Quantum theory of friction

Stephen M. Barnett[1] and James D. Cresser[2,3]
[1]*Department of Physics, University of Strathclyde, Glasgow G4 0NG, United Kingdom*
[2]*Department of Physics, Macquarie University, New South Wales 2109, Australia*
[3]*Australian Centre for Quantum Computer Technology, Macquarie University, Sydney, New South Wales 2109, Australia*
(Received 11 May 2005; published 17 August 2005)

We present a Markovian quantum theory of friction. Our approach is based on the idea that collisions between a Brownian particle and single molecules of the surrounding medium constitute, as far as the particle is concerned, instantaneous simultaneous measurements of its position and momentum.

DOI: 10.1103/PhysRevA.72.022107 PACS number(s): 03.65.Ta, 05.40.−a

I. INTRODUCTION

No physical system, whether classical or quantum, is ever truly isolated from its surroundings: "No man is an Iland, intire of itselfe" [1]. It will invariably be coupled, to a greater or lesser extent, to the effectively infinite number of degrees of freedom of its surrounding environment with which it will exchange energy and perhaps matter; i.e., it is an open system. Such systems exhibit irreversible dynamics, not to be observed for an isolated system, characterized by dissipation and random fluctuations of the system variables. The ubiquitous nature of open systems means that examples abound, but there is one particular example that has played a crucial role in the development of an understanding of open systems, both quantum and classical, and that is the motion of a particle subject to the randomly fluctuating influences of its surroundings, such as in the case of a macroscopic particle suspended in a liquid or gaseous environment. In this case, the particle is observed to follow a characteristic erratically varying path, usually referred to as Brownian motion, named after the botanist who famously described it. In addition, the particle experiences friction: any initial drift velocity of the particle is damped away.

It was Einstein and Smoluchowski [2] who independently realized the origin of Brownian motion as being the frequent random collisions between the particle and molecules making up its environment. Classical theories describing this behavior have been formulated in a number of ways, as elucidated, for instance, in [3]: in statistical terms, as in the original work of Einstein and Smoluchowski, leading to a Fokker-Planck equation description of the particle dynamics; by modeling the influences responsible for its erratic motion as externally applied classical stochastic forces, leading to a Langevin equation description; or through a classical treatment of the particle dynamics—for instance, by integrating over the microscopic degrees of freedom of the environment [4].

In the simplest case, the particle dynamics can be described by the (one-dimensional) Langevin equations

$$M\dot{x} = p, \quad \dot{p} = -\gamma p + F(t), \tag{1}$$

where M is the mass of the particle, γ is the damping coefficient, and $F(t)$ is a white noise term:

$$\langle F(t)\rangle = 0, \quad \langle F(t)F(t')\rangle = 2D_{pp}\delta(t-t'), \tag{2}$$

where the diffusion coefficient D_{pp}, given by

$$D_{pp} = M\gamma k_B T, \tag{3}$$

is such that the spread in steady-state momentum is given by the expected result of the equipartition theorem $\langle p^2(\infty)\rangle = Mk_BT$.

The analogous problem of *quantum* Brownian motion arises when both the particle and its surrounding environment are treated as quantum mechanical systems. The problem is a very important one, both because of its relevance in a wide range of physical circumstances and because the problem has proven to be a very fruitful model for developing means of investigating and understanding the properties of open quantum systems.

In addition to the dissipative dynamics found in the classical case—for instance, so-called Ohmic models yield operator equations of motion of the form of Eq. (1) for position and momentum—open quantum systems are found to have a number of distinctly nonclassical features. One of the most important of these is decoherence—the exceedingly rapid decay of quantum correlations—which is believed to play a crucial role in the emergence of the observed classical behavior of a quantum system [5]. In the case of Brownian motion, this would be the observed erratic motion of the particle. In a closely related way, the coupling of the system to the environment also means that their states become entangled, so that, in principle, information on the state of the system can be gained by observations performed on the environment. This perspective on the system-environment interaction plays a central role in providing an understanding of some aspects of the problem of measurement in quantum mechanics.

Among the various approaches that have been used in the study of these and other properties of open quantum systems, one that has proven to be extremely useful involves integrating out the environmental degrees of freedom, yielding an equation (the master equation) for the density operator $\hat{\varrho}(t)$ of the system. In the general case, this equation assumes the form of an integro-differential equation

$$\frac{d\hat{\varrho}}{dt} = \int_0^t \mathcal{L}(t-s)\hat{\varrho}(s)ds. \quad (4)$$

Because the current state of the system depends on its past history through the superoperator memory kernel $\mathcal{L}$, the dynamics of the system as described by this equation is loosely referred to as non-Markovian. A non-Markovian master equation is, under most circumstances, difficult if not impossible to solve and its physical interpretation is often obscure. The exception is when the memory of the system is exceedingly short, in which case the memory kernel is approximated as a δ function in time—the so-called Markov approximation—so that the future behavior of the system is determined solely by its current state; i.e., the dynamics is said to be Markovian.

Gorini *et al.* [6] and Lindblad [7] have shown, from a very general standpoint, that in order for the dynamics to be Markovian and in order that the trace, positivity (or more precisely complete positivity [8]), and Hermiticity of the reduced density operator of the system be preserved during its time evolution, the structure of the master equation must assume a particular form, usually referred to as the Lindblad form:

$$\frac{d\hat{\varrho}}{dt} = -\frac{i}{\hbar}\left[\hat{H},\hat{\varrho}\right] + \tfrac{1}{2}\sum_n \left[2\hat{L}_n\hat{\varrho}\hat{L}_n^\dagger - \hat{L}_n^\dagger\hat{L}_n\hat{\varrho} - \hat{\varrho}\hat{L}_n^\dagger\hat{L}_n\right], \quad (5)$$

where the $\hat{L}$ are system operators. As well as being substantially easier to solve than its non-Markovian relative, a Lindblad master equation has a structure that reflects the fact that the Markovian dynamics of the system is accompanied by an irreversible transfer of information on the state of the system to the environment. This information is then available for measurement by some suitable apparatus continuously monitoring the environment. The evolution of the density operator can then be decomposed into an ensemble of quantum trajectories [9–11] with each trajectory representing a possible evolutionary history of the state of the system conditioned by the outcomes of measurements made on the environment.

The derivation of a master equation for quantum Brownian in the early 1980s by Caldeira and Leggett [12] (see also the much earlier work of Agarwal [13] and also that of Unruh and Zurek [14]) was a crucial event in the development of the theory of open quantum systems, one which triggered an explosion of research. Their model has acquired the status of a paradigm of open quantum systems as it represents, among other things, an exactly solvable model of such a system. As such, it has been invaluable for the study of the above-mentioned phenomena associated with open systems, as well as serving as a model by which some of the fundamental features of the mathematical description of open quantum systems can be subject to intense scrutiny: in particular the structure of the master equation for the reduced density operator of the particle.

In their model, the particle was assumed to be linearly coupled to an environment modeled as a collection of simple harmonic oscillators. It is possible, for such a model, to integrate out the environmental degrees of freedom by use of the path-integral techniques of Feynman and Vernon [15]. The further step of deriving an exact (non-Markovian) master equation for the density operator $\hat{\varrho}$ of the particle is not at all straightforward and was not accomplished until quite some time later by Hu *et al.* [16]. Instead, what Caldeira and Leggett [12] were able to obtain was an approximate Markovian result in the limit of a high-temperature environment:

$$\frac{d\hat{\varrho}}{dt} = -\frac{i}{\hbar}\left[\frac{\hat{p}^2}{2M},\hat{\varrho}\right] - i\frac{\gamma}{2\hbar}[\hat{x},\{\hat{p},\hat{\varrho}\}] - \frac{D_{pp}}{\hbar^2}[\hat{x},[\hat{x},\hat{\varrho}]]. \quad (6)$$

Here, the particle-environment interaction is assumed to be "Ohmic"—i.e. such that the mean equations of motion of the particle reduce to the linear damping, classical result, Eq. (1), with the double-position commutator leading to the same momentum spread as given in the classical case. The mixed commutator-anticommutator term gives rise to the damping of the average energy of the particle and, hence, results in the Brownian particle coming into thermal equilibrium with its environment.

This Markovian master equation is significantly simpler than the exact result, but it is important to note that a Markovian "approximation" does not necessarily lead to a master equation of Lindblad form. The important situation in which the Markov approximation is satisfactory in this regard is quantum optics. There the interest is typically in systems with discrete energy levels (such as a two-level atom or a harmonic oscillator) that possess an oscillation frequency many orders of magnitude greater than other characteristic inverse time scales of the system induced by the (weak) coupling of the system to its environment. Under these conditions it is always possible to introduce an averaging procedure—the rotating-wave approximation—that leads to a Markovian master equation of Lindblad form. This is not the case with the Brownian motion of a free particle, however, as there is no characteristic frequency which means that the Markov approximation becomes a much more subtle affair. As pointed out by Stenholm [17], the Caldeira-Leggett master equation (6), though Markovian, is not of Lindblad form. This is physically unacceptable as it means that the density operator can be nonpositive during the early short-time evolution [17–19]. Moreover, deeper insight into the physics as would be provided by a measurement interpretation, e.g., via a quantum trajectory analysis of the dynamics of the Brownian particle is not possible.

As has been shown in [19,20], this master equation can be forced into Lindblad form by simply adding an extra double commutator term proportional to $[\hat{p},[\hat{p},\hat{\varrho}]]$ to the right-hand side of Eq. (6) to yield a "corrected" master equation

$$\frac{d\hat{\varrho}}{dt} = -\frac{i}{\hbar}\left[\frac{\hat{p}^2}{2M},\hat{\varrho}\right] - i\frac{\gamma}{2\hbar}[\hat{x},\{\hat{p},\hat{\varrho}\}] - \frac{D_{pp}}{\hbar^2}[\hat{x},[\hat{x},\hat{\varrho}]] - \frac{D_{xx}}{\hbar^2}[\hat{p},[\hat{p},\hat{\varrho}]]. \quad (7)$$

This will be of Lindblad form provided

$$D_{xx}D_{pp} \geqslant (\hbar\gamma/4)^2. \quad (8)$$

Using $D_{pp}=\gamma M k_B T$ leads to the requirement that

$$D_{xx} \geqslant \frac{\gamma\hbar^2}{16Mk_BT}. \tag{9}$$

In fact, Breuer and Petruccione [20] propose a "minimally invasive" correction by choosing the equal sign in Eq. (9). This *ad hoc* approach can be justified by use of a more refined version of the Markov approximation [21], yielding $D_{xx}=\gamma\hbar^2/12Mk_BT$—i.e., not the "minimally invasive value." The physical origin of the extra term, however, is not revealed by this analysis.

This extra term does not affect the mean equations of motion for position and momentum, but does lead to extra diffusion in position, which could be formally interpreted in the classical case as corresponding to Langevin equations of the form

$$M\dot{x} = p + G(t), \quad \dot{p} = -\gamma p + F(t). \tag{10}$$

There is now an additional noise term $G(t)$ with a correlation function

$$\overline{G(t)G(t')} = M^2 D_{xx}\delta(t-t'). \tag{11}$$

There is, of course, no classical physics justification for this noise source; its physical origins must be sought within the quantum features of the models that gives rise to the extra term.

Difficulties in obtaining a satisfactory Markovian master equation for quantum Brownian motion inspired another approach based on purely formal or phenomenological considerations. This approach, employed by Gallis [22] and Sandulescu and Scutaru [23], consists in choosing, *ad hoc*, suitable forms for the operators $\hat{L}_n$ in a Lindblad master equation for a single-particle system. This choice is constrained by the requirement that the mean equations of motion reduce to the usual classical equations of damped motion and that the evolution leads to the corresponding thermal equilibrium state. While this approach is not particularly revealing of the underlying physical processes, it nevertheless gives rise to classes of general forms that the master equation can have—the corrected master equation (7) being one. But it is obviously more satisfactory to be able to derive a master equation from a physical model. The Caldeira-Leggett is one such model, but quantum Brownian motion can be modeled in other ways. A physically reasonable alternative is to model more closely the underlying physical processes giving rise to Brownian motion—that is, the random collisions between the Brownian particle and the particles that make up its surrounding environment. The first study along these lines was that by Joos and Zeh [24], work which was later refined by Gallis and Fleming [25] and more recently by Hornberger and Sipe [26], all of whom obtained equations of the general forms predicted by the phenomenological approaches mentioned above [22,23]. The result is a Lindblad master equation describing the localization in space of the particle, though without damping of momentum. As a consequence the particle suffers thermal runaway [27]—its average energy undergoes a constant increase in time. This difficulty was overcome in the work of Diòsi [28], Dodd and Halliwell [29], and Vacchini [30] who extended the collisional model to include momentum damping and, in addition, deriving the extra position diffusion commutator term, along with more general expressions for the diffusion coefficients D_{xx} and D_{pp}.

The collisional approach brings into focus a different viewpoint on how to treat Brownian motion which avoids any specific model for the environment. If the Brownian particle in a state $\hat{\varrho}$ suffers a collision with an environmental particle initially in an (asymptotic) state $\hat{\varrho}_{\text{env}}$, uncorrelated with the state of the Brownian particle, then the collisional process can be understood as a mapping [31]

$$\hat{\varrho} \rightarrow \mathrm{Tr}_{\text{env}}[\hat{S}\hat{\varrho} \otimes \hat{\varrho}_{\text{env}}\hat{S}^\dagger] = \mathcal{S}\hat{\varrho}, \tag{12}$$

where the effect of the collision on the state of the Brownian particle is represented by the superoperator $\mathcal{S}$. The scattered environment particle will then carry away with it information on the position and momentum of the Brownian particle which at least in principle would be available for measurement by suitable detectors. By supposing that the collisions in effect perform a random sequence of measurements on the Brownian particle, it is possible to construct a Lindblad master equation for this particle. The dynamics of the particle, as represented by the master equation, then describes its response to the continuous intrusion of the measurement interaction.

Measurement-based master equations have been derived by Mensky and Stenholm [32], who used a restricted path integrals formalism which involved integrating over only those paths consistent with the measurement outcomes [33]. In other approaches models for position or joint position-momentum measurement apparatuses (not necessarily directly related to collisions) are introduced [34,35]. In the case of quantum Brownian motion, the information that could be gained by observations on the particle's environment would be its position and momentum. It is fundamental to quantum mechanics that both these quantities cannot be known with total precision at the same time. It is possible to imagine measurements of one or the other of these two quantities. However, it is not impossible to simultaneously measure position and momentum, but doing so necessarily introduces some degree of imprecision for both [36–39]. A quantum-limited measurement defines the best possible simultaneous measurement of both these quantities. If we are to understand the interaction of the particle with its environment as being responsible both for the damped dynamics of the particle and for feeding information on the position and momentum of the particle into the environment and if we are to represent the acquisition of this information by a suitably constructed Lindblad master equation, then we are immediately led to a particular form of this master equation. It is the derivation of this master equation and the analysis of some of its properties that constitute the main aims of this paper.

II. SIMULTANEOUS MEASUREMENT OF POSITION AND MOMENTUM

The relationship between measurements performed on the environment, the information gained on the state of the system, and the Lindblad structure of the master equation can be, in a sense, used to short-circuit the derivation of a master equation by avoiding a particular microscopic model for the particle plus environment. In applying this approach, the physical mechanism by which such measurements can be realized in practice—that is, the kind of system-environment interaction and the kind of measurement performed on the environment—is left unspecified though guidance is drawn from the kind of information that could, in principle, be extracted from the system of interest such as, for a Brownian particle, its position and/or momentum. Our approach to modeling friction is built on the idea that the viscous medium exerting the frictional force is, in effect, performing a generalized or "unsharp" simultaneous measurement of both the position and momentum of the particle [32,36–39]. Position and momentum are incompatible observables, represented by noncommuting operators, and it follows that such measurements are not of the conventional von Neumann type. The mathematical formalism for analyzing such measurements is the language of probability operator measures (POM's) [40] or positive operator-valued measures (POVM's) [38,41].

A. Quantum-limited measurements

We start with quantum-limited measurements, which may be described in terms of a POM with elements that are proportional to projectors onto Gaussian minimum uncertainty states [38,39]. The probability that a measurement of position and momentum yields results that lie between x and $x+dx$ for position and between p and $p+dp$ for momentum is

$$P_0(x,p)dxdp = \mathrm{Tr}[\hat{\pi}(x,p)\hat{\varrho}]dxdp, \tag{13}$$

where the Hermitean operators $\hat{\pi}(x,p)$ satisfy the conditions

$$\langle\psi|\hat{\pi}(x,p)|\psi\rangle \geqslant 0 \quad \forall\, |\psi\rangle, \tag{14}$$

$$\int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dp\, \hat{\pi}(x,p) = 1. \tag{15}$$

These correspond, respectively, to the physical requirements that the probability density be positive and normalized. Explicitly, for a quantum-limited measurement, we can write

$$\hat{\pi}(x,p) = \frac{1}{2\pi\hbar}|x,p\rangle\langle x,p|, \tag{16}$$

where $|x,p\rangle$ is a minimum uncertainty state fully characterized by the moments

$$\langle x,p|\hat{x}|x,p\rangle = x, \tag{17}$$

$$\langle x,p|\hat{p}|x,p\rangle = p, \tag{18}$$

$$\langle x,p|(\hat{x}-x)^2|x,p\rangle = \tfrac{1}{2}W^2, \tag{19}$$

$$\langle x,p|(\hat{p}-p)^2|x,p\rangle = \frac{\hbar^2}{2W^2}. \tag{20}$$

It is useful to note that we can write

$$|x,p\rangle = \hat{D}(x,p)|0,0\rangle, \tag{21}$$

where

$$\hat{D}(x,p) = \exp[i(p\hat{x} - \hat{p}x)/\hbar] \tag{22}$$

is the unitary Glauber displacement operator [43]. Hence we can write our POM elements as

$$\begin{aligned}\hat{\pi}(x,p) &= \hat{D}(x,p)\frac{1}{2\pi\hbar}|0,0\rangle\langle 0,0|\hat{D}^\dagger(x,p)\\ &= \hat{D}(x,p)\,\hat{\pi}(0,0)\hat{D}^\dagger(x,p),\end{aligned} \tag{23}$$

with

$$P_0(x,p) = \frac{1}{2\pi\hbar}\mathrm{Tr}[\hat{D}(x,p)|0,0\rangle\langle 0,0|\hat{D}^\dagger(x,p)\hat{\varrho}]. \tag{24}$$

Associated with the POM, in a way that is determined by the nature of the measurement process, we also have an effect [42]—that is, in general, pairs of operators $\hat{A}_\xi(x,p)$ and $\hat{A}_\xi^\dagger(x,p)$ that depend on a possibly continuous parameter ξ, in terms of which $\hat{\pi}$ can be expressed:

$$\hat{\pi}(x,p) = \sum_\xi \hat{A}_\xi^\dagger(x,p)\hat{A}_\xi(x,p), \tag{25}$$

such that the change in the state $\hat{\varrho}$ associated with the measurement outcome x,p is

$$\hat{\varrho} \to \sum_\xi \frac{\hat{A}_\xi(x,p)\hat{\varrho}\hat{A}_\xi^\dagger(x,p)}{\mathrm{Tr}[\hat{\varrho}\,\hat{\pi}(x,p)]}. \tag{26}$$

B. Imperfect measurements

Quantum-limited measurments, as presented above, represent an idealized limit. But as we cannot expect an observation of the surrounding environment to yield anything more than the most coarse-grained information concerning the position and momentum of the particle, a measurement-based quantum theory of friction requires *non-quantum-limited measurements*. We can construct a "smeared-out" probability distribution for the simultaneous measurement of position and momentum by convolving the ideal distribution $P_0(x,p)$ defined in Eq. (13) with a weighting factor $w(x,p)$ with the properties

$$w(x,p) \geqslant 0, \quad \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dp\, w(x,p) = 1, \tag{27}$$

to give a smeared-out probability distribution

$$P(x,p) = \int_{-\infty}^{+\infty} dx' \int_{-\infty}^{+\infty} dp'\, w(x',p')P_0(x+x',p+p'). \tag{28}$$

Using Eq. (24) this can be written as

$$P(x,p) = \frac{1}{2\pi\hbar}\int_{-\infty}^{+\infty} dx' \int_{-\infty}^{+\infty} dp' w(x',p') \mathrm{Tr}[\hat{D}(x+x',p+p') \times |0,0\rangle\langle 0,0|\hat{D}^\dagger(x+x',p+p')\hat{\varrho}] = \frac{1}{2\pi\hbar}\int_{-\infty}^{+\infty} dx' \int_{-\infty}^{+\infty} dp' w(x',p') \mathrm{Tr}[\hat{D}(x,p)|x',p'\rangle \times \langle x',p'|\hat{D}^\dagger(x,p)\hat{\varrho}] = \frac{1}{2\pi\hbar}\mathrm{Tr}[\hat{D}(x,p)\hat{\sigma}\hat{D}^\dagger(x,p)\hat{\varrho}], \tag{29}$$

where

$$\hat{\sigma} = \int_{-\infty}^{+\infty} dx' \int_{-\infty}^{+\infty} dp' w(x',p') |x',p'\rangle\langle x',p'|. \tag{30}$$

The weighting factor $w(x,p)$ is, by Eq. (27), a normalized probability distribution, so that expression [31] will define a density operator $\hat{\sigma}$. In fact, since the $|x,p\rangle$ form an overcomplete set of coherent states, Eq. (30) is just the Glauber-Sudarshan representation of $\hat{\sigma}$ [43].

Equation (29) defines a generalization of our POM elements to describe a less than quantum-limited measurement of position and momentum given by

$$\hat{\pi}(x,p) = \frac{1}{2\pi\hbar}\hat{D}(x,p)\hat{\sigma}\hat{D}^\dagger(x,p). \tag{31}$$

It follows that average of the measured results for x and p will be, for the particle in state $\hat{\varrho}$,

$$E(x) = \mathrm{Tr}[\hat{\varrho}\hat{x}] - \mathrm{Tr}[\hat{\sigma}\hat{x}], \tag{32}$$

$$E(p) = \mathrm{Tr}[\hat{\varrho}\hat{p}] - \mathrm{Tr}[\hat{\sigma}\hat{p}] \tag{33}$$

and their variances

$$\mathrm{Var}(x) = \Delta_\varrho x^2 + \Delta_\sigma x^2, \tag{34}$$

$$\mathrm{Var}(p) = \Delta_\varrho p^2 + \Delta_\sigma p^2, \tag{35}$$

where the subscripts correspond to the two density operators $\hat{\varrho}$ and $\hat{\sigma}$. If the measurements are to be unbiased, then the density operator $\hat{\sigma}$ should be such that $\mathrm{Tr}[\hat{\sigma}\hat{x}]=0$ and $\mathrm{Tr}[\hat{\sigma}\hat{p}]=0$. It is, however, the results for the variances that indicate how the smearing process leads us to a way in which less than ideal measurements can be represented. It is clear that the measured variances exceed their intrinsic values which would arise if either x or p were measured alone [in which case, $\mathrm{Var}(x)=\Delta_\varrho x^2$ and $\mathrm{Var}(p)=\Delta_\varrho p^2$]. For a quantum-limited measurement, $\hat{\sigma}$ would have to be a minimum uncertainty state, such as appears in Eq. (23), for which $\Delta_\sigma x \Delta_\sigma p = \frac{1}{2}\hbar$ [36,37]. Any other choice for $\hat{\sigma}$ would lead to larger measured variances corresponding to less accurate measurements. We would not expect the measurements associated with random interactions with an environment to be optimized, and this suggests that $\hat{\sigma}$ should correspond to a state with a large uncertainty product $\Delta_\sigma x \Delta_\sigma p \gg \frac{1}{2}\hbar$.

The simplest choice that can be made for $\hat{\sigma}$ is one for which $P(x,p)$ is completely characterized by the fewest number of parameters, which suggests choosing $w(x,p)$ to be a Gaussian:

$$w(x,p) = \frac{1}{2\pi\hbar\bar{n}}\exp\left[-\tfrac{1}{2}[(x/W)^2 + (Wp/\hbar)^2]/\bar{n}\right]; \tag{36}$$

i.e., we choose $\hat{\sigma}$ to be a Gaussian mixed state

$$\hat{\sigma} = \frac{\exp\left\{-\frac{1}{2}\lambda[(\hat{x}/W)^2 + (W\hat{p}/\hbar)^2]\right\}}{\mathrm{Tr}\left(\exp\left\{-\frac{1}{2}\lambda[(\hat{x}/W)^2 + (W\hat{p}/\hbar)^2]\right\}\right)} \tag{37}$$

where

$$\lambda = \ln(1 + 1/\bar{n}). \tag{38}$$

This state is characterized by the averages

$$\mathrm{Tr}[\hat{\sigma}\hat{x}] = 0 \text{ and } \mathrm{Tr}[\hat{\sigma}\hat{p}] = 0 \tag{39}$$

and the variances

$$\Delta_\sigma x^2 = \mathrm{Tr}[\hat{\sigma}\hat{x}^2] = W^2\left(\bar{n} + \frac{1}{2}\right), \tag{40}$$

$$\Delta_\sigma p^2 = \mathrm{Tr}[\hat{\sigma}\hat{p}^2] = \frac{\hbar^2}{W^2}\left(\bar{n} + \tfrac{1}{2}\right). \tag{41}$$

The variances above represent the increase of the variances in the measurements of position and momentum over and above the intrinsic variances $\Delta_\varrho x^2$ and $\Delta_\varrho p^2$. If $\bar{n}=0$, we regain the uncertainty-limited case, while for increasing $\bar{n}$, we have increasingly imprecise measurements.

Introducing the annihilation, creation, and number operators via

$$\hat{a} = (\hat{x}/W + iW\hat{p}/\hbar)/\sqrt{2}, \tag{42}$$

$$\hat{a}^\dagger = (\hat{x}/W - iW\hat{p}/\hbar)/\sqrt{2}, \tag{43}$$

$$\hat{n} = \hat{a}^\dagger\hat{a} = \tfrac{1}{2}[(\hat{x}/W)^2 + (W\hat{p}/\hbar)^2] - \tfrac{1}{2} \tag{44}$$

and the eigenstates $|n\rangle, n=0, 1, 2, \ldots$ of $\hat{n}$,

$$\hat{n}|n\rangle = n|n\rangle, \tag{45}$$

we then find that

$$\hat{\sigma} = (1 - e^{-\lambda})e^{-\lambda\hat{a}^\dagger\hat{a}} \tag{46}$$

$$= \frac{1}{1+\bar{n}}\sum_{n=0}^{\infty}\left(\frac{\bar{n}}{1+\bar{n}}\right)^n |n\rangle\langle n|. \tag{47}$$

Clearly $\bar{n}$ plays a role analogous to the mean photon number for a thermal single-mode state [43,44].

III. MASTER EQUATIONS FOR BROWNIAN MOTION

Using the results derived in the preceding section for a model of a less than quantum-limited simultaneous measurement of position and momentum, we will now construct and analyze a master equation for the Brownian motion and friction problem. The physical idea is that we model the friction in terms of collisions of the particle with background particles, atoms, and molecules, but we view these collisions as instantaneous simultaneous measurements of the position and momentum of the particle. The results of these measurements are presumably encoded on the state of the environment, but for our purposes it is enough simply for these collisions to have occurred. All that remains in order to write down our master equation is to construct suitable effects associated with our POM elements defined in Eq. (31). In doing so we are guided by our classical macroscopic understanding of Brownian motion.

A. Classical Brownian motion

The classical picture we have in mind is that of a particle of mass M whose free motion (in one dimension) is interrupted, at a rate R, by elastic collisions with particles of mass m. On the basis of this classical description, we can formulate a model of classical Brownian motion around which we will "wrap" the minimum of quantum mechanics to yield a quantum description of Brownian motion.

It is a simple exercise to show that a collision between the particle of interest with momentum p and an environmental particle of momentum ζ will result in a change in momentum of the Brownian motion particle given by

$$p \to \frac{M-m}{M+m}p + \frac{2M}{M+m}\zeta = Kp + \xi, \tag{48}$$

where we have introduced the parameter

$$K = \frac{M-m}{M+m} \tag{49}$$

so that

$$\xi = (1+K)\zeta. \tag{50}$$

As M would normally be greater that m, we have $0<K<1$. In fact, it would normally be the case that $M \gg m$ so that $K \sim 1-2m/M$. We should note that the collision also changes the momentum of the environment particle. If we were to monitor this particle, then this change, together with the position at which it occurred, would provide instantaneous information on the position and momentum of our Brownian particle. It is this feature that will lead us to describe collisions, in the quantum theory, as simultaneous measurements of position and momentum.

The momentum ζ will be a quantity which will vary randomly from one collision to the next—in fact, it will be assumed that ζ and hence ξ for different collisions will be uncorrelated. This is equivalent to assuming that a particle, once it has been scattered from the Brownian particle, does not interact with it again—i.e., there is no memory, and the process is Markovian.

In this classical picture, the Brownian particle will therefore undergo a random walk in which the free motion of the particle is interrupted at a rate R by collisions which result in a random change in momentum as described above, but with no instantaneous change in the position of the particle. Since we are assuming that these collisions are independent events, we can immediately derive a useful result in the steady state that can be compared with the quantum result derived later. Consider a sequence of collisions occurring at a rate R such that

$$\begin{aligned} p_1 &= Kp_0 + \xi_1, \\ p_2 &= Kp_1 + \xi_2 = K^2 p_0 + \xi_2 + K\xi_1, \\ &\vdots \\ p_N &= K^N p_0 + \sum_{n=1}^{N} K^{n-1}\xi_{N-n}. \end{aligned} \tag{51}$$

The assumption that the collisions are independent ($\overline{\xi_n \xi_m} = \overline{\xi^2}\delta_{nm}$) then leads to

$$\overline{p_N^2} = K^{2N}\overline{p_0^2} + \overline{\xi^2}\sum_{n=1}^{N} K^{2N(n-1)} = K^{2N}\overline{p_0^2} + \overline{\xi^2}\frac{1-K^{2N}}{1-K^2}. \tag{52}$$

Hence

$$\overline{p_\infty^2} = \frac{\overline{\xi^2}}{1-K^2}, \tag{53}$$

from which follows, using Eq. (50), that

$$\frac{\overline{p_\infty^2}}{2M} = \frac{\overline{\zeta^2}}{2m}; \tag{54}$$

i.e., the statistical independence of collisions is sufficient to guarantee that the Brownian particle and the environmental particles come to a common equilibrium mean kinetic energy. For the environment in thermal equilibrium, we expect at sufficiently high temperature T that the principle of equipartition will hold true—i.e.,

$$\overline{p_\infty^2} = Mk_B T, \tag{55}$$

from which follows

$$\overline{\zeta^2} = mk_B T, \tag{56}$$

where we have explicitly recognized that we are considering a one-dimensional system.

In the following, it is assumed that the environment is a gas of Boltzmann particles in thermal equilibrium at some temperature T so that ζ will be a Gaussian random quantity of mean zero and standard deviation $\overline{\zeta^2} = mk_B T$. The probability distribution of ξ will then be

$$\mathcal{P}(\xi) = \frac{e^{-\xi^2/2\overline{\xi^2}}}{\sqrt{2\pi\overline{\xi^2}}}; \tag{57}$$

i.e., ξ is of mean zero $\overline{\xi}=0$ of standard deviation

$$\overline{\xi^2} = (1-K^2)Mk_B T. \tag{58}$$

It is possible to construct a classical master equation for the joint position-momentum probability distribution for this jump process. If $P_{\mathrm{cl}}(x,p,t)$ is this probability distribution at time t, then at a later time $t+\Delta t$, the new distribution will be the sum of two terms, one associated with no collision occurring in the time interval $(t,t+\Delta t)$, the other associated with a collision occurring and the momentum of the particle undergoing a jump from p to $Kp+\xi$. The former term is weighted by the probability $1-R\Delta t$ for no collision to occur, the latter summed over all the possible values of ξ, and weighted by the probability $R\Delta t$ of a collision to occur. The result is

$$P_{\mathrm{cl}}(x,p,t+\Delta t) = (1-R\Delta t)(1+\Delta t\mathcal{H})P_{\mathrm{cl}}(x,p,t) + R\Delta t\int_{-\infty}^{+\infty}\mathcal{P}(p-Kp')P_{\mathrm{cl}}(x,p',t)dp', \tag{59}$$

where

$$\mathcal{H} = -\frac{p}{M}\frac{\partial}{\partial x} \tag{60}$$

is the generator of free evolution between collisions. In the limit of $\Delta t\to 0$, Eq. (59) becomes

$$\frac{\partial P_{\mathrm{cl}}}{\partial t} = -\frac{p}{M}\frac{\partial P_{\mathrm{cl}}}{\partial x} - RP_{\mathrm{cl}}(x,p,t) + R\int_{-\infty}^{+\infty}\mathcal{P}(p-Kp')P_{\mathrm{cl}}(x,p',t)dp'. \tag{61}$$

Of particular interest is the case of frequent $(R\to\infty)$ and weak $(K\to 1)$ collisions, the limit being taken in such a way that $R(1-K)=\gamma$, a constant. We find that Eq. (61) reduces to

$$\frac{\partial P_{\mathrm{cl}}}{\partial t} = -\frac{p}{M}\frac{\partial P_{\mathrm{cl}}}{\partial x} + \gamma\frac{\partial}{\partial p}(pP_{\mathrm{cl}}) + \gamma Mk_BT\frac{\partial^2 P_{\mathrm{cl}}}{\partial p^2}, \tag{62}$$

which is just Kramer's equation for Brownian motion [3]. A rigorous mathematical analysis of the simple model presented above can be found in Holley [46], which has been further generalized to three dimensions in Dürr *et al.* [45].

B. Quantum Brownian motion

In order to write down a quantum master equation, we are guided by the macroscopic picture just presented to construct suitable operators associated with the POM elements, Eq. (31). These operators are responsible for inducing a change in the state of the particle consequent on a collision occurring as well as being associated with an observed joint position-momentum result. This is accomplished by decomposing the POM $\hat{\pi}(x,p)$ as

$$\hat{\pi}(x,p) = \int_{-\infty}^{+\infty} d\xi\hat{A}_{\xi}^{\dagger}(x,p)\hat{A}_{\xi}(x,p), \tag{63}$$

so that the change in the state of the particle as a consequence of a collision is given by [compare Eq. (12)]

$$\hat{\varrho} \to \hat{A}_{\xi}(x,p)\hat{\varrho}\hat{A}_{\xi}^{\dagger}(x,p). \tag{64}$$

The operators $\hat{A}_{\xi}(x,p)$ are tailored so as to reflect the classical picture outlined above—i.e., that a collision results in no change in the instantaneous position of the particle, but the momentum changes by $p\to Kp+\xi$. This is the sense in which we are wrapping the minimum amount of quantum mechanics about the classical process. Thus we define the operators

$$\hat{A}_{\xi}(x,p) = \sqrt{\frac{\mathcal{P}(\xi)}{2\pi\hbar}}\hat{D}(x,Kp+\xi)\hat{\sigma}^{1/2}\hat{D}^{\dagger}(x,p), \tag{65}$$

which can be readily shown to satisfy Eq. (63). We see that these operators describe the effects of an imprecise joint measurement of position and momentum, with the particle undergoing no change in position as a consequence, but the momentum changing as in the classical case discussed above. This parallels the classical situation discussed in Sec. III A, and the master equation for the density operator $\hat{\varrho}$ of the particle can be derived in a way analogous to the derivation of Eq. (61).

If we suppose that the particle is moving in the presence of a potential $V(\hat{x})$ and suffers collisions at a rate R, then in a time interval $(t,t+\Delta t)$ the density operator $\hat{\varrho}$ of the particle will undergo unitary evolution, with probability $1-R\Delta t$, under the action of the Hamiltonian

$$\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{x}), \tag{66}$$

or with probability $R\Delta t$ undergo a change in state, Eq. (64), consequent on suffering a collision. Combining these two possibilities together, in a way completely analogous to the derivation of the classical result, Eq. (61), and summing over all the possible collisions that can occur gives, for the evolution of $\hat{\varrho}$,

$$\hat{\varrho}(t+\Delta t) = (1-R\Delta t)\left[\hat{\varrho} - \frac{i}{\hbar}[\hat{H},\hat{\varrho}]\Delta t\right] + R\Delta t\int_{-\infty}^{+\infty}d\xi\int_{-\infty}^{+\infty}dx\int_{-\infty}^{+\infty}dp\hat{A}_{\xi}(x,p)\hat{\varrho}\hat{A}_{\xi}^{\dagger}(x,p). \tag{67}$$

Taking the limit $\Delta t\to 0$ gives the result

$$\frac{d\hat{\varrho}}{dt} = -\frac{i}{\hbar}[\hat{H},\hat{\varrho}] - R\hat{\varrho} + R\int_{-\infty}^{+\infty}d\xi\int_{-\infty}^{+\infty}dx\int_{-\infty}^{+\infty}dp\hat{A}_{\xi}(x,p)\hat{\varrho}\hat{A}_{\xi}^{\dagger}(x,p). \tag{68}$$

This is our Markovian quantum theory of friction and is our main result. We devote the remainder of this paper to analyzing the physical content of this equation.

IV. PROPERTIES OF THE MASTER EQUATION

From the master equation (68) it is possible to derive equations for the moments of various important quantities in terms of which various limiting cases can be discussed. While it is a reasonably straightforward procedure to derive these equations, the calculations themselves are quite lengthy and we limit ourselves to giving equations of motions for the lowest-order moments:

$$\frac{d}{dt}\langle\hat{x}\rangle = \frac{\langle\hat{p}\rangle}{M}, \tag{69}$$

$$\frac{d}{dt}\langle\hat{p}\rangle = -\langle V'(\hat{x})\rangle - R(1-K)\langle\hat{p}\rangle, \tag{70}$$

$$\frac{d}{dt}\langle\hat{x}^2\rangle = \frac{1}{M}\langle\{\hat{x},\hat{p}\}\rangle + 2RW^2\Big[\bar{n} + \tfrac{1}{2} - \sqrt{\bar{n}(\bar{n}+1)}\,\Big], \tag{71}$$

$$\frac{d}{dt}\langle\hat{p}^2\rangle = -\langle\hat{p}V'(\hat{x}) + V'(\hat{x})\hat{p}\rangle - R(1-K^2)\langle\hat{p}^2\rangle + R\overline{\xi^2} + R\frac{\hbar^2}{W^2}\Big[(1+K^2)\big(\bar{n}+\tfrac{1}{2}\big) - 2K\sqrt{\bar{n}(\bar{n}+1)}\,\Big], \tag{72}$$

$$\frac{d}{dt}\langle\{\hat{x},\hat{p}\}\rangle = \frac{2}{M}\langle\hat{p}^2\rangle - 2\langle\hat{x}V'(\hat{x})\rangle - R(1-K)\langle\{\hat{x},\hat{p}\}\rangle. \tag{73}$$

Hereafter we will consider the original and simplest problem: that of a free particle subject to random fluctuations, which corresponds to setting $V(\hat{x})=0$.

A. Steady-state limit

For a free particle, the equations of motion for the mean position and momentum are immediately solvable:

$$\langle\hat{p}(t)\rangle = e^{-R(1-K)t}\langle\hat{p}(0)\rangle, \tag{74}$$

$$\langle\hat{x}(t)\rangle = \langle\hat{x}(0)\rangle + \frac{\langle\hat{p}(0)\rangle}{MR(1-K)}\big(1 - e^{-R(1-K)t}\big). \tag{75}$$

We see that the mean momentum is exponentially damped to zero at a rate given by the product $R(1-K)$ and that the long-time mean displacement is simply the initial velocity divided by $R(1-K)$. The remaining equilibrium properties of the particle are determined by taking the long-time limit of the remaining equations:

$$\langle\hat{p}^2\rangle_\infty = \frac{\overline{\xi^2}}{1-K^2} + \frac{\hbar^2\Big[(1+K^2)\big(\bar{n}+\tfrac{1}{2}\big) - 2K\sqrt{\bar{n}(\bar{n}+1)}\,\Big]}{W^2(1-K^2)}, \tag{76}$$

$$\langle\{\hat{x},\hat{p}\}\rangle_\infty = \frac{2}{R(1-K)M}\langle\hat{p}^2\rangle_\infty, \tag{77}$$

and

$$\frac{d}{dt}\langle\hat{x}^2\rangle\bigg|_\infty = \frac{2}{R(1-K)M^2}\langle\hat{p}^2\rangle_\infty + 2RW^2\Big[\bar{n} + \tfrac{1}{2} - \sqrt{\bar{n}(\bar{n}+1)}\Big]. \tag{78}$$

In the high-temperature limit, the term proportional to $\overline{\xi^2}$ will dominate, so that

$$\langle\hat{p}^2\rangle_\infty = \frac{\overline{\xi^2}}{1-K^2} = Mk_BT, \tag{79}$$

where Eq. (58) has been used. This result is in agreement with the classical result, Eq. (53); i.e., the equilibrium kinetic energy of the particle is given by

$$\frac{\langle\hat{p}^2\rangle_\infty}{2M} = \tfrac{1}{2}k_BT. \tag{80}$$

In addition, in the high-temperature limit, we have

$$\frac{d}{dt}\langle\hat{x}^2\rangle\bigg|_\infty = \frac{2k_BT}{R(1-K)M} = 2D, \tag{81}$$

where D is the diffusion coefficient. From Einstein's theory, D is given by [47]

$$D = \frac{k_BT}{6\pi\mu a}, \tag{82}$$

where μ is the viscosity of the medium and a is the radius of the particle. Comparison with the theory developed here then gives

$$MR(1-K) = 6\pi\mu a. \tag{83}$$

This is entirely reasonable as may be seen from Stokes's formula for the viscous force [48]:

$$F = -6\pi\mu a v, \tag{84}$$

where v is the velocity of the particle. As collisions occur at a rate R, resulting in a loss of momentum $(1-K)p$, the corresponding average force is

$$F = -R(1-K)p. \tag{85}$$

If we write $p=Mv$, then we find that the equality between the Eqs. (84) and (85) implies agreement with Eq. (83). This result is discussed further in the Conclusion (Sec. V).

B. Imprecise measurements

It is useful to consider these moment equations in the limit of very imprecise measurements—that is, when $\bar{n}\gg 1$, with the other parameters R and K held fixed. In this case, we find that Eqs. (71) and (72), for $\langle\hat{x}^2\rangle$ and $\langle\hat{p}^2\rangle$, respectively, become

$$\frac{d}{dt}\langle\hat{x}^2\rangle = \frac{1}{M}\langle\{\hat{x},\hat{p}\}\rangle + \frac{RW^2}{4\bar{n}}, \tag{86}$$

$$\frac{d}{dt}\langle\hat{p}^2\rangle = -R(1-K^2)\langle\hat{p}^2\rangle + R\overline{\xi^2} + \frac{\hbar^2}{W^2}R(1-K)^2\bar{n}. \tag{87}$$

We note that in this case, for a given R and K, the dispersion in the momentum results is dominated by the $\bar{n}$ contribution. It can also be noted that these results suggest that the limit of continuous observation, that is, $R\to\infty$, will exist provided that $K\to 1$—i.e., the fractional momentum lost per collision becomes vanishingly small, but such that $R(1-K)$ remains constant and such that $R/\bar{n}$ also remains constant—i.e., that the joint measurement of position and momentum becomes very imprecise. It is the limiting form of the master equation in this case that we will now consider.

C. Continuous measurement limit

The continuous measurement limit $R\to\infty$ requires both $K\to 1$ such that $R(1-K)=\gamma$ a constant and $R/\bar{n}$=constant, for a finite result to be obtained. In this limit, the moment equations become

$$\frac{d\langle\hat{x}\rangle}{dt} = \frac{\langle\hat{p}\rangle}{M}, \tag{88}$$

$$\frac{d\langle\hat{p}\rangle}{dt} = -\gamma\langle\hat{p}\rangle, \tag{89}$$

$$\frac{d\langle\hat{x}^2\rangle}{dt} = \frac{1}{M}\langle\{\hat{x},\hat{p}\}\rangle + \frac{RW^2}{4\bar{n}}, \tag{90}$$

$$\frac{d\langle\hat{p}^2\rangle}{dt} = -2\gamma\langle\hat{p}^2\rangle + 2\gamma M k_B T + \frac{\hbar^2}{W^2}\left[\frac{R}{4\bar{n}} + \frac{\gamma^2\bar{n}}{R}\right], \tag{91}$$

$$\frac{d\langle\{\hat{x},\hat{p}\}\rangle}{dt} = \frac{2}{M}\langle\hat{p}^2\rangle - \gamma\langle\{\hat{x},\hat{p}\}\rangle. \tag{92}$$

From this, the long-time limits of interest can be readily derived. First, we have the equilibrium mean kinetic energy of the Brownian particle:

$$\frac{\langle\hat{p}^2\rangle_\infty}{2M} = \tfrac{1}{2}k_B T + \frac{\hbar^2}{4\gamma M W^2}\left[\frac{R}{4\bar{n}} + \frac{\gamma^2\bar{n}}{R}\right]. \tag{93}$$

In order to estimate the magnitude of the quantum corrections occurring here relative to the usual classical equipartition result more detailed knowledge is required of the measurement-dependent parameters R and $\bar{n}$. This is also true of the rate at which the position of the particle diffuses:

$$\left.\frac{d\langle\hat{x}^2\rangle}{dt}\right|_\infty = \frac{2k_B T}{\gamma M} + \left[\frac{\hbar^2}{2\gamma^2 M^2 W^2} + W^2\right]\frac{R}{4\bar{n}} + \frac{\hbar^2}{2M^2W^2}\frac{\bar{n}}{R}, \tag{94}$$

where, once again, there are present additional terms that are dependent on the measurement process.

Finally we turn to the task of determining the master equation for the particle in the limit of continuous measurement. In order to carry carry this out, it is useful to rewrite the master equation (68) in another form in which the integrals over ξ, x, and p are carried out. In terms of superoperators $\mathcal{A}$ and $\mathcal{C}$ defined by

$$\mathcal{A}_p\hat{\chi} = \{\hat{p},\hat{\chi}\},$$

$$\mathcal{C}_x\hat{\chi} = [\hat{x},\hat{\chi}],$$

$$\mathcal{C}_p\hat{\chi} = [\hat{p},\hat{\chi}], \tag{95}$$

a very long calculation yields the following result:

$$\begin{aligned}\frac{d\hat{\varrho}}{dt} = &-\frac{i}{\hbar}\left[\frac{\hat{p}^2}{2M},\hat{\varrho}\right] - R\hat{\varrho} + R\exp\Big[-\tfrac{1}{2}\{(\mathcal{C}_x/W)^2\\ &- i\mathcal{C}_x\mathcal{A}_p/\hbar\}\ln K\Big]\exp[-\{\overline{\xi^2} + W_p^2\}\mathcal{C}_x^2/2\hbar^2K^2]\\ &\times\exp[-W_x^2\mathcal{C}_p^2/2\hbar^2]\hat{\varrho},\end{aligned} \tag{96}$$

where

$$W_x = W(\sqrt{\bar{n}+1} - \sqrt{\bar{n}}), \quad W_p = \frac{\hbar}{W}(\sqrt{\bar{n}+1} - K\sqrt{\bar{n}}). \tag{97}$$

We can use this form to explore the continuous measurement limit of $R\to\infty$. This limit will only exist if each of the three exponents is small. On expanding each exponential to first order, we find that the $-R\hat{\varrho}$ term is canceled to give

$$\begin{aligned}\frac{d\hat{\varrho}}{dt} = &-\frac{i}{\hbar}\left[\frac{\hat{p}^2}{2M},\hat{\varrho}\right] - \Big[\tfrac{1}{2}\{(\mathcal{C}_x/W)^2 - i\mathcal{C}_x\mathcal{A}_p/\hbar\}R\ln K + R\{\overline{\xi^2}\\ &+ W_p^2\}\mathcal{C}_x^2/2\hbar^2K^2 + RW_x^2\mathcal{C}_p^2/2\hbar^2\Big]\varrho,\end{aligned} \tag{98}$$

which requires $R\ln K, R\{\overline{\xi^2}+W_p^2\}/K^2$, and RW_x^2 all to be finite in the limit. This in turn requires $K\to 1$ such that $R(1-K), R\overline{\xi^2}/K^2, RW_p^2/K^2$, and RW_x^2 remain finite. Setting $R(1-K)=\gamma$, we then have

$$R\ln K \to -R(1-K) = -\gamma, \tag{99}$$

$$R\overline{\xi^2}/K^2 = \frac{R(1-K^2)}{K^2}Mk_BT \to 2\gamma M k_B T. \tag{100}$$

The third and fourth terms require $\bar{n}\sim R$, from which follows

$$\begin{aligned}RW_p^2/K^2 &= \hbar^2R(\sqrt{\bar{n}+1} - K\sqrt{\bar{n}})^2/K^2W^2\\ &\sim \hbar^2(\gamma^2\bar{n}/R + \gamma + R/4\bar{n})/2W^2,\end{aligned} \tag{101}$$

$$RW_x^2 = RW^2(\sqrt{\bar{n}+1} - \sqrt{\bar{n}})^2 \sim RW^2/4\bar{n}, \tag{102}$$

so that overall we find

$$\begin{aligned}\frac{d\hat{\varrho}}{dt} = &-\frac{i}{\hbar}\left[\frac{\hat{p}^2}{2M},\hat{\varrho}\right] - i\frac{\gamma}{2\hbar}[\hat{x},\{\hat{p},\hat{\varrho}\}] - \left\{\frac{\gamma M k_B T}{\hbar^2} + \frac{\gamma^2}{2W^2}\frac{\bar{n}}{R}\right.\\ &\left.+ \frac{1}{8W^2}\frac{R}{\bar{n}}\right\}[\hat{x},[\hat{x},\hat{\varrho}]] - \frac{W^2}{8\hbar^2}\frac{R}{\bar{n}}[\hat{p},[\hat{p},\hat{\varrho}]].\end{aligned} \tag{103}$$

This equation can be readily seen to be of Lindblad form with the coefficients D_{xx} and D_{pp} satisfying the inequality Eq. (8), though the inequality is not satisfied in the minimal sense.

It is worthwhile comparing this result with those obtained by [20,21,28,32]. We first note the appearance of a temperature-dependent coefficient in the momentum diffusion term, as found in the result, Eq. (7). This term leads to the correct equilibrium mean energy of the particle at high temperatures. But note that there is also a further contribution to momentum diffusion that has its origins in the measurements imposed on the particle, as well as a position diffusion also due to the measurements which, in terminology analogous to that used by Breuer and Petruccione [20], is the "minimum" required to guarantee a Lindblad form. We can note, finally, that in the absence of any damping (γ=0) this master equation reduces to that of Scott and Milburn [35] who obtained their result through an explicit analysis of the results of a particular model of joint measurement of position and momentum of a particle which, however, was not subject to thermal influences.

D. Physical interpretation

Our master equation (103) has a simple and appealing physical interpretation that emerges if we express the coefficients in terms of the variances $\Delta_\sigma x^2$ and $\Delta_\sigma p^2$ introduced in Sec. II B. If we recall that the master equation is valid in the limit $\bar{n} \gg 1$, then using Eqs. (40) and (41) leads us to rewrite Eq. (103) as

$$\frac{d\hat{\varrho}}{dt} = -\frac{i}{\hbar}\left[\frac{\hat{p}^2}{2M},\hat{\varrho}\right] - i\frac{\gamma}{2\hbar}[\hat{x},\{\hat{p},\hat{\varrho}\}] - \left\{\frac{\gamma}{\hbar^2}\left(Mk_BT + \frac{m}{M}\Delta_\sigma p^2\right) + \frac{R}{8\Delta_\sigma x^2}\right\}[\hat{x},[\hat{x},\hat{\varrho}]] - \frac{R}{8\Delta_\sigma p^2}[\hat{p},[\hat{p},\hat{\varrho}]], \tag{104}$$

where m is the mass of the colliding molecule (see Sec. III A). The first two terms are clearly associated with the free evolution of the particle and the damping of the momentum, respectively. The two double-commutator terms $[\hat{x},[\hat{x},\hat{\varrho}]]$ and $[\hat{p},[\hat{p},\hat{\varrho}]]$ are associated with and lead to diffusion of the momentum and position, respectively. The corresponding diffusion coefficients are

$$D_{pp} = \gamma\left(Mk_BT + \frac{m}{M}\Delta_\sigma p^2\right) + \frac{R\hbar^2}{8\Delta_\sigma x^2}, \tag{105}$$

$$D_{xx} = \frac{R\hbar^2}{8\Delta_\sigma p^2}, \tag{106}$$

which clearly satisfy the required inequality (8).

The variances $\Delta_\sigma x^2$ and $\Delta_\sigma p^2$ correspond to the uncertainties associated with the effective simultaneous measurement of position and momentum, over and above the particle's intrinsic variances. In the limit of very weak measurements, which we are considering, these account for essentially all of the imprecision in the measurement. It is clear that the term $R\hbar^2/8\Delta_\sigma x^2$ accounts for the diffusion in momentum associated with localizing the position to within $\Delta_\sigma x$. The term $R\hbar^2/8\Delta_\sigma p^2$ similarly accounts for diffusion in position induced by the momentum measurements.

We recognize the term Mk_BT as the high-temperature or classical value of the mean-squared momentum, and we can interpret $m\Delta_\sigma p^2/M$ as a low-temperature or quantum "zero-point energy" correction to this. To confirm this, we return to the simple description of a collision given in Sec. III A. In the collision the change in the momentum of the Brownian particle (48) is accompanied by a change in ζ, the momentum of the environmental particle:

$$\zeta \to \frac{m-M}{M+m}\zeta + \frac{2m}{M+m}p \approx -\zeta + \frac{2m}{M}p, \tag{107}$$

where we have used the fact that $M \gg m$. The net change in the momentum of the environment particle is

$$\delta\zeta \approx -2\zeta + \frac{2m}{M}p, \tag{108}$$

and it is this change that can be measured to provide our indirect measurement of the momentum of the Brownian particle. Momentum conservation holds both classically and in the quantum theory. Hence our ability to resolve differences in the momentum of our Brownian particle will be set by the intrinsic quantum or zero-point momentum fluctuations in ζ. Hence we have that

$$\Delta_\sigma p^2 = \left(\frac{M}{m}\right)^2\langle\hat{\zeta}^2\rangle_0, \tag{109}$$

where $\langle\hat{\zeta}^2\rangle_0$ is the zero-point mean-square momentum of an environment particle. It follows that the term $m\Delta_\sigma p^2/M$ is just what would be expected from the requirements of equipartition as

$$\frac{m}{M}\Delta_\sigma p^2 = \frac{M}{m}\langle\hat{\zeta}^2\rangle_0. \tag{110}$$

This means that the term $m\Delta_\sigma p^2/M$ produces a steady-state kinetic energy equal to the zero-point kinetic energy of an environment particle.

V. CONCLUSION

We have presented an analysis of quantum Brownian motion from a measurement theory perspective, built around a classical picture of Brownian motion as resulting from random collisions between the Brownian particle and particles in its surrounding environment. This classical picture yields Kramer's equation for the joint position-momentum distribution for the Brownian particle. In passing to the quantum version, the aim was to overcome issues, well known in various other models of quantum Brownian motion, related to obtaining a Markovian master equation of the correct Lindblad form. To this end, the theory was constructed, from the outset, to be of Lindblad form, by treating the collisions as generalized joint measurements of the position and momentum of the Brownian particle. In the limit of continuous measurement, we derived a master equation that gives damping of momentum, shows the presence of both momentum and position diffusion, has the correct equilibrium properties at high temperatures, and exhibits the presence of zero-point fluctuations of the bath on the Brownian particle.

The model also yields information on the observed paths of the Brownian particle. As the master equation is of Lindblad form, quantum trajectory methods can be used to generate the observed path of the Brownian particle. A complete description of the statistics of the measured path could also be obtained by deriving an equation analogous to the classical Kramer's equation (62) for the joint probability for the observed position and momentum of the particle. Indeed, the question of whether or not Kramer's equation can be derived in an appropriate continuous measurement limit needs to be investigated. Consideration of the measurement record is also important in another regard: the comparison made in Sec. IV A of the steady-state diffusion result obtained here with that of Einstein is based on using the quantum average of $\langle\hat{x}^2\rangle$ whereas it would be more logically consistent to make the comparison with the average over the *measured* values of x^2. The expectation, nevertheless, is that in the high-temperature limit, there ought not be any difference.

Further analysis of this model would involve investigating the temperature dependence of the various parameters with the aim in mind of obtaining a clear physical picture of, in particular, the $1/T$ dependence for the position diffusion term that arises through the "minimally invasive" procedure.

While inspired by the classical model introduced in Sec. III A, the quantum model developed here is one derived solely by taking a measurement point of view. Master equations for open systems can also be derived from a microscopic model which, provided the result is a Markovian master equation of Lindblad form, leads directly to a measurement interpretation. There are thus two perspectives here: the first one in which the gathering of information about a system under observation has primacy, the second in which the physical processes by which information is delivered to the environment is shown to arise via the details of the microscopic interaction between the system and its environment. The two perspectives ought to contain the same physics; i.e., the measurement master equation should be derivable from a microscopic model. A means by which this can be done, based on a detailed analysis of a quantum version of the classical model of Sec. III A, is under development.

In all the instances mentioned above the particle, apart from its interaction with its environment, is free. A further natural generalization therefore is to study the effects of any external potential, such as a harmonic oscillator potential, acting on the particle. Reults of this study are to be given elsewhere.

ACKNOWLEDGMENTS

We are grateful to John Jeffers for many helpful and informative discussions. This work was supported by the UK Engineering and Physical Sciences Research Council and by an Australian DEST IAP grant to support participation in the European Fifth Framework project QUPRODIS.

[1] J. Donne, *Devotions upon Emergent Occasions* Meditation XVII (1693).

[2] A. Einstein, Ann. d. Physik **17**, 549 (1905); M. von Smoluchowski, Ann. d. Physik **21**, 756 (1906).

[3] C. W. Gardiner, *Handbook of Stochastic Methods* (Springer-Verlag, Berlin, 1985).

[4] R. Zwanzig, J. Stat. Phys. **9**, 215 (1973).

[5] E. Joos, H. D. Zeh, C. Kiefer, D. Giulini, J. Kupsch, and I.-O. Stamatescu, *Decoherence and the Appearance of a Classical World in Quantum Theory*, 2nd ed. (Springer-Verlag, Berlin, 2003).

[6] V. Gorini, A. Kossakowski, and E. C. G. Sudarshan, J. Math. Phys. **17**, 821 (1976).

[7] G. Lindblad, Commun. Math. Phys. **48**, 119 (1976).

[8] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, England, 2000).

[9] H. J. Carmichael, *An Open Systems Approach to Quantum Optics*, Lecture Notes in Physics, New Series m: Monographs, Vol. m18 (Springer, Berlin, 1993).

[10] J. Dalibard, Y. Castin, and K. Mølmer, Phys. Rev. Lett. **68**, 580 (1992).

[11] M. B. Plenio and P. L. Knight, Rev. Mod. Phys. **70**, 101 (1995).

[12] A. O. Caldeira and A. J. Leggett, Physica A **121**, 587 (1983).

[13] G. S. Agarwal, Phys. Rev. A **4**, 739 (1971).

[14] W. G. Unruh and W. H. Zurek, Phys. Rev. D **40**, 1071 (1989).

[15] R. P. Feynman and F. L. Vernon, Jr., Ann. Phys. (N.Y.) **24**, 118 (1963).

[16] B. L. Hu, J. P. Paz, and Y. Zhang, Phys. Rev. D **45**, 2843 (1992).

[17] S. Stenholm, in *Quantum Dynamics of Simple Systems*, edited by G. L. Oppo *et al.* (Scottish Universities Summer Schools in Physics and Institute of Physics, Bristol, 1996), p. 304.

[18] V. Ambegaokar, Ber. Bunsenges. Phys. Chem. **95**, 400 (1991).

[19] W. J. Munro and C. W. Gardiner, Phys. Rev. A **53**, 2633 (1996).

[20] H.-P. Breuer and F. Petruccione, *The Theory of Open Quantum Systems* (Oxford University Press, Oxford, 2002), p. 179.

[21] L. Diòsi, Physica A **199**, 517 (1993).

[22] M. R. Gallis, Phys. Rev. A **48**, 1028 (1993).

[23] A. Sandulescu and H. Scutaru, Ann. Phys. (N.Y.) **173**, 277 (1987).

[24] E. Joos and H. D. Zeh, Z. Phys. B: Condens. Matter **59**, 223 (1985).

[25] M. R. Gallis and G. N. Fleming, Phys. Rev. A **42**, 38 (1990).

[26] K. Hornberger and J. E. Sipe, Phys. Rev. A **68**, 012105 (2003).

[27] L. E. Ballentine, Phys. Rev. A **43**, 9 (1991).

[28] L. Diòsi, Europhys. Lett. **30**, 63 (1995).

[29] P. J. Dodd and J. J. Halliwell, Phys. Rev. D **67**, 105018 (2003).

[30] B. Vacchini, Phys. Rev. Lett. **84**, 1374 (2000).

[31] S. Stenholm, Phys. Scr. **47**, 724 (1993).

[32] M. B. Mensky and S. Stenholm, Phys. Lett. A **308**, 243 (2003).

[33] M. B. Mensky, *Quantum Measurements and Decoherence* (Kluwer Academic, Dordrecht, 2000).

[34] C. M. Caves and G. J. Milburn, Phys. Rev. A **36**, 5543 (1987).

[35] A. J. Scott and G. J. Milburn, Phys. Rev. A **63**, 042101 (2001).

[36] E. Arthurs and J. L. Kelly, Jr., Bell Syst. Tech. J. **44**, 725 (1965).

[37] S. Stenholm, Ann. Phys. (N.Y.) **218**, 233 (1992).

[38] P. Busch, M. Grabowski, and P. Lahti, *Operational Quantum Physics* (Springer-Verlag, Berlin, 1995).

[39] N. G. Walker, J. Mod. Opt. **34**, 15 (1987).

[40] C. W. Helstrom, *Quantum Detection and Estimation Theory* (Academic, New York, 1975).

[41] A. Peres, *Quantum Theory: Concepts and Methods* (Kluwer Academic, Dordrecht, 1993).

[42] K. Kraus, *States, Effects and Operations* (Springer-Verlag, Berlin, 1983).

[43] S. M. Barnett and P. M. Radmore, *Methods in Theoretical Quantum Optics* (Oxford University Press, Oxford, 1997).

[44] There is no suggestion, of course, of a physical field mode or harmonic oscillator, but this representation allows us to exploit

the well-known harmonic-oscillator algebra to perform calculations.
[45] D. Dürr, S. Goldstein, and J. L. Lebowitz, Commun. Math. Phys. **78**, 507 (1981).
[46] R. Holley, Z. Wahrscheinlichkeitstheor. Verwandte Geb. **17**, 181 (1971).
[47] A. Einstein, *Investigations on the Theory of Brownian Movement* (Dover, New York, 1956).
[48] L. D. Landau and E. M. Lifschitz, *Fluid Mechanics*, 2nd ed. (Butterworth-Heinemann, Oxford, 1987), p. 61.