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ON THE NUMBER OF SIGN CHANGES OF HECKE EIGENVALUES OF NEWFORMS

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Abstract

We show that, for every x exceeding some explicit bound depending only on k and N, there are at least $C(k, N)x/\log^{17} x$ positive and negative coefficients $a(n)$ with $n \le x$ in the Fourier expansion of any non-zero cuspidal Hecke eigenform of even integral weight $k \ge 2$ and squarefree level N that is a newform, where $C(k, N)$ depends only on k and N. From this we deduce the existence of a sign change in a short interval.

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1. Introduction

Let f be a non-zero cusp form of even integral weight $k \ge 2$ and level N with real Fourier coefficients $a(n)$, $n \in \mathbb{N}$. We refer to [11] for basic definitions. It is well known that there are infinitely many $n \in \mathbb{N}$ such that $a(n) > 0$ as well as infinitely many n with $a(n) < 0$. For an extension of this result and a discussion of related questions, see [8] (compare also [2] in connection with binary theta functions).

If $N = 1$ and $k \equiv 2 \pmod 4$, then a result of Siegel [12] implies that the first sign change of $a(n)$ already occurs among the first $d(k) + 1$ coefficients, where $d(k)$ is the dimension of the space of cusp forms in question (see also [3]). On the other hand, if $N = 1$ and $k \equiv 0 \pmod 4$ or if $N > 1$, the method of Siegel [12] does not apply and thus a different approach, based on analytic number theory estimates, has been developed by Kohnen and Sengupta [9], which in turn is related to some ideas of Murty [10].

More precisely, let f be a fixed newform of weight k on the Hecke congruence subgroup

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbf{Z}) \middle| c \equiv 0 \pmod N \right\},$$

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which is a normalized Hecke eigenform. In particular, its Fourier coefficients $a(n)$, $n \in \mathbb{N}$, are the Hecke eigenvalues of f and $a(1) = 1$. Note that the $a(n)$ are real.

We assume throughout that N is squarefree.

As in [9], we note that it is quite reasonable to assume that $\gcd(n, N) = 1$ since the p-eigenvalues of f for $p|N$ are explicitly known.

In the following the implied constants in the symbols $\ll$ are always absolute and efficiently computable.

It is shown in [9] that for any $\varepsilon > 0$ there exist $n \in \mathbb{N}$ with $\gcd(n, N) = 1$ and such that

$$n \ll kN \exp\left(c\sqrt{\frac{\log N}{\log\log(N+2)}}\right)\log^{26+\varepsilon} k, \tag{1}$$

for which $a(n) < 0$, where c is an absolute constant and the implied constant depends only on ε. This bound has recently been improved by Iwaniec, Kohnen and Sengupta [7].

Here we show that the technique of [9] can in fact give a lower bound on the number of sign changes in a given interval $n \in [1, x]$. On the other hand, the approach of [7], which led to an improvement of (1), does not seem to apply immediately to the derivation of a lower bound on the number of sign changes.

To formulate our result, we introduce the divisor sums

$$\sigma_\alpha(N) = \sum_{d|N} d^\alpha.$$

Let $S_f^+(x)$ and $S_f^-(x)$ denote the number of positive integers $n \le x$ with $\gcd(n, N) = 1$ for which $a(n) > 0$ and $a(n) < 0$, respectively.

Theorem 1. *We have*

$$S_f^\pm(x) \gg \frac{x}{\sigma_{-1}(N)^4 \log^4(kN) \log^{17} x}$$

whenever $x \ge X(k, N)$, where

$$X(k, N) = Ck \max\{N\sigma_{-1}(N)^4\sigma_{-1/2}(N)^2 \log^8(kN),\ N^{1/2}\sigma_{-1}(N)^6 \log^{22}(kN)\},$$

for some absolute constant $C > 0$.

We also show that Theorem 1, coupled with a recent result of Alkan and Zaharescu [1], allows us to study sign changes in short intervals.

Theorem 2. *There are absolute constants $\eta < 1$ and $A > 0$ such that, for $y = x^\eta$,*

$$S_f^\pm(x + y) - S_f^\pm(x) > 0$$

whenever $x \ge (kN)^A$.

Let $T_f(x)$ denote the number of sign changes in the sequence $a(n)$ taken for consecutive positive integers $n \le x$ with $\gcd(n, N) = 1$, that is,

$$T_f(x) = \#\{n \le x \mid \operatorname{sign}(a(n)) \ne \operatorname{sign}(a(n+1)),\ \gcd(n, N) = 1\},$$

where, as usual,

$$\operatorname{sign}(a) = \begin{cases} -1 & \text{if } a < 0, \\ 0 & \text{if } a = 0, \\ 1 & \text{if } a > 0. \end{cases}$$

Splitting the interval $[1, x]$ into $x^{1-\eta}$ intervals of length $y = x^\eta$, we derive from Theorem 2 the following result.

COROLLARY 3. *There are absolute constants $\kappa > 0$ and $A > 0$ such that*

$$T_f(x) > x^\kappa$$

whenever $x \ge (kN)^A$.

2. Preparations

2.1. The idea of the proof We define the 'normalized' Hecke eigenvalues $\lambda(n)$ of f by the relation

$$a(n) = \lambda(n) n^{(k-1)/2}, \qquad n \in \mathbb{N}.$$

We now consider the sums

$$\vartheta_\nu(x) = \sum_{\substack{n \le x \\ \gcd(n,N)=1}} |\lambda(n)|^\nu \log^2(x/n) \quad \text{and} \quad \rho_\nu(x) = \sum_{\substack{n \le x \\ \gcd(n,N)=1}} \lambda(n)^\nu \log^2(x/n),$$

which we use only for $\nu = 1, 2, 3$.

By the Cauchy–Schwarz inequality,

$$\vartheta_2(x) \le \sqrt{\vartheta_1(x)\vartheta_3(x)}. \tag{2}$$

The proof of Theorem 1 is based on the observation that, if either $S_f^+(x)$ or $S_f^-(x)$ is small, then the sums $\vartheta_1(x)$ are close to the sum $|\rho_1(x)|$. But the known lower bound on $\vartheta_2(x)$ and the known upper bounds on $\rho_1(x)$ and $\vartheta_3(x)$ contradict (2).

The proof of Theorem 2 is based on the observation that Theorem 1 implies that, for any $\varepsilon > 0$ and a sufficiently large X, there are m and n with $X \le m < n \le X^{1+\varepsilon}$ which are close to each other and also satisfy

$$\gcd(mn, N) = 1, \qquad \lambda(m)\lambda(n) < 0.$$

After this selection of s with $\gcd(s, mnN) = 1$ in an appropriate interval (depending on m and n) and such that $\lambda(s) \ne 0$, the existence of which is implied by a result of [1], we can make sure that both sm and sn belong to the desired short interval and we also have

$$\lambda(sm)\lambda(sn) = \lambda(s)^2 \lambda(m)\lambda(n) < 0.$$

2.2. Some elementary bounds We need some elementary number theoretic estimates.

Recalling that N is squarefree we immediately obtain the following results.

LEMMA 4. *We have*

$$\prod_{p\mid N}(1+p^{-1}) = \sigma_{-1}(N).$$

LEMMA 5. *We have*

$$\prod_{p\mid N}(1-p^{-1/2}) \gg \frac{1}{\sigma_{-1}(N)\sigma_{-1/2}(N)}.$$

PROOF. Using the identity

$$\begin{aligned}\prod_{p\mid N}(1-p^{-1/2}) &= \prod_{p\mid N}(1-p^{-1})\prod_{p\mid N}(1+p^{-1/2})^{-1}\\ &= \prod_{p\mid N}(1-p^{-1})\sigma_{-1/2}(N)^{-1}\\ &= \prod_{p\mid N}(1-p^{-2})\prod_{p\mid N}(1+p^{-1})^{-1}\sigma_{-1/2}(N)^{-1}\\ &= \prod_{p\mid N}(1-p^{-2})\sigma_{-1}(N)^{-1}\sigma_{-1/2}(N)^{-1}\end{aligned}$$

yields the desired result. □

Let $\tau(n) = \sigma_0(n)$ be the number of positive integer divisors of n. We need the following well-known bounds (see [4, 6]).

LEMMA 6. *For any $z \geq 1$, we have*

$$\sum_{n\le z}\tau(n)^2 \ll z\log^3 z \quad \textit{and} \quad \sum_{n\le z}\tau(n)^3 \ll z\log^7 z.$$

2.3. Some bounds for sums $\vartheta_\nu(x)$ and $\rho_\nu(x)$ The following estimate is a combination of [9, Proposition 6] with a result of Goldfield, Hoffstein and Lieman [5] (which has also been used in [9]) as well as Lemmas 4 and 5.

LEMMA 7. *There are absolute constants $c_1, c_2 > 0$ such that the bound*

$$\vartheta_2(x) \geq \frac{c_1}{\sigma_{-1}(N)\log(kN)}x - c_2(kN)^{1/2}\log^3(kN)\sigma_{-1}(N)\sigma_{-1/2}(N)x^{1/2}$$

holds for every $x \geq 1$.

Using Lemma 4 instead of [9, Lemma 4] we can reformulate [9, Proposition 8] as the following.

LEMMA 8. *The bound*

$$\rho_1(x) \ll k^{1/2} N^{1/4} \log^2(kN)\sigma_{-1}(N)x^{1/2}$$

holds for every $x \geq 1$.

Finally, we need the following estimate.

LEMMA 9. *We have*

$$\vartheta_3(x) \ll x \log^7 x$$

for every $x \geq 1$.

PROOF. As in [9], we use the Deligne bound

$$|\lambda(n)| \leq \tau(n). \tag{3}$$

Now, by Lemma 6

$$\begin{aligned}\vartheta_3(x) &= \sum_{n\leq x} \tau(n)^3 \log^2(x/n) \ll \sum_{1\leq i\leq \log x+1} i^2 \sum_{x/e^i\leq n\leq x/e^{i-1}} \tau(n)^3 \\ &\ll \sum_{1\leq i\leq \log x+1} i^2 \sum_{n\leq x/e^{i-1}} \tau(n)^3 \ll x \log^7 x \sum_{1\leq i\leq \log x+1} i^2 e^{-i} \ll x \log^7 x,\end{aligned}$$

which finishes the proof. □

3. Proofs

3.1. Proof of Theorem 1 We note that there is an absolute constant $C_1 > 0$ such that, if we put

$$X_1(k, N) = C_1 kN\sigma_{-1}(N)^4\sigma_{-1/2}(N)^2 \log^8(kN),$$

then Lemma 7 implies that the bound

$$\vartheta_2(x) \gg \frac{x}{\sigma_{-1}(N) \log(kN)} \tag{4}$$

holds for $x \geq X_1(k, N)$. Using (4) together with Lemma 9 and (2) we see that

$$\vartheta_1(x) \gg \frac{x}{\sigma_{-1}(N)^2 \log^2(kN) \log^7 x} \tag{5}$$

for $x \geq X_1(k, N)$. Let

$$A_f^+(x) = \sum_{\substack{n\leq x,\\ \gcd(n,N)=1\\ \lambda(n)>0}} \lambda(n) \log^2(x/n),$$

$$A_f^-(x) = -\sum_{\substack{n\leq x,\\ \gcd(n,N)=1\\ \lambda(n)<0}} \lambda(n) \log^2(x/n).$$

Then by Lemma 8,

$$A_f^+(x) - A_f^-(x) = \rho_1(x) \ll k^{1/2} N^{1/4} \log^2(kN)\sigma_{-1}(N)x^{1/2}. \tag{6}$$

From (5), one has

$$A_f^+(x) + A_f^-(x) = \vartheta_1(x) \gg \frac{x}{\sigma_{-1}(N)^2 \log^2(kN) \log^7 x}. \tag{7}$$

We see that (6) and (7) imply that

$$\min\{A_f^+(x), A_f^-(x)\} \gg \frac{x}{\sigma_{-1}(N)^2 \log^2(kN) \log^7 x}$$

for $x \geq X_2(k, N)$, where

$$X_2(k, N) = C_2 k N^{1/2} \sigma_{-1}(N)^6 \log^{22}(kN),$$

and C_2 is large enough.

By (3) and the Cauchy inequality

$$(A_f^+(x))^2 \leq S_f^+(x) \sum_{n \leq x} \tau^2(n) \log^4(x/n). \tag{8}$$

Using Lemma 6 and applying the same argument as in Lemma 9, we derive

$$\sum_{n \leq x} \tau^2(n) \log^4(x/n) \ll x \log^3 x,$$

which implies the desired bound for $S_f^+(x)$. The case of $S_f^-(x)$ is fully analogous.

3.2. Proof of Theorem 2 Note that, as is well known, f cannot have complex multiplication since by our assumption N is squarefree. Therefore, by [1, Theorem 1], there are some absolute positive constants α and β such that, for a sufficiently large real Z and any integer $M \geq 1$ with $M \leq Z^\beta$, there exists $s \in [Z, Z + Z^\alpha]$ with $\lambda(s) \neq 0$ and $s \equiv 1 \pmod{M}$.

Define

$$X = (x^\beta/N)^{1/(4+2\beta)}.$$

By Theorem 1, for $x \geq (kN)^A$ with a sufficiently large A (such that $X \geq X(k, N)$), there are m and n with $X \leq m < n < X^2$ and also with

$$\gcd(mn, N) = 1, \quad \lambda(m)\lambda(n) < 0.$$

From [1, Theorem 1] we conclude that we can assume that

$$n \leq m + X^\gamma.$$

For some $\gamma < 1$ (provided x is large enough).

We now put $Z = x/m$ and $M = mnN$. One immediately verifies that $M \le Z^{\beta}$ for the above choice of X. Thus, by [1], we can find $s \in [Z, Z + Z^{\alpha}]$ with $\lambda(s) \neq 0$ and $s \equiv 1 \pmod{M}$. In particular, since $\gcd(s, nmN) = 1$ then, as we have noted before,

$$\lambda(sm)\lambda(sn) = \lambda(s)^2\lambda(m)\lambda(n) < 0.$$

We also have

$$\begin{aligned} x \le sm < sn \le (Z + Z^{\alpha})(m + X^{\gamma}) &= x + ZX^{\gamma} + (m + X^{\gamma})Z^{\alpha} \\ &\le x + m^{\gamma}Z + 2mZ^{\alpha} \end{aligned}$$

(since $m \ge X$) and, after simple calculations, the result follows.

4. Remarks

Using the 'individual' bounds

$$\sigma_{-1}(N) \ll \log\log(N+2), \qquad \sigma_{-1/2}(N) \ll \exp\left(\frac{\sqrt{\log N}}{\log\log(N+2)}\right),$$

as well as the bounds 'on average'

$$\frac{1}{M}\sum_{N \le M} \sigma_{-1}(N) \ll \frac{1}{M}\sum_{N \le M} \sigma_{-1/2}(N) \ll 1,$$

which can easily be derived from prime number theory using standard methods of estimating multiplicative functions (see [4, 6]), one can obtain more simplified forms of Theorem 1.

Finally we note that it would be very interesting to obtain an explicit value for the constant η in the bound of Theorem 2.

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