2\frac{m_i}{e}\mathbf{v}_\pm. \tag{9}$$

Defining $\partial \mathbf{E}_\pm/\partial t = -i\gamma\mathbf{E}_\pm$, one sees from Eq. (9) that the generation of $\mathbf{E}_\pm$ occurs over

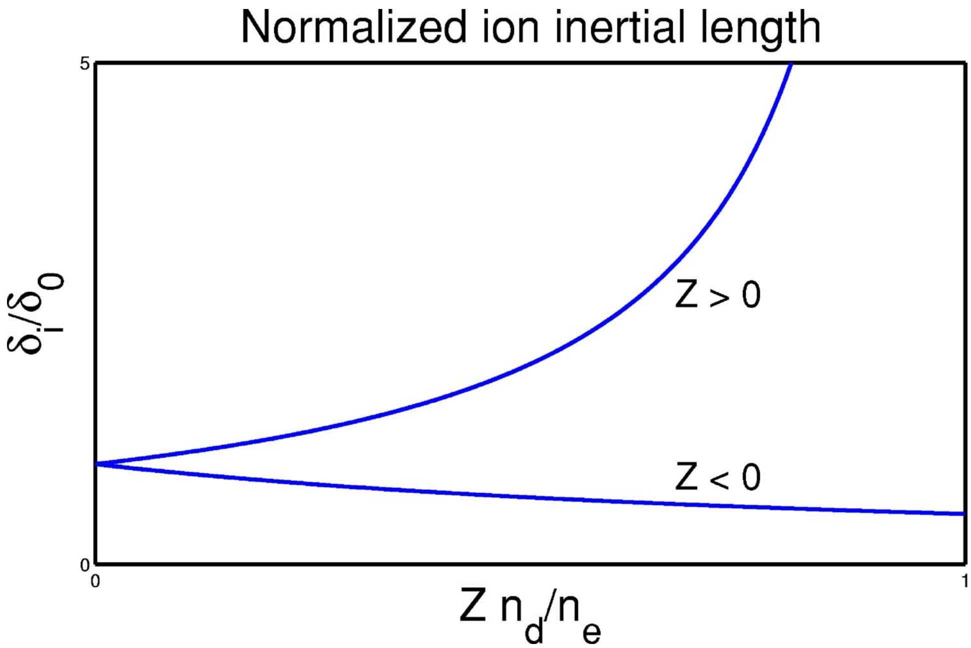

FIG. 1. Plot of δ_i/δ_0 ($\delta_0=V_A/\omega_{ci}$) against Zn_d/n_e. For positively charged grains $Z>0$, with increasing Zn_d/n_e, the ion inertial scale becomes very large. For negatively charged grains $Z<0$, owing to the smallness of δ_i, the Hall term may not be important since $\delta_i/\delta_0 \ll L$.

$$\frac{\gamma}{\omega_{pi}} \sim \frac{1}{(1-Zn_d/n_e)}\frac{\omega_{pi}}{\omega_{ci}}. \tag{10}$$

Here $\omega_{pi}=\sqrt{4\pi n_i e^2/m_i}$ is the ion-plasma frequency. The electric field generation is caused by the charge separation due to gyration of the plasma particles against the fixed dust background. Thus, it is clear from Eq. (10) that the time over which this field evolves is proportional to the ratio of the plasma and the cyclotron frequencies and inversely proportional to the quasineutrality factor $(1-Zn_d/n_e)$.

When grains form a positively charged stationary background, then for $Zn_d \sim n_e$, $\gamma/\omega_{pi} \gg 1$. Due to the presence of such an "instantaneous electric field," plasma will rotate at $\Omega_r \equiv (Zn_d/n_e)\omega_{ci}$, caused by the drift of the gyration center of the plasma fluid against the stationary dust background. As will be shown later [discussion below Eq. (13)] in the presence of a positively charged stationary background, the ion inertial length can become arbitrarily large if $Zn_d \sim n_e$. This results in the Hall MHD operating on scales that can become comparable to the system size.

Note that the ideal MHD description of the two component plasma assumes that the relative drift between electrons and ions are absent, i.e., $\mathbf{v}_e=\mathbf{v}_i$. The reason for the Hall effect in such a plasma is due to the relative drift between the plasma particles, since $\mathbf{E}+\mathbf{v}_e\times\mathbf{B}/c=0$, can be written as $\mathbf{E}+\mathbf{v}_i\times\mathbf{B}/c=\mathbf{J}\times\mathbf{B}/en_e$ in the presence of the drift. However, in a dusty plasma, or for that matter in any multicomponent plasma, since $n_e \neq n_i$, even when relative drift between the plasma particles are absent, i.e., $\mathbf{v}_e=\mathbf{v}_i$, owing to the presence of a third, charged component, the Hall effect will always be present. This can be seen from Eq. (6) since due to $(1-Zn_d/n_e)$ factor, $\mathbf{v}_e=\mathbf{v}_i$ does not imply $\mathbf{J}=0$. The Hall effect disappears in such a plasma if in addition to the absence of relative drift one also demand $Zn_d \ll n_e$. In the astrophysical environment[3] generally $Zn_d \leq n_e$ and thus, Hall drift will always be present. In a multicomponent plasma, the Hall effect can be caused either by the relative drift between the plasma particles or due to the presence of another charged component. This is the reason why the Hall MHD description of a multicomponent dusty plasma could be the proper description of the magnetized dust dynamics in many astrophysical situations.[15,16]

Taking the curl of the electron momentum Eq. (2) and making use of Maxwell's equation, the induction equation can be written as

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times \left[\left(1-\frac{Zn_d}{n_e}\right)(\mathbf{v}\times\mathbf{B}) - \left(\frac{\mathbf{J}\times\mathbf{B}}{en_e}\right)\right]. \tag{11}$$

One sees from (11) that the ideal-MHD limit correspond to $c\nabla\times\mathbf{B}/4\pi e n_e \to 0$. The ratio of the convective and Hall terms can be defined as

$$\frac{c}{V_A}\frac{\left(1-\dfrac{Zn_d}{n_e}\right)^{-1}}{4\pi e n_e}|\nabla\times\mathbf{B}| \sim \frac{1}{L}\left(1-\frac{Zn_d}{n_e}\right)^{-1}\frac{V_A}{\omega_{ci}}. \tag{12}$$

Here $V_A=B_0/\sqrt{4\pi\rho}$ is the Alfvén speed and $\mathbf{v}$ is normalized by V_A. Defining $\omega_{ci}=eB/(m_i c)$ as the ion-cyclotron frequency, the ion-inertial length δ_i can be written as

$$\delta_i = \frac{1}{\left(1-\dfrac{Zn_d}{n_e}\right)}\frac{V_A}{\omega_{ci}}. \tag{13}$$

Note that when $Z=0$, Eq. (13) reduces to $\delta_0=V_A/\omega_{ci}$, which is the ion skin depth of a two-component plasma. The presence of the charged grains considerably modifies this scale. The plot δ_i/δ_0 against Zn_d/n_e is shown in Fig. 1. It is seen from the figure that when $Zn_d \ll n_e$, the Hall effect has no large scale role for both positively $(Z>0)$ as well as negatively $(Z<0)$ charged grains. However, for positively charged grains if $Zn_d \sim n_e$, the inertial scale becomes very large and the Hall MHD is the only proper description of such a plasma.

It is clear [Eq. (12)] that the ratio of the convective to the Hall term is $\sim\delta_i/L$ and depending upon this ratio, the Hall effect's importance to the dynamics is determined. In the absence of the charged grains, this ratio is very small and the Hall MHD description on large scale is not necessary. For a negatively charged grain too, when $|Z|n_d \sim n_e$, the Hall description on the large scale L ($\delta_i/L \ll 1$) is not required. When $Zn_d \ll n_e$, the Hall term is not important even for a positively charged grain since $\delta_i \ll L$. However, if $Zn_d \sim n_e$, for positively charged grains, the ion-inertial length becomes arbitrary large and the Hall description of such a plasma must be considered. Therefore, there are two factors that determines if the MHD or the Hall MHD is the suitable description of the dynamics: (a) the sign of the grain charge, and, (b) whether most of the ionic charges are carried by the grains, i.e., $Zn_d \sim n_e$. Therefore, a general statement about the Hall MHD description of a dusty plasma can be misleading.

The rate of change of magnetic flux $\Phi = \int \mathbf{B}\cdot\mathbf{ds}$ is

$$\frac{d\Phi}{dt} = \int\int \left[\frac{\partial \mathbf{B}}{\partial t} - \nabla \times (\mathbf{v}\times\mathbf{B})\right]\cdot \mathbf{ds}. \tag{14}$$

Making use of Eq. (11) in (14) and utilizing (2) and (5) one gets

$$\frac{d}{dt}\int\int \mathbf{\Omega}\cdot\mathbf{ds} = \int\int \nabla\left(\frac{Zn_d}{n_e}\right)\times\left(\mathbf{v}_e\times\hat{\mathbf{B}}\right)\cdot\mathbf{ds}. \tag{15}$$

The generalized flux, $\mathbf{\Omega}$, that is a combination of magnetic flux and fluid vorticity, is given as

$$\mathbf{\Omega} = \left(1 - \frac{Zn_d}{n_e}\right)\left[\hat{\mathbf{B}} + \frac{\nabla\times\mathbf{v}}{\omega_{ci}}\right]. \tag{16}$$

Note from Eq. (15) that in the absence of density or charge gradients, i.e., when $\nabla(Zn_d/n_e)=0$, or, when the electron guiding center drift $\mathbf{v}_e = c\mathbf{E}\times\mathbf{B}/B^2$ is small, the factor $(1-Zn_d/n_e)$ in the generalized vorticity drops out of Eq. (15) and thus does not play any role in the vorticity conservation. However, in the presence of the gradients, or when the guiding center motion cannot be ignored, the factor $(1-Zn_d/n_e)$ starts affecting the evolution of the generalized vorticity. Note that in a two-component Hall-MHD framework, when grains are either neutral or are altogether absent ($Zn_d=0$), the generalized flux is always conserved, $\int\mathbf{\Omega}\cdot ds = \text{Const}$. In a dusty plasma however, only under special conditions the generalized flux is conserved. For example when the ion-inertia is unimportant, i.e., $d \log \mathbf{v}/dt \ll \omega_{ci}$, ions are also frozen in the fluid (since $\mathbf{E}+\mathbf{v}\times\mathbf{B}/c=0$) and the generalized flux is mainly due to the magnetic part of the flux, in Eq. (16). In the opposite limit, when ion inertia breaks the symmetry between the plasma components, it is the matter vorticity flux that is frozen in the fluid. In a dusty medium generally there will be an interplay between the field and the matter vorticities, and it is the combined quantity $\mathbf{A}+\mathbf{v}/\omega_{ci}$ that is conserved when grain charge fluctuation is unimportant. Here $\hat{\mathbf{B}} = \nabla\times\mathbf{A}$. Note that the matter vorticity term ($\sim\nabla\times\mathbf{v}$) in Eq. (16) is due only to the Hall term in the induction equation.

To summarize, it is clear from Eq. (15) that the magnetic flux is unlikely to be conserved in a dusty plasma. As the parameter Zn_d/n_e can vary either due to the density or charge gradients, the flux conservation in a dusty plasma is unlikely to hold. A fundamental reason for the flux diffusion is the presence of the charged grains. How strong or weak the flux nonconservation is depends on the steepness of the gradient. For example, in an interstellar shock, owing to the presence of steep gradients, the change in the flux will be maximum. Therefore, one may say that charged grain acts like a source or sink of the flux depending upon the direction of the gradients and the polarity of the grain charge.

The set of Eqs. (4), (5), and (11) along with a relation between $\mathbf{E}$ and $\mathbf{B}$ will provide a closed set, needed to study the dynamics of a cold, dusty plasma. This can be provided by the Maxwell's equation

$$\nabla\times\mathbf{E} = -\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t}. \tag{17}$$

Thus, one can investigate the cold dusty plasma dynamics with the help of the set of (4), (5), (11), and (17).

III. MHD WAVES IN A DUSTY MEDIUM

The wave properties of a cold, dusty, homogeneous medium in the absence of any flow ($\mathbf{v}=0$) is investigated. The linearized Eqs. (4), (5), (11), and (17) around $\mathbf{B}$, ρ, P with $\mathbf{E}=0$ can be written as

$$\frac{\partial\delta\rho}{\partial t} + \nabla\cdot(\rho\delta\mathbf{v}) = \mathbf{0}, \tag{18}$$

$$\rho\frac{\partial\delta\mathbf{v}}{\partial t} = -Zen_d\delta\mathbf{E} + \frac{\delta\mathbf{J}\times\mathbf{B}}{\mathbf{c}}, \tag{19}$$

$$\frac{\partial\delta\mathbf{B}}{\partial t} = R\nabla\times(\delta\mathbf{v}\times\mathbf{B}) - \nabla\times\frac{\delta\mathbf{J}\times\mathbf{B}}{en_e}, \tag{20}$$

$$\nabla\times\delta\mathbf{E} = -\frac{1}{c}\frac{\partial\delta\mathbf{B}}{\partial t}, \quad \nabla\times\delta\mathbf{B} = \frac{4\pi}{c}\delta\mathbf{J}. \tag{21}$$

Here $R=(1-Zn_d/n_e)$. The linearized quantities δf are proportional to the $\exp(i\omega t + i\mathbf{k}\cdot\mathbf{x})$ where ω is the angular frequency and $\mathbf{k}$ is the wave vector. The linearized equations are

$$i\omega\delta\rho + i\rho\mathbf{k}\cdot\delta\mathbf{v} = 0, \tag{22}$$

$$i\omega\delta\mathbf{v} = -\frac{Zen_d}{\rho}\delta\mathbf{E} + \frac{i[(\mathbf{k}\cdot\mathbf{B})\delta\mathbf{B} - (\mathbf{B}\cdot\delta\mathbf{B})\mathbf{k}]}{4\pi\rho}. \tag{23}$$

By taking into account $\mathbf{k}\cdot\delta\mathbf{B}=\mathbf{0}$, and, assuming plasma quasineutrality $\mathbf{k}\cdot\delta\mathbf{E}=\mathbf{0}$, one writes

$$\mathbf{k}\cdot\delta\mathbf{v} = \frac{-k^2}{4\pi\rho\omega}(\mathbf{B}\cdot\delta\mathbf{B}). \tag{24}$$

Making use of $\delta\mathbf{E} = (\omega/ck^2)\mathbf{k}\times\delta\mathbf{B}$, Eq. (23) can be written as

$$\omega \delta \mathbf{v} = \frac{1}{4\pi\rho}[(\mathbf{k}\cdot\mathbf{B})\delta\mathbf{B} - (\mathbf{B}\cdot\delta\mathbf{B})\mathbf{k}] + i\frac{Zen_d}{\rho}\left(\frac{\omega}{ck^2}\right)(\mathbf{k}\times\delta\mathbf{B}). \tag{25}$$

The induction Eq. (20) can be written as

$$i\omega\delta\mathbf{B} = iR[(\mathbf{k}\cdot\mathbf{B})\delta\mathbf{v} - (\mathbf{k}\cdot\delta\mathbf{v})\mathbf{B}] + \frac{\omega_A^2\mu\hat{\mathbf{k}}\times\delta\mathbf{B}}{\omega_{ci}}. \tag{26}$$

Here $\hat{\mathbf{k}}=\mathbf{k}/|\mathbf{k}|$, $\hat{\mathbf{B}}=\mathbf{B}/|\mathbf{B}|$, $\hat{\mathbf{k}}\cdot\hat{\mathbf{B}}=\cos\theta\equiv\mu$, $\omega_A^2=k^2V_A^2$, $V_A^2=B^2/(4\pi\rho)$. Making use of Eqs. (24) and (25), the induction Eq. (26) can be written as

$$(\omega^2 - R\mu^2\omega_A^2)\delta\mathbf{B} = R\omega_A^2(\hat{\mathbf{B}}\cdot\delta\mathbf{B})[\hat{\mathbf{B}} - \hat{\mathbf{k}}\mu] + i\omega\omega_{ci}\mu\left[R\frac{Zn_d}{n_i} - \left(\frac{\omega_A}{\omega_{ci}}\right)^2\right](\hat{\mathbf{k}}\times\delta\mathbf{B}). \tag{27}$$

Assuming $\delta\mathbf{B}\|\mathbf{B}$, the following dispersion relation (DR) can be derived by dotting (27) with $\hat{\mathbf{B}}$:

$$\omega^2 = R\omega_A^2. \tag{28}$$

One readily recognizes this as the Alfvén mode. The factor R in Eq. (28) is due to the presence of the dust component. For positively charged grains, this factor becomes very small when $Zn_d\sim n_e$ and the Alfvén wave may not exist in such a medium. However, for a negatively charged grains, the Alfvén wave can be excited at $1.4\omega_A$ when $Zn_d\sim n_e$.

Next consider a case when magnetic perturbation is perpendicular to the ambient magnetic field, $\delta\mathbf{B}\perp\mathbf{B}$. Then DR becomes

$$\omega^2 = R\omega_A^2\mu^2 \pm \omega\omega_{ci}\mu\left[R\frac{Zn_d}{n_i} - \left(\frac{\omega_A}{\omega_{ci}}\right)^2\right]. \tag{29}$$

One recognizes in Eq. (29) a mixture of the Alfvén and ion-cyclotron modes. When $Zn_d\ll n_e$, the dispersion relation (29) gives the familiar Alfvén wave in the $\omega_A\ll\omega_{ci}$ limit

$$\omega^2 \simeq \omega_A^2\cos^2\theta, \tag{30}$$

and, in the opposite case, when $\omega_{ci}\ll\omega_A$, one gets the electrostatic ($\nabla\times\delta\mathbf{E}\approx\mathbf{0}$) ion-cyclotron, $\omega^2\simeq\omega_{ci}^2\cos^2\theta$ and the electromagnetic whistler modes, $\omega^2\simeq\omega_A^2/\omega_{ci}\cos^2\theta$. The waves in a dusty medium are no different than in a two component electron-ion plasma[17] in the $Zn_d\ll n_e$ $(R\to 1)$ limit. When $Zn_d\sim n_e$ $(R\simeq 0)$, the normal mode of the medium are whistler waves. This is expected since the ion-inertial scale becomes very large when $Zn_d\sim n_e$ and the Hall effect will be important on a large scale. When $Zn_d\neq n_e$ the dispersion relation becomes

$$\omega^2 \simeq \mu^2\left(\frac{Zn_d}{n_i}\right)^2\omega_{ci}^2 \equiv \mu^2\Omega_r^2. \tag{31}$$

Note from Eq. (31) that the electrostatic cyclotron mode is the most dominant mode in an immobile positive dust background. Recall that the cause of excitation of the ion cyclotron mode in a two-component plasma is the Hall term[17] $\mathbf{J}\times\mathbf{B}/(en_e)$. The excitation of this mode is related to the finite ion Larmor radius effect, i.e., to the "electric field" over the ion gyroradius. Thus, it is not surprising that dusty plasma can support such a wave since the excitation of the electrostatic mode can be attributed to the dominant role of the electric field which is generated when ions move against a fixed dust background, i.e., the electric field appears due to the quasineutrality constraint. Therefore, unlike the ion inertial scale of a two component plasma, it is the combination of the Debye scale and the gyroradius over which the dusty electrostatic ion-cyclotron mode will operate. Clearly then the mechanism of the dusty ion-cyclotron mode excitation is quite different from the two-component case. The role of such an electric field in rotating plasma at a frequency Ω_r is well known.[14] Furthermore, since Hall effect may operate on the large scale in a dusty medium, the resonance at this frequency can play important role in heating of the plasma.

IV. HALL INSTABILITY

The simplest case in which the ion and electron inertia can be ignored and dust is considered stationary, can be subject to the Hall instability in the presence of the inhomogeneities. Since Hall drift is nondissipative in nature, such an instability will cause the transfer of energy from the large scale to the small scale until finally dissipation takes over. The Hall instability has been proposed as a viable mechanism for the magnetic energy redistribution in the neutron star crusts.[18,19] However, unlike the neutron star where convective motion is presumably absent in the crystallized crust, in a dusty plasma the convection of the plasma fluid is neglected in the zero inertia limit.

Using a local analysis, here we show that a new instability may exist in the dusty plasma due to the Hall effect. The free energy for the growth of this instability comes from the inhomogeneous background.

Adding the electron and ion momentum Eqs. (2) and (3), one gets the following induction equation

$$\frac{\partial\mathbf{B}}{\partial t} = -\nabla\times\left[\left(\frac{\mathbf{J}\times\mathbf{B}}{Zen_d}\right)\right]. \tag{32}$$

An equilibrium field $\mathbf{B}=B(z)\hat{\mathbf{x}}$ is assumed. Then $\mathbf{J}=B(z)'\hat{\mathbf{y}}$ and $\nabla\times[\mathbf{J}\times\mathbf{B}]=0$ is satisfied. The linearized Eq. (32) is given as

$$\frac{\partial\delta\mathbf{B}}{\partial t} = -\frac{\nabla\times[\delta\mathbf{J}\times\mathbf{B}+\mathbf{J}\times\delta\mathbf{B}]}{Zen_d}. \tag{33}$$

Confining to the plane wave solution with $\mathbf{k}=k\hat{x}$, after some straightforward algebra, the following differential equation in δB_y can be derived:

$$\frac{d^2\delta B_y}{dz^2} + \alpha^2\delta B_y = 0, \tag{34}$$

where

$$\alpha^2 = \left(\frac{Zn_d}{n_e}\frac{\omega}{\omega_A}\frac{1}{\delta_i}\right)^2 - \left(k^2 + \frac{B''}{B}\right). \tag{35}$$

Here B'' implies the second derivative in z. For B''/B =const, the solution of the above equation is given as

$$\delta B_y = C_1 \exp(i\alpha z) + C_2 \exp(-i\alpha z). \tag{36}$$

For an imaginary α, the fluctuation will become unstable. The condition for an imaginary α implies

$$\frac{Zn_d}{n_e}\frac{\omega}{\omega_A}\frac{1}{\delta_i} < \sqrt{k^2 + \frac{B''}{B}}. \tag{37}$$

Thus for $(k^2 + B''/B) < 0$, the growth rate of the instability becomes

$$\omega_i \approx \left(\frac{\omega_A^2}{\omega_{ci}}\right)\left(\frac{Zn_d}{n_e}\right)^{-1}\sqrt{\left|k^2 + \frac{B''}{B}\right|}. \tag{38}$$

Here $\omega_i = \mathrm{Im}[\omega]$. One may conclude that the presence of charged dust/impurity background may excite the Hall instability in a dusty plasma. When $Zn_d \sim n_e$, the instability grows at a frequency close to the whistler frequency. For a general magnetic field profile, the coefficient of the differential Eq. (35) is a function of z and for a periodic profile, the equation forms a subclass of the Floquet equation, namely the Mathieu's equation. The solution of the above equation with proper boundary condition needs to be investigated in specific circumstances.

If the grain charge fluctuates due to the presence of the density gradient, then an additional term in the induction equation $\nabla n \times \mathbf{J} \times \mathbf{B}$ needs to be considered. This will modify the dispersion relation and the growth rate of the Hall instability will change.

V. SUMMARY AND DISCUSSION

(i) Charged grains of all sizes, ranging from nm to cm, are ubiquitous in the interstellar and interplanetary medium.[20] The motion of the plasma particles against the fixed charged dusty background will invariably give rise to the "charge separation" and thus the induced electric field will evolve over ω_{ci}/ω_{pi}. If the number density of the charged grains Zn_d are comparable to the electron number density n_e, then for positively charged grains, the generation of the field is instantaneous since $\gamma \to \infty$. For negatively charged grains, the time over which field evolves is $\sim \omega_{ci}/\omega_{pi}^2$.

(ii) The Hall MHD description of a dusty plasma is not necessary in the $|Z|n_d \ll n_e$ limit. For negatively charged grains the Hall description is not necessary even when $|Z|n_d \sim n_e$. However, for positively charged dusty background, the ion inertial scale may become arbitrary large. Therefore, with increasing Zn_d/n_e, the Hall description is the only correct description of such a dusty plasma.

(iii) The generalized flux, that is a combination of the matter vorticities and field, is unlikely to be conserved in a dusty medium. This could be due to the presence of charge or density gradient in such a plasma.

(iv) The excitation of the normal mode in a dusty medium is significantly modified in comparison with the two component plasma. Depending upon the sign of the grain charge, electrostatic or electromagnetic mode will be excited in such a medium. For example for a positively charged dusty background, the Alfvén mode may not exist when $Zn_d \sim n_e$. Only for negatively charged grains, the Alfvén mode will exist. For positively charged grains, a modified ion-cyclotron mode, at a frequency Ω_r will be excited in the system.

(v) Dusty medium is subjected to the Hall instability in the presence of the magnetic inhomogeneities. The growth rate of this instability is proportional to the whistler frequency. Since this instability is nondissipative in nature, this may cause the redistribution of the magnetic energy from large to small scales. Thus dust/impurity can be the facilitator of the direct energy cascade in a turbulent plasma medium.

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