The solitary wave propagation in a collisional dusty plasma

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The present work investigates the nonlinear wave properties of a collisional dusty plasma consisting of electrons, ions, and charged grains. It is shown that the collisional dusty medium is inherently dispersive in nature when the electrons and ions are strongly magnetized. In such a plasma, when the fluctuation wavelength is larger than the dust skin depth, the balance between the dispersion and nonlinearity leads to the modified Korteweg–de Vries equation. It is quite plausible that the soliton solutions of the nonlinear equation may explain some of the large scale structures observed in various space plasma settings. © *2008 American Institute of Physics*. [DOI: 10.1063/1.2967491]

I. INTRODUCTION

Most of the space and astrophysical plasmas, such as those in cometary tails, planetary rings, interstellar molecular clouds, and solar nebulae, are dusty. Not only the charged grains of various size and mass are present in such a medium but the coupling of these grains to the ambient magnetic field determines the normal mode behavior of this medium. Although charging of the grain could be due to several competing physical processes occurring simultaneously, inelastic collision of the plasma particles with the grain is one of the most important charging mechanisms. The plasma-grain collision causes not only the damping of the high frequency waves but also facilitates the excitation and propagation of the low frequency fluctuations in the medium. For example, if the plasma-dust collision frequency is higher than the dynamical frequency of interest then the low frequency mode will propagate in the medium without dissipation and the role of the collision will be confined to moving the plasma and the dust particles together as a single medium. In such a scenario collision helps the propagation of waves in the medium without damping. However, in the opposite, high frequency limit, when the plasma-grain collision frequency is smaller than the frequency of interest, collision will cause the damping of the waves due to irreversible loss of free energy. These properties of collisions are not dusty plasma specific but are generic to any multicomponent plasma medium.[1–3]

The study of the magnetohydrodynamic (MHD) waves is thought to play an important role in the space plasmas. For example, it is thought that the Alfvén waves are a possible source of turbulence in the interstellar medium.[4] When the amplitude of these waves is large, resulting nonlinear waves may provide the initial conditions to many astrophysical problems. It is well known that the interplay between dispersion and nonlinearity gives rise to the nonlinear solitary structures.[5–23] The standard MHD waves have degeneracy; i.e., two wave propagation velocities coincide when the wave normal direction is parallel to the ambient background magnetic field. For fast modes (when the sound velocity is less than the Alfvén velocity), the nonlinear Korteweg–de Vries (KdV) equation can be derived for an oblique wave normal case,[7,8] whereas for intermediate waves, the modified Korteweg–de Vries (MKdV) can be derived.[9] When the wave degeneracy is retained and the wave normal is quasiparallel to the ambient magnetic field, instead of two separate KdV and MKdV equations, two coupled equations for the confluent MHD modes can be combined to give derivative nonlinear Schrödinger (DNLS) equation.[6] Such waves have been studied for last several decades in space plasmas.

As noted above, the undamped propagation of the low frequency fluctuations in a collisional medium is possible. Therefore, it is important to investigate the nonlinear wave propagation in such a medium owing to its potential applications to various space plasma settings. Although the motivation of this work is to discuss the results in the context of space plasmas, to keep the analysis simple, the role of neutral is completely neglected. Furthermore, the large dispersion in mass, charge, and size of the grain, which can significantly influence the propagation of the waves, is as well neglected. We shall keep these limitations in mind while discussing the results.

The nonlinear wave properties of collisional dusty plasma have been investigated in the recent past.[23] It was shown that when the electrons and ions are highly magnetized, the multifluid set of equations reduces to the Hall MHD equations in the low frequency limit.[24] The nonlinear propagation of the waves in such a collisional, magnetized medium is described by the DNLS equation.[23] The present work investigates the nonlinear wave properties of collisional dusty plasma when the electrons and ions are strongly magnetized and the dust skin depth (which is a ratio of the Alfvén speed in the dusty medium to the dust cyclotron frequency) is very small in comparison with the fluctuation wavelength. For such wavelengths, dust will be frozen along with the plasma particles in the magnetized fluid. We show that the balance between the dispersion and nonlinearity in such a scenario leads to the MKdV equation.

The basic set of equations is discussed in Sec. II. It is

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shown that when the electrons and ions are magnetized, the relative drift between the charged grains and the plasma particles (electrons and ions) gives rise to the Hall diffusion in the medium. Further, the ratio of the convective and the Hall term suggest that the Hall term in the induction equation can be ignored if the characteristic scale size of the system is larger than the dust skin depth and thus, the multifluid description of a collisional dusty plasma can be reduced to the single fluid description which is very similar to the ideal MHD. In Sec. III employing reductive perturbation technique, MKdV equation is derived. In Sec. IV a discussion and summary of the results are given.

II. BASIC MODEL

We shall assume that the dusty plasma consists of the electrons, ions and charged grains. The dynamics of such a plasma is given in terms of continuity and momentum equations for respective species with a suitable closure model; viz., an equation of state. The continuity equation is

$$\frac{\partial \rho_j}{\partial t} + \nabla \cdot (\rho_j \mathbf{v}_j) = 0. \tag{1}$$

Here, ρ_j is the mass density and $\mathbf{v}_j$ is the velocity, and j stands for electrons, ions, and grains. The momentum equations are

$$0 = -\nabla P_e - en_e\left(\mathbf{E}' + \frac{\mathbf{v}_e \times \mathbf{B}}{c}\right) - \rho_e \nu_{ed} \mathbf{v}_e, \tag{2}$$

$$0 = -\nabla P_i + en_i\left(\mathbf{E}' + \frac{\mathbf{v}_i \times \mathbf{B}}{c}\right) - \rho_i \nu_{id} \mathbf{v}_i, \tag{3}$$

$$\rho_d \frac{d\mathbf{v}_d}{dt} = -\nabla P_d + Zen_d \mathbf{E}' + \sum_{j=e,i} \rho_j \nu_{jd} \mathbf{v}_j. \tag{4}$$

Here, $\mathbf{E}' = \mathbf{E} + \mathbf{v}_d \times \mathbf{B}/c$ is the electric field in the dust frame with $\mathbf{E}$ and $\mathbf{B}$ as the electric and magnetic fields, respectively, e is the electric charge, Z is the number of charge on the grain, n_j is the number density, and $\nu_{jd} = n_d \langle \sigma v \rangle_{jd}$ is the collision frequency of the dust with jth species. Equations (2)–(4) on the right-hand side has pressure gradient term, Lorentz force term, and collisional momentum exchange terms. The electron and ion inertia have been neglected in Eqs. (2) and (3) since $m_e, m_i \ll m_d$. Furthermore, we wish to investigate very low frequency waves; i.e., $\omega \lesssim \nu_{id}$.

We shall define mass density of the bulk fluid as $\rho = \rho_e + \rho_i + \rho_d \approx \rho_d$. Then the bulk velocity $\mathbf{v} = (\rho_i \mathbf{v}_i + \rho_e \mathbf{v}_e + \rho_d \mathbf{v}_d)/\rho \approx \mathbf{v}_d$. The continuity equation [summing up Eq. (1)] for the bulk fluid becomes

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0. \tag{5}$$

The momentum equation can be derived by adding Eqs. (2)–(4):

$$\rho \frac{d\mathbf{v}}{dt} = -\nabla P + \frac{\mathbf{J} \times \mathbf{B}}{c}. \tag{6}$$

Here, $P = P_e + P_i + P_d$ is the total plasma pressure and $\mathbf{J} = -en_e \mathbf{v}_e + en_i \mathbf{v}_i$ is the current density in the dust frame. Equation (6) is similar to the ideal-MHD momentum equation as collision provides the gluing of the various components. One notes that the collision could make all the plasma components move together because we are considering only the low frequency fluctuations. However, for the high frequency fluctuations, ion and electron inertia could play an important role in the medium, and momentum equations of a three-component collisional plasma cannot be reduced to the above simple form.

In order to derive the induction equation, we note that Eqs. (2) and (3) can be inverted to yield

$$\mathbf{v_{i\perp}} = \frac{\frac{c}{B}\beta_i\left(\mathbf{E}'_\perp - \frac{\nabla_\perp P_i}{en_i}\right) + \beta_i^2 \frac{c\mathbf{E}' \times \mathbf{B}}{B^2} - \beta_i \frac{\nabla P_i \times \hat{\mathbf{B}}}{\rho_i \nu_{id}}}{(1 + \beta_i^2)},$$

$$\mathbf{v_{e\perp}} = \frac{-\frac{c}{B}\beta_e\left(\mathbf{E}'_\perp + \frac{\nabla_\perp P_e}{en_e}\right) + \beta_e^2 \frac{c\mathbf{E}' \times \mathbf{B}}{B^2} + \beta_e \frac{\nabla P_e \times \hat{\mathbf{B}}}{\rho_e \nu_{ed}}}{(1 + \beta_e^2)},$$

$$\tag{7}$$

where $\hat{\mathbf{B}} = \mathbf{B}/B$, and

$$\mathbf{v_{i\parallel}} = \beta_i \frac{c\mathbf{E}'_\parallel}{B} - \frac{\nabla_\parallel P_i}{\rho_i \nu_{id}}, \quad \mathbf{v_{e\parallel}} = -\beta_e \frac{c\mathbf{E}'_\parallel}{B} - \frac{\nabla_\parallel P_e}{\rho_e \nu_{ed}}. \tag{8}$$

Here, $\beta_j = \omega_{cj}/\nu_{jd}$ is the plasma Hall parameter and is a measure of magnetization of the electrons and ions. For example, when plasma cyclotron frequency ω_{cj} dominates the plasma-dust collision frequency, i.e., $\beta_e \gg 1$, and $\beta_i \gg 1$, the Hall term ($\sim \mathbf{E} \times \mathbf{B}$) will dominate in Eq. (7). Physically, in the dust frame, $\beta_j \gg 1$ implies that the plasma drift due to collision is very small compared to the transverse gyration of the plasma particles across the magnetic field. This results in the plasma particles going away or coming close locally to a stationary observer in the dust frame resulting in a time-dependent Hall electric field. This Hall field is generated over plasma-cyclotron time scale and depending upon the sign of the grain charge, the Hall scale can become arbitrary large.[24]

In the $\beta_j \gg 1$ limit, the relative drift between electrons and ions are small, i.e., $\mathbf{v}_e \approx \mathbf{v}_i$, and thus the electron velocity $\mathbf{v}_e$ can be written as $\mathbf{v}_e = -\mathbf{J}/Zen_d$. Taking the curl of Eq. (2) $\mathbf{E}' + \mathbf{v}_e \times \mathbf{B}/c \approx 0$ and making use of the Maxwell equation $c\nabla \times \mathbf{E} = \partial_t \mathbf{B}$, the induction equation can be written as

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times \left[(\mathbf{v} \times \mathbf{B}) - \left(\frac{\mathbf{J} \times \mathbf{B}}{Zen_d}\right) \right]. \tag{9}$$

We see from the induction equation (9) that the ideal-MHD limit corresponds to $\nabla \times \mathbf{B}/Zen_d \to 0$. The ratio of the Hall to the convection term is

$$\left(\frac{\mathbf{J} \times \mathbf{B}}{Zen_d}\right)\left(\frac{1}{\mathbf{v} \times \mathbf{B}}\right) \sim \frac{1}{L}\frac{V_A}{\omega_{cd}} \sim \frac{\delta_d}{L}, \tag{10}$$

where $V_A = V_A = B / \sqrt{4\pi\rho}$ is the Alfvén speed, $\delta_d = V_A / \omega_{cd}$ is the dust skin depth, and L is the characteristic length of the system. Therefore, for the fluctuations of wavelength λ $(\sim L)$ larger than the dust skin depth, the contribution of the Hall term in the induction equation can be neglected and the magnetic field can be assumed frozen in the dusty fluid.

The wave properties of the medium will be investigated by assuming $\delta_d \ll L$; i.e., neglecting the Hall term in the induction equation (9) and utilizing Eqs. (5) and (6) along with an equation of state

$$\frac{d}{dt}\left(\frac{P}{\rho^\gamma}\right) = 0. \tag{11}$$

III. WAVES IN THE MEDIUM

To investigate the wave properties of the medium, we shall assume that wave propagation is one dimensional and wave vector $\mathbf{k} = k\hat{x}$. Further, we shall assume that all physical quantities depend on x only and, $\partial/\partial y = \partial/\partial z = 0$. Then for $\mathbf{B} = (B_x, B_y, B_z)$ and $\mathbf{v} = (u_x, u_y, u_z)$, we obtain

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u_x) = 0,$$

$$\rho \frac{du_x}{dt} = -\frac{\partial P}{\partial x} - \frac{1}{8\pi}\frac{\partial}{\partial x}(B_y^2 + B_z^2),$$

$$\rho \frac{du_y}{dt} = \frac{B_x}{4\pi}\frac{\partial B_y}{\partial x}, \quad \rho \frac{du_z}{dt} = \frac{B_x}{4\pi}\frac{\partial B_x}{\partial x}, \quad B_x = \text{const}, \tag{12}$$

$$\frac{\partial B_y}{\partial t} = \frac{\partial}{\partial x}[u_y B_x - u_x B_y], \quad \frac{\partial B_z}{\partial t} = \frac{\partial}{\partial x}[u_z B_x - u_x B_z],$$

$$\frac{dP}{dt} + \gamma P \frac{\partial u_x}{\partial x} = 0.$$

The behavior of the linear waves in the system of equations (12) is described briefly. Linearizing the above set of equations about the uniform equilibrium state $\mathbf{v} = 0$, $\rho = \rho_0$, $P = P_0$, $\mathbf{B}_0 = (\mathbf{B}_{x0}, \mathbf{B}_{y0}, \mathbf{B}_{z0})$ and Fourier analyzing fluctuations as $\sim \exp[i(\omega t - kx)]$, one obtains the following dispersion relation:

$$(V_p^2 - V_A^2 \cos^2 \phi)[V_p^4 - (V_A^2 + C_s^2)V_p^2 + C_s^2 V_A^2 \cos^2 \phi] = 0, \tag{13}$$

where $V_p = \omega/k$ is the phase velocity of the waves, $V_A = B_0 / \sqrt{4\pi\rho_0}$ is the Alfvén speed, $C_s^2 = \gamma P_0 / \rho_0$ is the sound speed, and $\cos \phi = (\mathbf{k} \cdot \mathbf{B})/kB_0$.

Equation (13) admits the Alfvén mode $V_p = \pm V_A \cos \phi$ and the fast and slow magnetosonic modes

$$V_{f,s}^2 = \frac{1}{2}\{V_A^2 + C_s^2 \pm [(V_A^2 + C_s^2)^2 - 4C_s^2 V_A^2 \cos^2 \phi]^{1/2}\}. \tag{14}$$

The oblique wave normal case for fast modes (when the sound velocity is less than the Alfvén velocity), nonlinear Korteweg–de Vries (KdV) equation was derived for a collisionless cold plasma.[7,8] By using multiple scale technique,

the analysis of Refs. 7 and 8 can be easily extended to finite beta (a ratio of the matter to the field pressure) model. Introducing the stretched variable $\xi = x - \cos \phi t$, $\tau = \epsilon \cos \phi t$, and expanding the normalized (with respect to background quantities B_0, ρ_0, P_0) dependent variables as

$$\rho = 1 + \epsilon^{1/2}\rho^1 + \cdots, \quad u_x = \epsilon^{1/2}u_x^1 + \cdots,$$

$$u_y = \epsilon^{1/2}u_y^1 + \cdots, \quad u_z = u_z^0 + \epsilon^{1/2}u_z^1 + \cdots,$$

$$B_y = B_y^0 + \epsilon^{1/2}B_y^1 + \cdots, \quad B_z = B_z^0 + \epsilon^{1/2}B_z^1 + \cdots,$$

$$P = 1 + \epsilon^{1/2}P^1 + \cdots,$$

and following the similar analysis as given in Ref. 9, one can obtain the modified Korteweg–de Vries (MKdV) equation which includes the effect of finite plasma beta

$$\frac{\partial u_x^1}{\partial \tau} + A_1(u_x^1)^2 \frac{\partial u_x^1}{\partial \xi} + A_2 \frac{\partial^3 u_x^1}{\partial \xi^3} = 0, \tag{15}$$

where

$$A_1 = \frac{3}{2}\frac{\cos^2 \phi - \beta/2}{V_A \sin 2\phi \sin \phi}, \quad A_2 = \frac{\cos^2 \phi - \beta/2}{2\sin^2 \phi}. \tag{16}$$

Here, $\beta = 2C_s^2 / V_A^2$ is the ratio of the plasma to the field pressures. Equation (15) can be integrated in a stationary frame $\eta = \xi - A_1 V_s \tau$ (here, V_s is the velocity of the solitary wave). This gives the nonlinear ordinary differential equation of the form

$$\frac{A_2}{2}\left(\frac{du_x^1}{d\eta}\right)^2 + \frac{A_1}{12}(u_x^1)^2[(u_x^1)^2 - 6V_s] = 0, \tag{17}$$

where the boundary condition at infinity $u_x^1(\pm\infty) = u_{x\eta}^1(\pm\infty) = u_{x\eta\eta}^1 = 0$ have been used. For the solitary wave traveling to the right $(V_S > 0)$, the solution of the MKdV equation (15) is

$$u_x^1 = A \, \text{sech}[\eta/\Delta], \tag{18}$$

The amplitude and the width of the solitary wave [Eq. (18)] is given by

$$A = \sqrt{\frac{6V_s}{A_1}}, \quad \Delta = \sqrt{\frac{A_2}{V_s}}. \tag{19}$$

We note that MKdV equation has been derived for the oblique propagation. If the wave is propagating parallel to the field, i.e., $(\mathbf{k} \cdot \mathbf{B})/kB_0 = 1$, Eq. (15) is no longer valid since the denominator of A_1 and A_2 is zero in this case. For the Alfvén waves in a dispersive medium, one can derive a derivative nonlinear equation.[23]

IV. DISCUSSION

The space and astrophysical plasmas are generally cold, magnetized plasmas with $\beta \gg 1$. Thus, for $A_1 \sim \beta/V_A$ and $A_2 \sim \beta$, the amplitude of the wave $A \sim \sqrt{V_s V_A}$ is $\sim V_A$ for $V_s \sim V_A$. Although neutral dynamics has been overlooked in the present work, it is instructive to calculate the soliton width for the astrophysical plasmas. In very dense collapsing molecular cloud cores where neutral density $\gtrsim 10^6$ cm^{-3}, the

charged grains are more numerous than the electrons and ions.[25,26] We shall assume the commonly accepted ratio of the dust to neutral mass density $\rho_d/\rho_n=0.01$. Thus, $m_d=10^{-15}$ g gives $n_d=10^{-11}n_n$ when $m_n=m_p$. Assuming a dense molecular cloud with $n_n=10^6$ cm^{-3}, one gets $n_d=10^{-5}$ cm^{-3}. For a typical mGauss field $B=10^{-3}$ G, the amplitude of the solitary wave is $V_A\sim0.1$ km/s. The width of the soliton wave will be $\sim\sqrt{L/\beta}$ for $V_s\sim V_A$. Thus, for $L\sim$ few parsecs, the width of the nonlinear structure could be $1/10$ of a parsec for $\beta=100$. Therefore, the existence of large scale solitary structure ($\sim0.1-0.01$ pc) is plausible in the medium. This inference at best hints at the possibility of soliton structure. A more refined calculation including the neutral dynamics is necessary. Therefore, given the limitation of the present model, results about the existence of solitary structure in the astrophysical plasmas are at best speculative.

To summarize, we have shown that in the low frequency limit, when the electrons and ions are highly magnetized, the multifluid set of equations reduces to the set of equations which are very similar to the single fluid ideal-MHD equations. The nonlinear propagation of the waves in the magnetized, collisional dusty medium can be described by the MKdV equation.

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[1] B. S. Tanenbaum and D. Mintzer, Phys. Fluids **5**, 1226 (1962).
[2] R. Kulsrud and W. P. Pearce, Astrophys. J. **156**, 445 (1969).
[3] B. P. Pandey and M. Wardle, Mon. Not. R. Astron. Soc. **371**, 1014 (2006).
[4] J. Arons and C. E. Max, Astrophys. J., Lett. Ed. **196**, L77 (1975).
[5] D. J. Kaup and A. C. Newell, J. Math. Phys. **19**, 798 (1978).
[6] E. Mjolhus and T. Hada, in *Nonlinear Waves and Chaos in Space Plasmas*, edited by T. Hada and H. Matsumoto (TERRAPUB, Tokyo, 1997), p. 121.
[7] T. Kakutani, H. Ono, T. Taniuti, and C. C. Wei, J. Phys. Soc. Jpn. **24**, 1159 (1968).
[8] H. Kever and G. K. Morikawa, Phys. Fluids **12**, 2090 (1969).
[9] T. Kakutani and H. Ono, J. Phys. Soc. Jpn. **26**, 1305 (1969).
[10] T. Taniuti and H. Washimi, Phys. Rev. Lett. **21**, 209 (1968).
[11] T. Taniuti and N. Yajima, J. Math. Phys. **10**, 1369 (1969).
[12] A. Hasegawa, Phys. Fluids **15**, 870 (1972).
[13] T. Kawata, N. Kobayashi, and H. Inoue, J. Phys. Soc. Jpn. **46**, 1008 (1979).
[14] P. K. Shukla and L. Stenflo, Astrophys. Space Sci. **190**, 23 (1992).
[15] T. Hada, Geophys. Res. Lett. **20**, 2415, DOI: 10.1029/93GL02704 (1993).
[16] F. Verheest, *Waves in Dusty Space Plasmas* (Kluwer, Norwell, 2000), Chap. 6.
[17] M. S. Ruderman, J. Plasma Phys. **67**, 271 (2002).
[18] P. K. Shukla and F. Verheest, Astron. Astrophys. **401**, 849 (2003).
[19] I. Kourakis and P. K. Shukla, J. Phys. A **36**, 11901 (2003).
[20] I. Kourakis and P. K. Shukla, Nonlinear Processes Geophys. **12**, 407 (2005).
[21] Y. Nariyuki and M. Hada, Nonlinear Processes Geophys. **13**, 425 (2006).
[22] I. Kourakis, F. Verheest, and N. F. Cramer, Phys. Plasmas **14**, 022306 (2007).
[23] B. P. Pandey, S. V. Vladimirov, and A. Samarian, Phys. Plasmas **15**, 053705 (2008).
[24] B. P. Pandey and J. Vranjes, Phys. Plasmas **13**, 122106 (2006).
[25] R. Nishi, T. Nakano, and T. Umebayashi, Astrophys. J. **368**, 181 (1991).
[26] M. Wardle and C. Ng, Mon. Not. R. Astron. Soc. **303**, 239 (1999).