

**Quantum causal histories in the light of quantum information**Etera R. Livine<sup>1,\*</sup> and Daniel R. Terno<sup>2,3,†</sup><sup>1</sup>*Laboratoire de Physique, ENS Lyon, CNRS UMR 5672, 46 Allée d'Italie, 69364 Lyon Cedex 07, France*<sup>2</sup>*Perimeter Institute, 31 Caroline St, Waterloo, Ontario, Canada N2L 2Y5*<sup>3</sup>*Centre for Quantum Computer Technology, Department of Physics, Division of Information and Communication Sciences Macquarie University, Sydney NSW 2109, Australia*

(Received 12 December 2006; published 2 April 2007)

We use techniques of quantum information theory to analyze the quantum causal histories approach to quantum gravity. While it is consistent to introduce closed timelike curves (CTCs), they cannot generically carry independent degrees of freedom. Moreover, if the effective dynamics of the chronology-respecting part of the system is linear, it should be completely decoupled from the CTCs. In the absence of a CTC, not all causal structures admit the introduction of quantum mechanics. It is possible for those and only those causal structures that can be represented as quantum computational networks. Dynamics of the subsystems should not be unitary or even completely positive. However, we show that other commonly made assumptions ensure the complete positivity of the reduced dynamics.

DOI: [10.1103/PhysRevD.75.084001](https://doi.org/10.1103/PhysRevD.75.084001)

PACS numbers: 04.60.-m, 03.67.-a

**I. INTRODUCTION**

The quest for a quantum theory of gravity produced a variety of approaches that include string theory, loop quantum gravity, spin foams, causal sets, and causal dynamical triangulations. The successful theory has to provide a coherent structure that accommodates both classical relativity and quantum mechanics. In some appropriate limit it should produce the familiar physical phenomena on the flat spacetime background and make predictions on the kind and magnitude of the departures from this picture.

Those goals have not yet been achieved by any of the approaches, but these attempts have brought many insights and led to a better understanding of the problem's complexity. The quantum causal histories (QCHs) approach [1] to the quantization of gravity is a background-independent formalism that satisfies many of the conditions that are argued for by the above models. The idea is to use a causal set to describe the causal structure and a quantum theory is introduced through the assignment of finite-dimensional Hilbert spaces to the elementary events. Originally motivated by quantum cosmology [2], and providing a description of the causal spin foam models [3], QCHs make a direct contact with quantum computation and quantum information theory [4] in general.

Quantum computers are discrete systems that evolve by local interactions, and every such system can be simulated efficiently on a quantum computer [4,5]. The approaches to quantum gravity, and QCH, in particular, depict it as a discrete and local quantum theory. Hence it should be describable as a quantum computation [5].

We apply the quantum-informational considerations to several questions in QCH. First, we consider closed timelike curves and the modifications in quantum mechanics

that they may cause. Next, we show that the only causal histories that are compatible with quantum mechanics are those that can be represented as quantum computational networks. Finally we deal with the evolution of the subnetworks and the role of completely positive maps.

The remainder of this section is devoted to the review of the necessary concepts. We begin with a brief outline of QCHs in the Hilbert space language, roughly following [1]. If a spacetime is time-orientable and has no closed timelike curves, then its causal structure can be completely described as a partial order relation on its points. The relation  $x \preceq y$  is defined if there exists a future-directed nonspacelike curve from  $x$  to  $y$ . It is transitive, and the absence of CTCs means that  $x \preceq y$  and  $y \preceq x$  are simultaneously true if and only if  $x = y$ . Those two conditions make the relation “ $\preceq$ ” into a partial order.

A discrete analogue of a smooth chronology-respecting spacetime is a causal set  $\mathcal{C}$ , which is a locally finite and partially ordered set. That is, for any two events  $x, y \in \mathcal{C}$ , there exist (at most) finitely many events  $z \in \mathcal{C}$  such that  $x \preceq z \preceq y$ . If the events  $x$  and  $y$  are not related, i.e., neither  $x \preceq y$  nor  $y \preceq x$  holds, then they are spacelike separated, this fact being denoted as  $x \sim y$ . At the discrete level there is no distinction between causal and chronological [6] entities. An acausal set is a subset  $\xi \in \mathcal{C}$  such that all events in it are spacelike separated from one another. Then the maximal acausal sets are the discrete analogues of spacelike hypersurfaces.

A causal set can be represented by a directed graph of elementary relations, as on Fig. 1. Its vertices are the points of  $\mathcal{C}$ , while the edges  $x \rightarrow y$  represent the elementary causal relations, namely  $x \preceq y$  without any intermediate  $z$  such that  $x \preceq z \preceq y$ .

A future-directed path is a sequence of events such that there exists an edge from each event to the next. It is an analogue of a future-directed nonspacelike curve. A future-directed path is future (past) inextendible if there exists no

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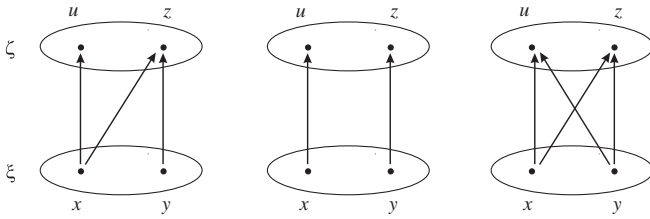


FIG. 1. Three possible causal histories.

event in  $\mathcal{C}$  which is in the future (past) of the entire path. Then one can define complete future and complete past of an event. An acausal set  $\xi$  is a complete future of an event  $x$  if  $\xi$  intersects any future-inextendible future-directed path that starts at  $x$ , and a complete past is defined similarly. If an acausal set  $\zeta$  is a complete future of an acausal set  $\xi$  and at the same time the set  $\xi$  is a complete past of  $\zeta$ , then the sets form a complete pair,  $\xi \preceq \zeta$ . The resulting structure is a discrete, locally finite, causal pre-spacetime [1].

A local quantum structure on it is introduced by attaching a finite-dimensional Hilbert space  $\mathcal{H}(x)$  to every event  $x \in \mathcal{C}$ . For only two spacelike separated events  $x$  and  $y$  the composite state space is  $\mathcal{H}(x, y) = \mathcal{H}(x) \otimes \mathcal{H}(y)$ , with an obvious generalization to larger sets. In ordinary quantum mechanics (of closed systems) time evolution is a unitary map of Hilbert spaces. In a QCH approach one introduces a unitary evolution between complete pairs of acausal sets  $\xi$  and  $\zeta$ , where, e.g.,  $\zeta$  is the complete future of  $\xi$ ,  $\xi \preceq \zeta$ . One can think of a complete pair as successive Cauchy surfaces of an isolated component of spacetime, or of all spacetime, with a unitary map  $U$  relating  $\mathcal{H}(\xi)$  and  $\mathcal{H}(\zeta)$ .

Hence a QCH consists of a causal set  $\mathcal{C}$ , a finite-dimensional Hilbert space  $\mathcal{H}(x)$  at every  $x \in \mathcal{C}$  and a unitary map

$$U(\xi, \zeta): \mathcal{H}(\xi) \rightarrow \mathcal{H}(\zeta), \quad (1)$$

for any complete pair  $\xi \preceq \zeta$ . The maps have a natural composition property

$$U(s, \zeta)U(\xi, s) = U(\xi, \zeta), \quad \text{for } \xi \preceq s \preceq \zeta. \quad (2)$$

Different possible causal relations between complete sets are shown on Fig. 1.

Quantum circuits [4], as the one depicted on Fig. 2, represent sequences of unitary operations (that are called gates) that are performed on one or several quantum wires, which represent distinct discrete quantum systems. Usually those systems are qubits—two-dimensional quantum systems. The wires carry information around the circuit, and the conventional direction from left to right may correspond to the passage of time, or to the information carrier moving from one location to another. Quantum circuits do not contain loops, so there is no feedback from one part of the circuit to another. Quantum states cannot be cloned, so the wires do not split up. There are special symbols for

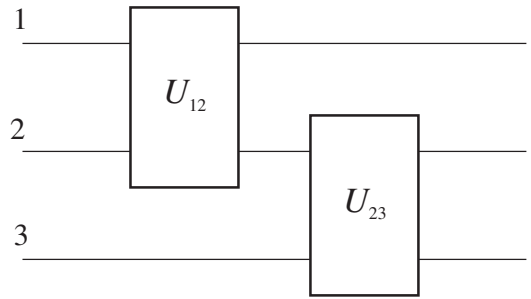


FIG. 2. A quantum circuit with two gates.

particular sets of gates, but through this paper we use only a generic  $n$ -partite box symbol.

Evolution of an open quantum system is nonunitary. When the initial correlations with the environment can be ignored, the resulting dynamics is completely positive (CP) [4,7,8]. This is a crucial property: a linear map  $T(\rho)$  is called positive if it transforms any positive matrix  $\rho$  (namely, one without negative eigenvalues) into another positive matrix. It is called completely positive if  $T \otimes \mathbb{1}$  acting on a bipartite state  $\rho$  produces a another bipartite state, i.e., a positive trace-one operator. For instance, complex conjugation of  $\rho$  (whose meaning is time reversal) is a positive map, since it preserves the eigenvalues of the Hermitian matrix. However, it is not completely positive and as such is used to identify entangled states [4,7]. CP maps are the integral part of the toolbox of quantum information. Later we discuss their role in QCHs.

The rest of this paper is organized as follows. In the next section we discuss CTCs. Section III deals with the possible types of causal relations that permit to introduce quantum mechanics. Evolution of subsets of the complete pairs is subject of Sec. IV. In Sec. V we discuss the relation between local and global information about quantum states.

## II. STRUCTURE: ABSENCE OF CAUSAL LOOPS

Solutions of Einstein equations with CTCs have been known for a long time, and they reemerged to the public eye after the introduction of traversable wormholes [9,10]. A discrete setting [11] makes it easier to analyze potential paradoxes that result from the presence of CTCs.

The basic assumption in construction of a QCH is that the set  $\mathcal{C}$  is partially ordered, i.e. it contains no CTCs. Which kind of quantum mechanics, if any, results from allowing the systems in the chronology-respecting and chronology-violating regions to interact restricts the allowed CTCs, regardless of other assumptions. The rest of the properties, both from the point of view of a local observer and a global “superobserver,” depend on the additional hypotheses that are described below.

We use Fig. 3 to illustrate possible definitions. The diagram contains a CTC  $\mathcal{W}$  that is made of the four-point set  $W, w_1, \dots, w_4 \in W$  and the edges between them. This is

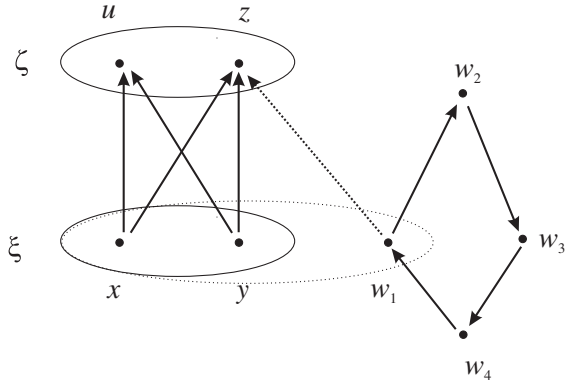


FIG. 3. The complete past of the set  $\zeta = \{u, z\}$ .

a chronology-violating set, since both statements  $w_i \preceq w_j$  and  $w_i \succeq w_j$  are true for all points of  $W$ .

There are two possible ways in which  $W$  can be embedded into a larger diagram. If there are no causal relations between the normal (chronology-respecting) region and the points of the CTC, then it is disconnected from the rest of the set and can be simply ignored. On the other hand, points of the CTC may be allowed to influence the chronology-respecting region, and may be influenced by it, similar to the continuous case [9].

The minimal extension of the rules is to allow points of a CTC into an acausal set if they are causally disconnected from the rest of the set. Figure 3 represents a part of one possible causal structure. Since any single point of  $W$  is spacelike separated from the points of  $\xi$ , a set  $\bar{\xi} = \xi \cup \{w_1\}$  is also an acausal set. Similarly, other acausal sets are  $\bar{\xi}^I = \xi \cup \{w_2\}$ ,  $\varsigma = \{u, w_2\}$ , etc. The loop  $W$  is not future-inextendible, since  $z \succeq W$ . As a result, the complete future of  $w_1$  (or any other point of  $W$ ) is an acausal set  $\zeta$ . To avoid the question of how to interpret the definition of the past-inextendible path, assume that the Fig. 3 is a part of a larger causal structure that is shown on Fig. 4(a).

It is easy to see that the set  $\xi$  is the complete past of  $\zeta$ , hence the two sets form a complete pair. When we introduce quantum mechanics, the rest of the points of  $W$  are irrelevant, and the effective diagram is represented on Fig. 5. On the other hand, the point  $w_1$  (and together with

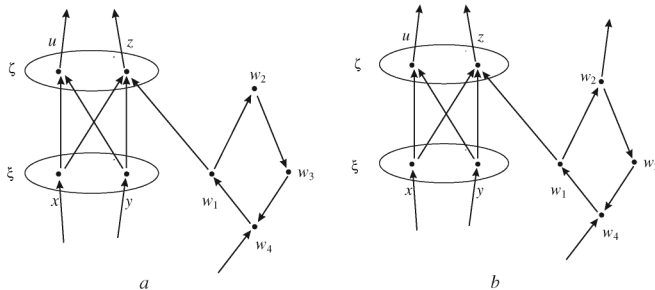


FIG. 4. Two ways of embedding the diagram of Fig. 3 into a bigger set.

it the set  $\bar{\xi}$ ) on Fig. 4(b) has no complete future at all, since there is no acausal set that intersects both future-directed future-inextendible paths that start at  $w_1$ . In this case quantum mechanics as described in Sec. I is impossible.

Hence if the complete pairs of the (generalized) acausal sets are necessary to introduce quantum mechanics, only CTCs that are compatible with the existence of such pairs are allowed. In this case a single representative of the loop is picked and treated as part of a standard partially ordered causal structure, and the existence of CTCs has no consequence.

It looks more natural to relax a demand of the acausality in the definitions of the past and future sets [11,12]. One allows a single point from each CTC into such an ‘‘acausal’’ set, so the points may be causally related only through the loop. For example, the set  $\bar{\zeta} = \zeta \cup \{w_2\}$  may be taken to be the complete future of  $\bar{\xi}$ , which is then its complete past.

To avoid the inconsistency of quantum theory in this model one must impose a self-consistency requirement [11]. Since the points of  $W$  belong to the complete past of  $\zeta$ , the (reduced) state on all events  $w_i$  should be the same. That means, if  $\rho_w$  is the state in the CTC region at the ‘‘temporal origin’’ at  $w_1$  it should be the same ‘‘after’’ the evolution  $U$ .

Since the preparation procedure is possible at any point of the chronology-respecting region, the state on  $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$  is taken to be a direct product state [11]. Here  $\mathcal{H}_A$  stands for the Hilbert space of the chronology-respecting set, and  $\mathcal{H}_B$  for the CTCs. Then the evolution [e.g.,  $U: \mathcal{H}(\bar{\xi}) \rightarrow \mathcal{H}(\bar{\zeta})$  of the above example] is supplemented by the consistency condition

$$\rho_B = \text{tr}_A[U(\rho_A \otimes \rho_B)U^\dagger]. \tag{3}$$

It was shown that there is always at least one solution for this self-consistency equation and in some cases  $\rho_B$  belongs to a continuous family of the solutions [11,12].

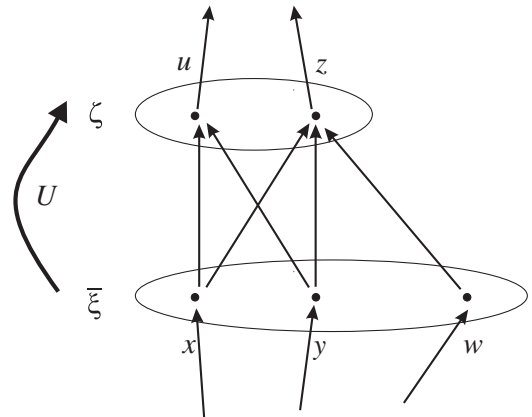


FIG. 5. Effective causal structure that disregards the CTC altogether.

However, we now show that for generic  $U$  and  $\rho_A$  the solution is unique.

Any state  $\rho$  on  $d_A \times d_B$ -dimensional space  $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$  can be decomposed as

$$\rho = \frac{1}{d_A d_B} \left( \sigma_A^0 \otimes \sigma_B^0 + \sum_i \alpha_i \sigma_A^i \otimes \sigma_B^0 + \sum_j \beta_j \sigma_A^0 \otimes \sigma_B^j + \sum_{i,j} \gamma_{ij} \sigma_A^i \otimes \sigma_B^j \right). \quad (4)$$

Here  $\sigma_X^i$  represent generators of  $SU(d_X)$ ,  $\sigma^0 = \mathbb{1}$  and the real vectors  $\vec{\alpha}$  and  $\vec{\beta}$  of the size  $d_A^2 - 1$  and  $d_B^2 - 1$ , respectively, are the generalized Bloch vectors of the reduced density operators. If the state is a direct product  $\rho_A \otimes \rho_B$ , then  $\gamma_{ij} = \alpha_i \beta_j$ . Denote the action of  $U$  as

$$U \sigma_A^\alpha \otimes \sigma_B^\beta U^\dagger = \sum_{\mu,\nu} s_{\alpha\beta}^{\mu\nu} \sigma_A^\mu \otimes \sigma_B^\nu, \quad (5)$$

$$\alpha, \mu = 0, \dots, d_A^2 - 1, \quad \beta, \nu = 0, \dots, d_B^2 - 1.$$

The consistency condition is a linear system

$$\sum_i \beta_i \left( \delta_{im} - s_{0i}^{0m} - \sum_j \alpha_j s_{ji}^{0m} \right) = \sum_i \alpha_i s_{i0}^{0m}. \quad (6)$$

Its solution is not unique if and only if

$$\Delta(\vec{\alpha}) = \det \left\| \delta_{im} - s_{0i}^{0m} - \sum_j \alpha_j s_{ji}^{0m} \right\| = 0. \quad (7)$$

To have a  $\Delta(\vec{\alpha})$  for a generic  $\rho_A$  (apart from a finite set of states), two equations should be satisfied simultaneously:

$$\det \|\delta_{im} - s_{0i}^{0m}\| = 0, \quad (8)$$

and

$$\sum_j \alpha_j s_{ji}^{0m} = 0. \quad (9)$$

Since the last equation holds for all but a finite set of  $\alpha_j$ 's it means that

$$s_{ji}^{0m} = 0, \quad \forall j = 1, \dots, d_A^2 - 1. \quad (10)$$

Hence the solution of a self-consistency equation is not unique only for a subset of lower dimensionality of the set of all unitaries on  $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ .

This points to a crucial difference with the usual quantum theory. The Hilbert space  $\mathcal{H}_B$  on a CTC generically carries no independent degrees of freedom: the states  $\rho_B$  are uniquely determined from  $U$  and  $\rho_A$ , apart from the set of a measure zero.

From Eq. (6) it follows that  $\rho_B$  is a rational function of  $\rho_A$  and so a generic  $U$

$$\rho'_A = \text{tr}_B[U(\rho_A \otimes \rho_B)U^\dagger] \quad (11)$$

leads to a nonlinear evolution. Actually, from the point of view of a local observer to whom only the chronology-

respecting set is accessible, any evolution other than  $U_A \otimes \mathbb{1}_B$  produces a nonlinear local dynamics. Writing explicitly

$$\rho'_A = \frac{1}{d_A} \left( \sigma_A^0 + \left( \sum_i \alpha_i s_{i0}^{m0} + \sum_j \beta_j (\vec{\alpha}) s_{0j}^{m0} + \sum_{ij} \alpha_i \beta_j (\vec{\alpha}) s_{ij}^{m0} \right) \sigma_A^m \right), \quad (12)$$

one sees that since for a generic  $\rho_A$  the consistent  $\vec{\beta}$  is a nonlinear function of  $\vec{\alpha}$ , the linear evolution on  $\mathcal{H}_A$  is possible only if

$$s_{0j}^{m0} = 0, \quad s_{ij}^{m0} = 0, \quad \forall j = 1, \dots, d_B^2 - 1. \quad (13)$$

As a result, the state of  $\mathcal{H}_A$  is independent of the ‘‘environment,’’ hence its evolution is unitary and  $U_A \otimes \mathbb{1}_B$ .

Ignoring a ‘‘conspiracy’’ question about a mechanism that prepares the state  $\rho_B$  in accordance with the state  $\rho_A$  in the causally disconnected region, the nonlinearity of the resulting evolution is an observable consequence of a non-trivial interaction with CTC states.

The consistency condition also restricts the number of causal links that a CTC may have with the chronology-respecting part of the diagram. As it was argued above, in the generic case the state of the loop is uniquely determined by  $\rho_A$  and  $U$ . That means if there is a second causal link with the loop, the consistency condition now becomes

$$\rho_B = \text{tr}_A[V \rho_{AB} V^\dagger] = \text{tr}_A[V U \rho_A \otimes \rho_B U^\dagger V^\dagger], \quad (14)$$

which holds for the set of measure zero of the bipartite unitary transformations or may have no solution at all. Even when the state of the loop is not determined uniquely, every imposition of the consistency condition reduces the dimensionality of the set of consistent states  $\rho_B$  at least by 1. Hence, if we are ready to accept the lower dimensional set of the possible evolutions, a CTC cannot have more that  $d_B^2 - 1$  links with the chronology-respecting part of the graph, while if one insists on generic  $U$ , there could be only a single link.

We conclude the following. Within the strict interpretation of the acausal surfaces, CTCs either prevent the introduction of quantum theory altogether, or lead to no (observable) changes by contributing an extra subspace to the ordinary chronology-respecting system. Under the relaxed rules à la Deutsch [11], the demand of linearity of the dynamics leads to a decoupling of a chronology-respecting region and CTCs. If a nonlinearity is accepted, then the CTC region does not carry independent degrees of freedom and only a certain amount of causal relations between the regions may be allowed.

### III. QUANTUM HISTORIES AND THE OBSERVER-DEPENDENCE OF CAUSAL LINKS

From now on we work with a causal set  $\mathcal{C}$ , which has no timelike loops. Nevertheless, it turns out that not all causal



histories are consistent with quantum mechanics. Note that the precise meaning of arrows in the relation  $x \preceq y$  is as follows. For  $x \in \xi$  and  $y \in \zeta$  the existence of  $U(\xi, \zeta)$  implies that if there is no future-directed path between  $x$  and  $y$ , i.e.,  $x \sim y$ , then the reduced final state on  $\mathcal{H}(y)$ ,  $\rho_y^\zeta = \text{tr}_{\xi \setminus y} \rho^\xi = \text{tr}_{\xi \setminus y} U \rho^\xi U^\dagger$  is independent of the initial reduced state  $\rho_x^\xi$ .

We deal first with the situation when it is possible to identify the initial and final Hilbert spaces pointwise, e.g., on Fig. 1 we assume that  $d_x \equiv \dim \mathcal{H}(x) = d_u \equiv \dim \mathcal{H}(u)$ , etc. A general case is considered at the end of this section.

Consider three possible causal relations that are presented on Fig. 1. Despite its intuitive appeal the causal history (a) is incompatible with the defining Eq. (1). It can be observed on a simple example of two qubits. Label the basis of each of the spaces by  $|0\rangle, |1\rangle$ . The controlled-NOT (CNOT) unitary operation on two qubits [4] apparently fits the described scheme: the value of the qubit  $x$  remains the same, while the qubit  $y$  may be flipped. The action of CNOT on this basis is given in the first two columns of the table below. However, in the basis  $|\pm\rangle = (|0\rangle \pm |1\rangle)/\sqrt{2}$ , the same unitary evolution should be depicted with the arrow going from  $y$  to  $u$ .

$ \psi\rangle$	$U \psi\rangle$	$ \psi\rangle$	$U \psi\rangle$
00	00	++	++
01	01	-+	-+
10	11	+-	--
11	10	--	+-

In general if an operation on  $\mathcal{H}(y)$  is controlled by the state on  $\mathcal{H}(x)$  which remains unchanged,

$$U(|\psi\rangle \otimes |\phi\rangle) = |\psi\rangle \otimes V_\psi |\phi\rangle, \quad (15)$$

for all states  $|\phi\rangle$ , then it is actually independent of  $|\psi\rangle$ ,  $V_\psi \equiv V$ . Indeed, consider two possible initial states  $|\psi\rangle|\phi\rangle$  and  $|\psi'\rangle|\phi\rangle$ . Their overlap is preserved under  $U$ , hence

$$\langle \phi | V_\psi^\dagger V_{\psi'} |\phi\rangle = 1, \quad (16)$$

for an arbitrary state  $|\phi\rangle$ . Hence  $V_\psi = V_{\psi'}$ .

As a result, only the second and the third histories of Fig. 1 are consistent with existence of an arbitrary unitary evolution on causal sets. The alternative is to introduce an external (classical) observer who is restricted to do measurements in a (given) particular basis, say  $(0, 1)$  as in the above example. The CNOT example shows that the observer will prescribe different causal relations depending on its choice of measurement basis. Then, by restricting the allowed unitary evolutions between the complete sets, the asymmetric causal structure is made compatible with quantum mechanics.

Adhering to the latter option is not only too restrictive, but not always possible. Consider the projectors  $P_0$  and  $P_1$ ,  $P_0 + P_1 = \mathbb{1}$ , on the singlet (spin-0) and triplet (spin-1) states of  $\mathbb{C}^2 \otimes \mathbb{C}^2$ . Then for generic values of the parameters  $\alpha, \beta$  there is no basis in which the unitary

$$U = e^{i\alpha} P_0 + e^{i\beta} P_1, \quad (17)$$

can be represented as in Eq. (15). This particular set of unitary operators is relevant to computational universe models [5] and to loop quantum gravity [13].

From the above discussion it follows that we have to work with the symmetric diagrams that take into account mutual influence of quantum systems. This is automatically taken into account in the dual picture, where edges represent quantum systems and vertices denote the interactions. It is obtained graphically by turning points of the causal set diagram into lines, and the arrows between the points into boxes that cover the lines. Applying the same transformation once more, we obtained a symmetrized version of the initial diagram. For example, if one starts from the diagram (a) on Fig. 1 two duality transformations bring it to the right diagram on Fig. 6. It is the standard quantum network picture of quantum information theory [4]. This remains true also for more complicated diagrams, the complete surfaces of which are considered as bipartite systems.

However, having three or more events in the complete surfaces allows for more complicated structures, as shown on Fig. 7. The diagram 7(b) and its mirror image with  $1 \sim 3^*, 3 \preceq 1^*$  are consistent with the symmetry requirement. Any bipartite splitting ( $1$  and  $2$  vs  $3$ , etc.) results in a symmetric coarse-grained diagram.

*Proposition 1.* The diagram 7(b) admits a unitary evolution  $U$  if and only if

$$U = U_{23} U_{12}, \quad [U_{23}, U_{12}] \neq 0, \quad (18)$$

where  $U_{23}$  and  $U_{12}$  are the bipartite unitary operators that act on  $\mathcal{H}_2 \otimes \mathcal{H}_3$  and  $\mathcal{H}_1 \otimes \mathcal{H}_2$ , respectively.

First we prove the sufficient condition, namely, that  $1 \preceq 3^*$  and  $3 \sim 1^*$ . While from the circuit diagram of Fig. 2 that gives a graphic decomposition of  $U$  the latter statement is obvious, it is worthwhile to consider a formal proof. Since linear operators form a Hilbert space with a Hilbert-Schmidt inner product, a bipartite unitary operator can be written as

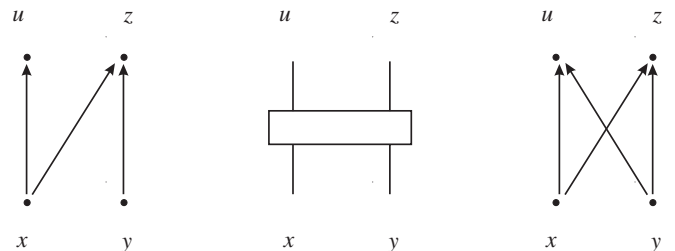
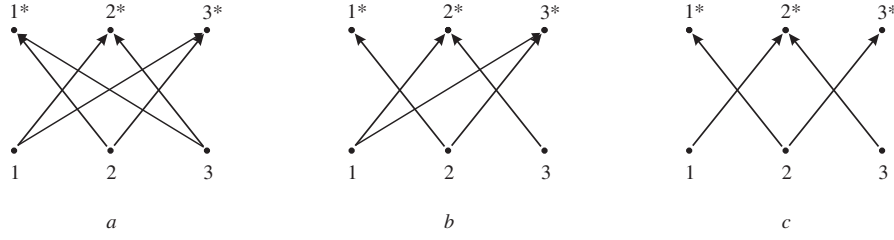


FIG. 6. Two consecutive duality transformation.


 FIG. 7. Possible causal links at the one step of the evolution of a tripartite system. The trivial links such as  $1 \rightarrow 1^*$  are not shown.

$$U_{AB} = \sum_{\mu} \alpha_{\mu} A_{\mu} \otimes B_{\mu}, \quad (19)$$

where the operators  $\{A_{\mu}\}$  and  $\{B_{\mu}\}$  form the orthogonal bases,

$$\text{tr} A_{\mu}^{\dagger} A_{\nu} = d_A \delta_{\mu\nu}, \quad \text{tr} B_{\mu}^{\dagger} B_{\nu} = d_B \delta_{\mu\nu}, \quad (20)$$

and the Schmidt coefficients satisfy

$$\alpha_{\mu} > 0, \quad \sum_{\mu} |\alpha_{\mu}|^2 = 1, \quad (21)$$

and there is no more than  $\min(d_A, d_B)$  terms. Hence

$$U_{12} = \sum_{\mu} \alpha_{\mu} A_{\mu} \otimes B_{\mu} \otimes \mathbb{1}, \quad U_{23} = \mathbb{1} \otimes \sum_{\nu} C_{\nu} \otimes D_{\nu}. \quad (22)$$

It is enough to consider initial pure product state [7], so

$$U(|\phi\rangle_1 \otimes |\varphi\rangle_2 \otimes |\psi\rangle_3) = \sum_{\mu\nu} \alpha_{\mu} \beta_{\nu} A_{\mu} |\phi\rangle_1 \otimes C_{\nu} B_{\mu} |\varphi\rangle_2 \otimes D_{\nu} |\psi\rangle_3. \quad (23)$$

The reduced density matrix

$$\rho'_1 = \sum_{\mu\nu\mu'\nu'} \alpha_{\mu} \alpha_{\mu'} \beta_{\nu} \beta_{\nu'} \times \text{tr}(B_{\mu} |\varphi\rangle \langle \varphi| B_{\mu'}^{\dagger} C_{\nu'}^{\dagger} C_{\nu}) \times \langle \psi | D_{\nu'}^{\dagger} D_{\nu} | \psi \rangle A_{\mu} |\phi\rangle \langle \phi| A_{\mu'}^{\dagger}. \quad (24)$$

Expanding the trace and using  $\langle k | \langle l | U_{23}^{\dagger} U_{23} | k' \rangle | l' \rangle = \delta_{kk'} \delta_{ll'}$ , we get

$$\rho'_1 = \sum_{\mu\mu'} \alpha_{\mu} \alpha_{\mu'} \langle \varphi | B_{\mu'}^{\dagger} B_{\mu} | \varphi \rangle A_{\mu} |\phi\rangle \langle \phi| A_{\mu'}^{\dagger}, \quad (25)$$

as expected. On the other hand, since  $[C_{\nu}, B_{\mu}] \neq 0$  at least for some  $\mu$  and  $\nu$ ,

$$\begin{aligned} \rho'_3 &= \sum_{\nu\nu'} \beta_{\nu} \beta_{\nu'} \langle \varphi | C_{\nu'}^{\dagger} C_{\nu} | \varphi \rangle D_{\nu} |\psi\rangle \langle \psi| D_{\nu'}^{\dagger} \\ &+ \sum_{\mu\nu\mu'\nu'} \langle \phi | A_{\mu'}^{\dagger} A_{\mu} | \phi \rangle \langle \varphi | O(\mu, \mu', \nu, \nu') | \varphi \rangle \\ &\times D_{\nu} |\psi\rangle \langle \psi| D_{\nu'}^{\dagger}, \end{aligned} \quad (26)$$

where the operator  $O$  is built from the Schmidt basis operators and their commutators. Since this operator is

nonzero, its expectation is nonzero at least for some  $|\varphi\rangle_2$ , and the state  $\rho'_3$  indeed depends on  $|\phi\rangle_1$ .

To establish the necessary condition we show that any other form of  $U$  leads to  $3 \leq 1^*$  diagram. The commuting  $U_{12}$  and  $U_{23}$  correspond to 7(c), as shown below. Any tripartite unitary can be decomposed [4] as certain products of the bipartite unitary operators. If it is not of the form of Eq. (18), then

$$U = V U'_{12} U_{23} U_{12}, \quad (27)$$

where  $V$  is some unitary (which may be equal to the identity) and  $U'_{12}$  acts on  $\mathcal{H}_1 \otimes \mathcal{H}_2$ . The relation  $1 \leq 3^*$  follows from the sufficient condition applied to  $U_{23} U_{12}$ , and the relation  $3 \leq 1^*$  is established by the factor  $U'_{12} U_{23}$ .

*Proposition 2.* The diagram 7(c) admits a unitary evolution  $U$  if and only if

$$U = U_{23} U_{12}, \quad [U_{23}, U_{12}] = 0. \quad (28)$$

The sufficient part follows from Eq. (26), since in this case  $O = 0$ . Proposition 1 guarantees that any other form of  $U$  introduces additional causal links.

It should be noted that the diagrams with less-than-maximal number of links correspond to the zero-volume subset of the set of all unitary operators. For example, a generic three-qubit unitary is characterized by 64 parameters, while two two-qubit unitaries have at most 32.

Figure 8 gives an example of a graph where it is impossible to identify the spaces of the individual events. However, since all the spaces are finite-dimensional, their dimensions can be uniquely represented as a monomial  $p_1^{n_1} \dots p_k^{n_k}$ , where the numbers  $p_i$  are prime and the integers  $n_i \geq 1$ . Since the dimensionality of the Hilbert spaces on the complete surfaces are equal, it is possible to decom-

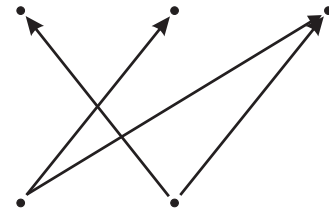


FIG. 8. In a generic causal history it is impossible to associate fixed Hilbert spaces to several individual events (points). This leads to the natural question: how to define/characterize individual events?

pose them into fictitious subspaces of the dimensions that are prime powers. Those subspaces can be identified and the above consistency requirements can be applied to them.

Our discussion established the following

*Theorem 1.* A discrete, locally finite, causal pre-spacetime structure admits a unitary evolution between its acausal surfaces if and only if it can be represented as a quantum computational network.

This quantum computational network, or more simply quantum circuit, describes the dual causal set: events (points or boxes) are the unitary evolutions and objects/states are represented as arrows (or lines) between the events. As we already saw in the example of the CNOT gate, this quantum circuit picture allows a better representation of the causal relations. The bipartite system case was shown on Fig. 6. As for a tripartite system as on Fig. 7, we have once again *a priori* three lines coming in a big box representing the unitary evolution and then the same three lines coming out. This picture can nevertheless be refined for the more peculiar causal structures 7(b) and 7(c). Indeed we will now have a couple of 2-lines boxes, one for the evolution of the system  $1 \otimes 2$  and one for the system  $2 \otimes 3$ , as shown on Fig. 2. For the diagram 7(c), the order of these two boxes does not matter since the two unitaries commute with each other. On the other hand, the history 7(b) gives *a priori* a precise time ordering of the two boxes, (12) coming before (23), and no change of measurement basis can lead to a causal arrow between 3 and 1\*.

#### IV. COMPLETELY POSITIVE MAPS

Unitary maps on the complete pairs tell very little about the causal relations between the acausal sets. To overcome this difficulty completely positive (CP) trace-preserving dynamics of the subsystems that are associated with parts of complete surfaces of QCH was postulated [1,14]. A trace-preserving map of density matrices is dual to a unital (i.e., taking  $\mathbb{1}$  to  $\mathbb{1}$ ) map on observables.

*A priori* there is no reason to assume complete positivity of the reduced dynamics. Indeed, it is known that prior correlations (not necessarily entanglement) between the subsystems may lead to a non-completely positive evolution. Hence enforcing this requirement excludes many types of dynamics that establish correlations between the subsystems. The requirement that the reduced dynamics is unital, i.e., the maximally mixed state on  $\mathcal{H}(x)$  is a fixed point of the map, is more natural [1]. Indeed, an average over all CP maps is the map  $\Lambda$ , which transforms the whole state space into the total mixture, i.e.,  $\Lambda(\rho) \propto \mathbb{1}$ . On the other hand, in the absence of measurement data an arbitrary test state at the output can be estimated as a total mixture. Consequently, the resulting reconstructed map is trivially unital [15,16].

The reduced dynamics can be discussed as follows. Consider a complete pair of acausal sets that are related by a unitary evolution  $U$  and let a part of the acausal  $\xi$

carry the Hilbert space  $\mathcal{H}(\xi_A)$ . When one inquires about possible reduced dynamics on it, that means following the evolution for all possible initial states on  $\mathcal{H}(\xi_A)$ , while keeping the reduced state of the remaining subsystem  $\mathcal{H}(\xi_B)$  and the correlations between the subsystems fixed. Then the evolution of a  $d_A$ -dimensional state  $\rho_A$  will be given by a linear, possibly non-CP, map. In particular [8], under the action of a unitary  $U$  on an extended system, the reduced dynamics of  $\rho_A$  is given by an affine map, with its linear part being a CP map and the traceless part related to the initial correlations between the system  $A$  and the ancilla  $B$  [8]. More precisely,

$$\Xi(\rho_A) = \sum_k M_k \rho_A M_k^\dagger + \sum_{ijl} c_{l,ij} \Gamma_{ij} \sigma_l, \quad (29)$$

where the first expression on the right-hand side is a completely positive map  $\Lambda(\rho_A)$  that is written in Kraus decomposition form, the coefficients  $c_{l,ij}$  depend on  $U$  alone, and

$$\Gamma_{ij} = (\gamma_{ij} - \alpha_i \beta_j) / d_A d_B, \quad (30)$$

is the correlation tensor [17], and the matrices  $\sigma_l$  are the generators of  $SU(d_B)$ .

Any CP map  $\Lambda$  can be decomposed into a unital CP map  $\Lambda_0$  and the constant term  $\sum_l a_l \sigma_l$ . Then if  $\Xi(\mathbb{1}) = \mathbb{1}$ , the constant parts of the map cancel, and one is left with

$$\Xi(\rho_A) = \Lambda_0(\rho), \quad (31)$$

i.e., with a completely positive evolution. It should be noted that the requirement of unital evolution of the reduced subsystem imposes a simultaneous constraint on both allowed global unitaries  $U$  and the initial states.

#### V. FROM GLOBAL STATES TO LOCAL INFORMATION AND BACK

Choosing an acausal set is analogous to fixing a leaf in the foliation in the continuum case. On such a slice it is meaningful to define reduced density matrices, as in the previous section. However, an inverse procedure (reconstructing  $\rho_\xi$  from  $\rho_{\xi_A}$  and  $\rho_{\xi_B}$ ) is not unique. As it is seen from Eq. (4), an additional information on correlations is required. For example, in the two-qubit case a generic density matrix is specified by 15 real parameters, while the reduced density matrices specify only six of them. Even for a pure state  $\rho_\xi$  there is more information in  $\Gamma$  than just the degree of entanglement. All the maximally entangled states have  $\alpha_i = \beta_j = 0$ , but are distinguished by different elements of  $\Gamma$ .

Dynamics can help in reducing redundancy of the possible state reconstruction. In addition, a change of the Lorentz frame that induces a change of (semiglobal) foliations, is analogous to the choice of a different partition into acausal sets.

If we consider an evolution of  $\xi_A$  that is a subset of the acausal set  $\xi$  into  $\zeta_A$  of the acausal set  $\zeta$ , which may be also taken as the subsets of the acausal sets  $\xi'$  and  $\zeta'$  respectively, the reduced dynamics  $\Xi(\rho_A)$  should be the same. Given the reduced density matrices knowledge of the local evolution constrains their possible embedding into the global states on  $\mathcal{H}(\xi)$  and  $\mathcal{H}(\zeta)$ . To summarize the situation, the Lorentzian structure of the space-time is defined through the definition of spacelike foliations of the causal history and how the quantum states living on these foliations are related by Lorentz boosts. The quantum states associated to each spacelike hypersurface are not uniquely determined by the local information, i.e., the reduced density matrices, but we need the global information contained in the correlations between these density matrices. These correlations are *a priori* induced by the precise dynamics (unitarity operators) on the causal set. At the end of the day, more work is needed in order to be able to define precisely how the dynamics determines the action of Lorentz transformations of quantum states on the causal set. This is left for future investigation.

To conclude, we have discussed the quantum causal histories as introduced in [1,2]. These are basically causal sets dressed with Hilbert spaces (on the nodes) and (completely positive) operators on the arrows describing the evolution of the quantum states along the causal set. To start with, we have addressed the issue of closed timelike loops and shown that they carry no relevant extra degrees of freedom. Then we have noticed that the causal links of generic quantum causal histories are observer-dependent:

they depend on the basis of measurements chosen by the observer. To get rid of this observer dependence, we must add causal links reflecting the way the dynamics entangle the various systems. Moreover, restrictions on the measurement bases also restrict the allowed unitary evolutions. At this level, it appears more natural to switch to a dual picture where causal histories are described in terms of quantum computational networks (quantum circuits): at the end of the day, these are the only causal histories that allow the introduction of quantum mechanics without imposing restrictions on the measurements. We have also discussed the requirement of complete positivity of the evolution operators. This is quite a restrictive requirement. However, it results from a standard posing of the initial-value problem and the assumption that the maximally mixed state remains such in the course of evolution. Finally, we discussed the action of Lorentz boosts on such quantum causal histories and explained that it involves understanding how the global quantum states can be deduced from the states (reduced density matrices) of the subsystems. This is a hard problem which involves understanding how the causal set dynamics creates correlations between the subsystems and which we leave as an open issue.

#### ACKNOWLEDGMENTS

We thank Steve Bartlett, Fotini Markopoulou, Lee Smolin and Bill Unruh for stimulating discussions. Part of the work by ERL was performed at the Perimeter Institute.

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