Creation, Manipulation, and Detection of Abelian and Non-Abelian Anyons in Optical Lattices

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Anyons are particlelike excitations of strongly correlated phases of matter with fractional statistics, characterized by nontrivial changes in the wave function, generalizing Bose and Fermi statistics, when two of them are interchanged. This can be used to perform quantum computations [A. Yu. Kitaev, Ann. Phys. (N.Y.) **303**, 2 (2003)]. We show how to simulate the creation and manipulation of Abelian and non-Abelian anyons in topological lattice models using trapped atoms in optical lattices. Our proposal, feasible with present technology, requires an ancilla particle which can undergo single-particle gates, be moved close to each constituent of the lattice and undergo a simple quantum gate, and be detected.

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The quest for physical systems where anyons [1] can be observed has concentrated so far in effectively 2d materials with topological order [2]. Abelian anyons, whose interchange generates a nontrivial phase in the wave function, exist in the fractional quantum Hall effect. Non-Abelian anyons, whose interchange effects full unitary gates on the wave function, are expected at certain filling fractions [3] (see recent experimental progress in [4]). In spin lattice systems, anyons can appear as low-lying excitations of topologically ordered ground states (see, e.g., [5–7]). Several implementations of lattice models with anyonic excitations have been put forward [8–14]. Those involving atoms or molecules in optical lattices are especially attractive, given recent experimental progress [15]. Specifically, Kitaev's honeycomb lattice model [16] can be engineered [11,12], and anyonic interferometry in its Abelian phase can be performed with cavity-mediated *global* string operations [13] or using individual addressing to braid excitations [17] (but due to the perturbative nature of the effective Hamiltonian in this model, the visibility of anyonic interferometry is degraded [13,18]).

Here we propose a novel scheme to create topologically ordered states, generate and braid anyons, and detect their statistics for any setup based on particles in optical lattices. We use a lattice of particles of species A to build the topological code and an ancilla of different species B that can be moved independently and brought close to any A particle to perform controlled operations on the code [19]. Preparing the ancilla in superposition states, making it interact with appropriate code particles, and measuring its state, the following can be achieved: creation of a topological state, or a general error-correcting code (ECC); creation, braiding, and measurement (fusion) of anyons, all operations needed to perform topological quantum computation (TQC) by braiding; and anyonic interferometry, allowing direct observation of anyonic statistics. Note that (i) by using an ancilla with different quantum states to perform the manipulation of the anyons, tasks can be carried out that are not possible using classical (e.g., laser) manipulation of anyons (without single-particle addressability, all proposed methods lack the power of ours); (ii) there is no need in principle of single-particle addressability, especially to perform proof-of-principle experiments; (iii) it is based on successfully demonstrated technologies [20–22]; (iv) it is the first realistic protocol for simulating universal TQC in an atomic, molecular, optical system [while engineering the microscopic Hamiltonian to build topological protection may be some time off (though see [23]), our method works independently of the existence of the background Hamiltonian].

We consider 2d lattices loaded with atoms or molecules, e.g., ^{87}Rb. The ancilla, e.g., ^{23}Na, can be moved independently using a laser potential not affecting Rb atoms (see Fig. 1). These are now routinely loaded in optical lattices, and in the Mott insulator state one can have extended regions with one particle per site [22,24]. Single particles can also be loaded in optical potentials and moved without decoherence [21]. Our scheme can be extended to layered 3d configurations [25]. We first consider the toric code Hamiltonian [5], with Abelian anyonic excitations only, as a toy model, but our scheme is basically model-independent; later, we apply it to the $D(S_3)$ quantum double model [5], which has non-Abelian anyons and is universal for TQC [26].

When the ancilla is brought close to a code atom, they experience a 2-qubit unitary $U_Z = |0\rangle_a\langle 0| \otimes I + |1\rangle_a\langle 1| \otimes Z$ between their internal levels (X, Y, and Z are Pauli operators). This gate can be implemented by cold collisions [27] or any other means [28,29]. Single qubit operations can be applied to the ancilla without having to address it, due to the different level structure of code atoms. Gates like $U_X = |0\rangle_a\langle 0| \otimes I + |1\rangle_a\langle 1| \otimes X$ can be implemented by applying appropriate gates before and after U_Z. The internal ancilla state (measurement of Z_a) can also be detected with standard techniques without having to address it or affecting the code.

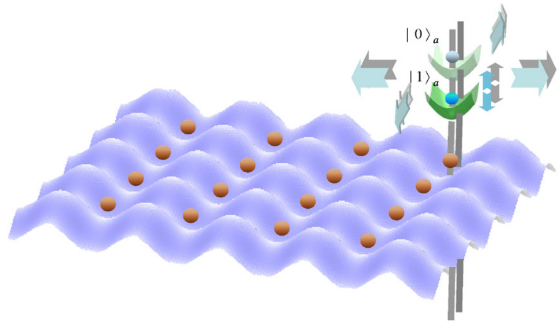

FIG. 1 (color online). $2d$ optical lattice in the $x - y$ plane loaded with one atom per site. Atoms are in different Zeeman levels $|0\rangle$ and $|1\rangle$ in their ground electronic state, storing quantum information. An ancillary atom of a different species in its electronic ground state, with relevant Zeeman levels $|0\rangle_a$ and $|1\rangle_a$, is trapped using laser standing waves along the three directions. Lasers in the $x - y$ plane are far detuned from the fine structure splitting: The potential controlling horizontal movement of the ancilla does not depend on its internal state. The laser propagating along the z direction is tuned in between the fine-split excited P levels: The potential controlling the vertical movement depends on the internal state [27,28].

We next show how to create Kitaev's toric code [5] with these tools. The code is defined as the ground level of a stabilizer Hamiltonian on a square lattice of qubits, realized as Rb atoms, at the edges of a square lattice. The Hamiltonian $H = -\sum_v A_v - \sum_p B_p$ is the sum of mutually commuting stabilizers $A_v = \prod_{i\in v} X_i$ and $B_p = \prod_{i\in p} Z_i$, where v runs over all vertices and p over plaquettes and products involve the qubits surrounding the vertices or plaquettes. We can associate the presence or absence of particlelike excitations at the plaquettes (magnetic defects) and vertices (electric defects) with the fulfillment or not of the ground level conditions $A_v = +1$ and $B_p = +1$. Plaquette and vertex excitations are thus characterized by $A_v = -1$ and $B_p = -1$ and appear at the ends of strings of Z and X operators applied on a ground state. These "particles" turn out to have nontrivial (anyonic) exchange statistics due to the anticommutation of the X and Z Pauli operators; namely, the wave function gets multiplied by -1 when a vertex particle winds around a region containing a single plaquette excitation. Detection of this phase change is possible via interference experiments involving superpositions of states with and without anyons. Moreover, the degeneracy of the code allows one to interpret it in terms of a set of logical qubits whose Z and X operators are given in terms of chains of Z's and X's.

We work with a rectangular surface with smooth and rough boundaries [25], with appropriate 3-body vertex and plaquette operators along the boundary providing for a two-dimensional code space: One logical qubit is encoded as the eigenvalue of a chain of Z's along an edge path connecting the rough boundaries. The code space is spanned by $+1$ coeigenstates of the stabilizers. To create a state $|\Psi\rangle$ in the code, we start with a well-defined state $|0\rangle^{\otimes N}$ ($+1$ eigenstate of each B_p) and measure the A stabilizers sequentially, from left to right and top to bottom, using the ancilla. If each outcome is $+1$, our goal is achieved, since $|\Psi\rangle \propto \prod_v(1 + A_v)|0\rangle^{\otimes N}$ [30]. If -1 is obtained, we can correct by applying Z_b to qubit b at the bottom of the vertex using again the ancilla, since $Z_b(1 - A_v)|0\rangle^{\otimes N} = (1 + A_v)|0\rangle^{\otimes N}$; Z_b is applied to a qubit that has not been measured yet (for the last row, Z_b can be applied to the rightmost qubit). Once we have measured all vertices and thus prepared the state, we can measure all stabilizers to detect errors and apply error-correcting X's or Z's to the corresponding qubits by using the ancilla. We could have started at another state, measured all operators in any order, and then corrected errors in this way to prepare the desired state (this can be used to prepare the target state in models beyond Kitaev's). We now show how plaquette and vertex measurements, as well as X's and Z's, are performed using the ancilla. To measure A_v, we prepare the ancilla in state $|+\rangle_a \propto |0\rangle_a + |1\rangle_a$, move it to each qubit in the vertex, and apply U_X each time. Then we apply a Hadamard gate to the ancilla and measure X_a. If the result is ± 1, we have applied $\langle \pm | \prod_{i\in v} U_X | + \rangle = (1 \pm A_v)$ to the qubits at the vertex, thus performing the desired measurement. B_p is measured by substituting U_Z for U_X. To apply X (Z) to a qubit, we prepare the ancilla in state $|1\rangle_a$, approach it to that qubit, and apply U_X (U_Z). Once the toric code state is prepared, operations within the code are performed by applying strings of operators, using the ancilla in state $|1\rangle_a$, and applying U_X or U_Z sequentially on the desired qubits by bringing them close to the ancilla. To measure string operators, prepare the ancilla in state $|+\rangle$, follow the same sequence, and measure X_a at the end. The toric code has two kinds of elementary excitations [5]: pairs of anyons in frustrated vertices (electric defects, $A_v = -1$) and in frustrated plaquettes (magnetic defects, $B_p = -1$), with mutual Abelian anyonic statistics. They can be created by applying Z or X to a given qubit and can be moved, braided, and fused together by applying these operators along a given path using the ancilla. Superpositions of states with and without vortices, or where they are in different places (see Fig. 2), can be created, allowing one to observe fractional statistics: The simplest interference experiment is shown in Fig. 3. On how to infer anyonic statistics from interference experiments, see [31].

We now outline the preparation and manipulation of anyons in a non-Abelian setting universal for quantum computation [26]: the quantum double model $D(S_3)$ based on the group of permutations of three elements S_3 [5] (see a brief discussion in [31]; the full construction will be fully given in [32]). It is a lattice model generalizing the toric code, where local degrees of freedom live at the (oriented) edges of the lattice with orthonormal bases $\{|g\rangle\}$ are labeled by the six group elements $g \in S_3$. The Hamiltonian, also of the form $H = -\sum_v A_v - \sum_p B_p$, has commuting vertex and plaquette stabilizers imposing constraints on the ground states. Their violation define particlelike excita-

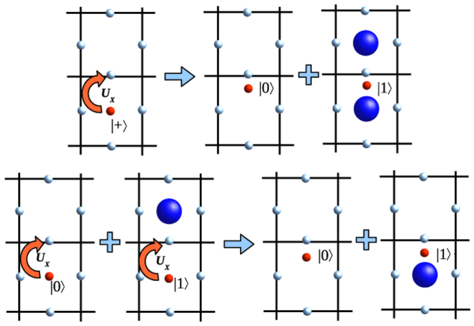

FIG. 2 (color online). Top: Creation of a superposition of the vacuum and a pair of magnetic defects. On top of the ground state (code atoms shown as light circles), an ancilla (dark) is initialized as $|+\rangle$ and brought close to a code atom, effecting $U_X = |0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes X$ and creating two magnetic defects (big blobs) in the adjacent plaquettes in the superposition component where the ancilla is in state $|1\rangle$, while the code remains in the ground state in sector $|0\rangle$. Bottom: Anyon transport. The ancilla interacts via U_X with a code atom between a plaquette satisfying the ground state condition and one of the plaquettes containing a magnetic defect in a superposition sector, transferring the anyon to the first plaquette.

tions (anyons) with topological charges (electric, magnetic, and dyonic), with non-Abelian fusion and braiding rules. Creation, transport, and fusion of anyons can be achieved generalizing the controlled-NOT operations of the toric code to controlled left and right group multiplications: $\mathcal{U}_h^{L,R} = |0\rangle_B\langle 0| \otimes I_A + |1\rangle_B\langle 1| \otimes (\sigma_h^{L,R})_A$, with $\sigma_h^L|g\rangle = |hg\rangle$, $\sigma_h^R|g\rangle = |gh\rangle$. The local degrees of freedom for the $D(S_3)$ model are qudits of six dimensions, and their six basis elements can be encoded into ground electronic hyperfine states of an alkali atom with enough levels, used as code lattice A. To create, transport, and fuse pure electric and magnetic charge states (enough to simulate universal TQC [26]), a 6-state ancilla species B is especially appropriate (see Fig. 4). As in the toric code, vertex operators can be measured using ancilla-assisted operations to prepare the ground state (see [31]). With one single ancilla, which need not be spatially addressed, our algorithm requires $\mathcal{O}(nm)$ steps in an $n \times m$ region; with an auxiliary lattice with one ancilla per face of the code lattice, assuming addressability, it can be parallelized to depth $\mathcal{O}(n + m)$ (essentially optimal [33]; see [31]).

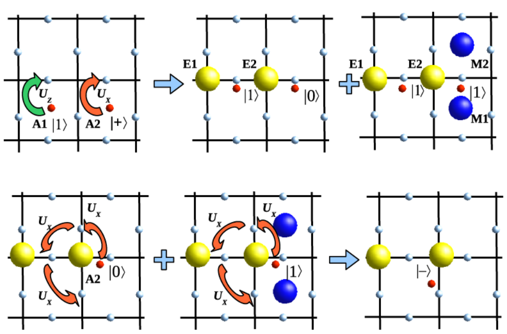

FIG. 3 (color online). Minimal interferometry experiment for the toric code. Top: Ancilla A1, initialized as $|1\rangle$, unconditionally creates a pair E1, E2 of electric defects at neighboring vertices by application of $U_Z = |0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes Z$. Ancilla A2, initialized as $|+\rangle$, creates a superposition of the presence and the absence of two magnetic defects M1 and M2 at neighboring plaquettes by application of U_X. Bottom: M1 is wound around E2 by sequential U_X interaction of ancilla A2 with the code atoms surrounding E2. M1 and M2 are eventually reannihilated, bringing both sectors to the ground state with a relative minus sign: $|\mathrm{GS}\rangle_{\mathrm{code}} \otimes |1\rangle_{\mathrm{A1}} \otimes |-\rangle_{\mathrm{A2}}$; i.e., a phase -1 is generated in the sector where braiding of defects takes place. In this case, the interferometry results can be read from the ancilla lattice by a local projective measurement on A2.

This scheme is independent of the method used to construct the topological state. It can be built by cooling an atomic ensemble interacting via an engineered topological Hamiltonian, providing in principle topological protection to the code except for anyonic manipulations, which should take the system to excited levels in a controlled way (as needed to perform TQC as such; on how to simulate relevant Hamiltonians, see [11,12]). But it can also be constructed by the above procedure using ancillas to impose stabilizer constraints, enough to perform proof-of-principle interference experiments, a worthy goal by itself. This also allows fault-tolerant quantum computation with general ECCs (topological codes are excellent ECCs; local operators do not mix topological sectors, and string operators mixing them can be efficiently implemented).

The arrangement of ancillas is flexible. One ancilla, individually manipulable [21], suffices in principle. Varying degrees of parallelization are possible: The ground state can be constructed with sequential operations on one column of the sample (parallel to smooth boundaries) at a time, so as to cover the whole sample; in interferometry, commuting operations can be done simultaneously; some parallelization can be introduced with a coarser optical potential for the ancilla than for the code.

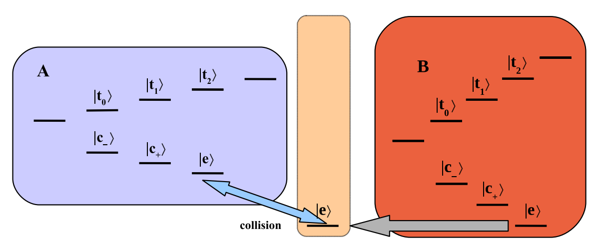

FIG. 4 (color online). Level structure for the $D(S_3)$ model. The elements of S_3 are encoded into ground electronic hyperfine states of a trapped alkali atom A (^{87}Rb or ^{23}Na, with 8 levels, or ^{133}Cs, with 16 levels). A 6-state ancilla B of a different species is used to control operations. Ancilla states can be moved independently: Bringing $|e\rangle_B$ close to a code atom A and coupling it to $|e\rangle_A$, a collisional phase gate $\mathcal{Z}_e = I_A \otimes I_B - 2|e\rangle_B \times \langle e| \otimes |e\rangle_A\langle e|$ is obtained; together with simultaneous 1-particle operations $\mathcal{V}_{\text{all A}} = \bigotimes_A v_A$ on the code, it provides all controlled operations. Indeed, $\mathcal{V}_{\text{all A}}^\dagger \mathcal{Z}_e \mathcal{V}_{\text{all A}}$ yields controlled phase gates for any code state with $v_A = |g\rangle_A\langle e| + |e\rangle_A\langle g|$ and transpositions with $v_A \propto (|e\rangle + |g\rangle)_A\langle e| + (|e\rangle - |g\rangle)_A\langle g|$ (these can be composed to obtain any controlled group multiplication).

This method shares common problems of optical lattice schemes of quantum computation, in particular, spontaneous emission. Essentially, only the vertical direction of the ancilla is close to resonance; lifetimes of seconds can be reached by tuning the laser between the fine structure levels and can be enhanced by restricting manipulation of the vertical dynamics to the (short) times the ancilla is close to a code atom. The ancilla can be repumped after an operation, allowing one to repeat the tasks and detect errors. Controlled logic by cold collisions [27] requires cooling the system to the physical ground state, possible for both the code (in the 1st Bloch band, the ground state of the local potential) and the ancilla. Rydberg gates [28,29] based on dipole-dipole interactions eliminate this condition. Ancilla-code interactions must break the code in a controlled way, not creating (superpositions of) stray anyons spoiling the quantum memory: Theoretical analysis [34] and experimental results [35] suggest that excellent control and small decoherence rates are achievable. Then the implementation benefits from the added protection of topological codes (or, in general, ECCs). With a bias magnetic field, arbitrary single qudit unitaries can be realized using frequency and polarization selectivity of microwave or Raman laser pulses. Collisional gates can be realized using trap-induced shape resonances [36] or using Raman pulses to map code and ancilla ground states ($|e\rangle_{A,B}$ in Fig. 4) to a vibrational excited state of each lattice well, evolving by a collisional phase and mapping back [37].

The experimental techniques required by our method are (i) independent trapping of two particle species A and B with different laser potentials; (ii) diluting the population of species B so that each particle is individually addressable; (iii) bringing species A to a Mott insulator phase; (iv) initialization of species A in a product state $|0\rangle^{\otimes N}$; (v) single-particle gates on species B; (vi) simultaneous gates on all particles of species A; (vii) independent transport of internal states of single particles of species B so as to effect cold collisions with particles of species A. Additionally, large scale simulations would require the ability to recool to vibrational states. Each one of these techniques has been demonstrated experimentally; bringing them together will pose an interesting experimental challenge.

A method based on an ancilla species thus allows one to perform all TQC tasks on a given topological state, independently of the way in which this state is constructed, in optical lattices; e.g., universal TQC based on braiding can be performed on top of the ground state of the $D(S_3)$ model. This scheme is likely to prove the most practical and general way to perform TQC in optical lattices. Large computations will face a steep scaling (a problem not exclusive of topological settings), but observing interference phenomena and applying gates by anyon braiding is feasible with today's technology. This method can also be used advantageously in general ECCs.

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[1] F. Wilczek, Phys. Rev. Lett. **48**, 1144 (1982); **49**, 957 (1982).
[2] X.-G. Wen, Adv. Phys. **44**, 405 (1995).
[3] G. Moore and N. Read, Nucl. Phys. **B360**, 362 (1991).
[4] M. Dolev *et al.*, Nature (London) **452**, 829 (2008).
[5] A. Yu. Kitaev, Ann. Phys. (N.Y.) **303**, 2 (2003).
[6] M. A. Levin and X.-G. Wen, Phys. Rev. B **71**, 045110 (2005).
[7] P. Fendley and E. Fradkin, Phys. Rev. B **72**, 024412 (2005).
[8] B. Douçot, L. B. Ioffe, and J. Vidal, Phys. Rev. B **69**, 214501 (2004).
[9] Zh.-Y. Xue, Z. D. Wang, and Sh.-L. Zhu, Phys. Rev. A **77**, 024301 (2008).
[10] B. Paredes and I. Bloch, Phys. Rev. A **77**, 023603 (2008).
[11] L. M. Duan, E. Demler, and M. D. Lukin, Phys. Rev. Lett. **91**, 090402 (2003).
[12] A. Micheli, G. K. Brennen, and P. Zoller, Nature Phys. **2**, 341 (2006).
[13] L. Jiang *et al.*, Nature Phys. **4**, 482 (2008).
[14] Y. J. Han, R. Raussendorf, and L.-M. Duan, Phys. Rev. Lett. **98**, 150404 (2007).
[15] I. Bloch, J. Dalibard, and W. Zwerger, Rev. Mod. Phys. **80**, 885 (2008).
[16] A. Yu. Kitaev, Ann. Phys. (N.Y.) **321**, 2 (2006).
[17] Ch. Zhang, V. W. Scarola, S. Tewari, and S. Das Sarma, Proc. Natl. Acad. Sci. U.S.A. **104**, 18 415 (2007).
[18] S. Dusuel, K. P. Schmidt, and J. Vidal, Phys. Rev. Lett. **100**, 177204 (2008).
[19] Using auxiliary particles as "moving heads" was already proposed in J. I. Cirac and P. Zoller, Nature (London) **404**, 579 (2000); K. G. H. Vollbrecht, E. Solano, and J. I. Cirac, Phys. Rev. Lett. **93**, 220502 (2004); T. Calarco *et al.*, Phys. Rev. A **70**, 012306 (2004).
[20] M. Anderlini *et al.*, Nature (London) **448**, 452 (2007).
[21] S. Kuhr *et al.*, Phys. Rev. Lett. **91**, 213002 (2003).
[22] M. Greiner *et al.*, Nature (London) **415**, 39 (2002).
[23] D. Bacon, Phys. Rev. A **78**, 042324 (2008).
[24] D. Jaksch *et al.*, Phys. Rev. Lett. **81**, 3108 (1998).
[25] E. Dennis *et al.*, J. Math. Phys. (N.Y.) **43**, 4452 (2002).
[26] C. Mochon, Phys. Rev. A **69**, 032306 (2004).
[27] D. Jaksch *et al.*, Phys. Rev. Lett. **82**, 1975 (1999).
[28] G. K. Brennen *et al.*, Phys. Rev. Lett. **82**, 1060 (1999).
[29] M. D. Lukin *et al.*, Phys. Rev. Lett. **87**, 037901 (2001).
[30] F. Verstraete, M. M. Wolf, D. Pérez-García, and J. I. Cirac, Phys. Rev. Lett. **96**, 220601 (2006).
[31] See the appendix of the online version of this Letter, arXiv:0802.3163.
[32] G. K. Brennen, M. Aguado, J. I. Cirac, F. Verstraete (to be published).
[33] S. Bravyi, M. B. Hastings, and F. Verstraete, Phys. Rev. Lett. **97**, 050401 (2006).
[34] E. Charron *et al.*, Phys. Rev. A **74**, 012308 (2006).
[35] D. Schrader *et al.*, Phys. Rev. Lett. **93**, 150501 (2004).
[36] R. Stock, E. L. Bolda, and I. H. Deutsch, Phys. Rev. Lett. **91**, 183201 (2003).
[37] F. W. Strauch *et al.*, Phys. Rev. A **77**, 050304(R) (2008).