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TOPOLOGICAL FUNCTORS AS TOTAL CATEGORIES

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ABSTRACT. A notion of central importance in categorical topology is that of topological functor. A faithful functor $\mathcal{E} \to \mathcal{B}$ is called topological if it admits cartesian liftings of all (possibly large) families of arrows; the basic example is the forgetful functor **Top** $\to$ **Set**. A topological functor $\mathcal{E} \to 1$ is the same thing as a (large) complete preorder, and the general topological functor $\mathcal{E} \to \mathcal{B}$ is intuitively thought of as a "complete preorder relative to $\mathcal{B}$". We make this intuition precise by considering an enrichment base $\mathcal{Q}_{\mathcal{B}}$ such that $\mathcal{Q}_{\mathcal{B}}$-enriched categories are faithful functors into $\mathcal{B}$, and show that, in this context, a faithful functor is topological if and only if it is total (=totally cocomplete) in the sense of Street–Walters. We also consider the MacNeille completion of a faithful functor to a topological one, first described by Herrlich, and show that it may be obtained as an instance of Isbell's generalised notion of MacNeille completion for enriched categories.

1. Introduction

One of the more inconvenient facts in mathematics is that of the relatively bad behaviour of the category **Top** of topological spaces: though complete, cocomplete and extensive, it is not regular, coherent, locally presentable, or (locally) cartesian closed. Many authors have thus been led to propose replacing the category of spaces by some other category which either embeds **Top** or embeds into **Top** in a reasonable manner, but which possesses some of the desirable properties that **Top** itself lacks; some examples are the categories of quasitopological spaces, approach spaces, convergence spaces, uniformity spaces, nearness spaces, filter spaces, epitopological spaces, Kelley spaces, compact Hausdorff spaces, Δ-generated spaces, or of sheaves on some small subcategory of **Top**.

In attempting to impose some kind of order on this proliferation of notions, a useful organising framework is that of *categorical topology* [7]; a key insight of which is that categories of space-like structures are most fruitfully studied not as categories *simpliciter*, but as categories equipped with a faithful functor $\mathcal{A} \to$ **Set**. Desirable properties of a category of space-like structures can be re-expressed as properties of this functor, and the process of replacing **Top** by a category with such properties then becomes that of adjoining those desirable properties to the usual forgetful functor **Top** $\to$ **Set**, or some subfunctor thereof; see [16] for an overview.

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In general, categorical topology studies faithful functors not just into **Set**, but into an arbitrary base category $\mathcal{B}$. When $\mathcal{B} = 1$, such functors are simply preorders; this motivates the step of regarding a general faithful functor $p\colon \mathcal{E} \to \mathcal{B}$ as a "preorder relative to $\mathcal{B}$", and many aspects of categorical topology can be seen as as elucidation of this idea. Of particular importance are the "complete preorders relative to $\mathcal{B}$", the so-called *topological functors* [11, 1, 19]. A faithful functor is topological if it admits a generalisation of the characterising property of a Grothendieck fibration in which one may form cartesian liftings not just of single arrows, but of arbitrary (possibly large) families of them.

The intuition that topological functors are relativised complete preorders can be seen in many places throughout the literature: for example, in the various completion processes by which a faithful functor may be turned into a topological one [8], which correspond to the constructions by which a poset may be turned into a complete lattice, and indeed reduce to these constructions when $\mathcal{B} = 1$. However, nowhere is this basic intuition wholly justified. The objective of this paper is to rectify this by way of enriched category theory: given a category $\mathcal{B}$, we describe an enrichment base for which the enriched categories are faithful functors into $\mathcal{B}$, and the enriched categories which are *total* (= totally cocomplete)—in a sense to be recalled below—are the topological functors into $\mathcal{B}$.

It is worth saying a few words about the kind of enrichment base we will require. Most enrichments, as in [13], involve a monoidal category $\mathcal{V}$, with a $\mathcal{V}$-category then having homs which are objects of $\mathcal{V}$. For example, when $\mathcal{V}$ is the monoidal poset $(\mathbf{2}, \wedge, \top)$, $\mathcal{V}$-categories are precisely preorders, and so we may regard order theory as a particular kind of enriched category theory. In the early 1980's, Walters [26, 27] realised the value of a more general kind of enrichment (first suggested by Bénabou [3]) based on a bicategory $\mathcal{W}$; in a $\mathcal{W}$-category, each object is typed by an object of $\mathcal{W}$, while the homs are appropriately-typed *morphisms* of $\mathcal{W}$. This kind of enrichment includes the more familiar kind on regarding a monoidal category $\mathcal{V}$ as a one-object bicategory.

Walters' application for this notion was to sheaf theory: for each site, he describes a bicategory $\mathcal{W}$ such that Cauchy-complete, symmetric, skeletal $\mathcal{W}$-categories are precisely sheaves on that site. More generally, posets internal to sheaves correspond to Cauchy-complete skeletal $\mathcal{W}$-categories[1] and so we may regard order theory internal to a topos as another kind of enriched category theory. Both the monoidal poset $(\mathbf{2}, \wedge, \top)$ (seen as a one-object bicategory) and Walters' bases for enrichment are instances of a general class of well-behaved bicategories, the *quantaloids* [20], whose enriched category theory [24, 25] behaves like non-standard order theory. A quantaloid is a bicategory whose homs are complete posets, and whose composition preserves joins in each variable. The enrichment bases which give rise to faithful functors over a base $\mathcal{B}$ are also quantaloids, and this justifies our viewing their theory as non-standard order theory: our main result can then be seen as saying that, for this non-standard order theory, the complete preorders are the topological functors into $\mathcal{B}$.

Beyond proving our basic equivalence, we give an application to *MacNeille completions.*

[1] Although the general Cauchy-complete $\mathcal{W}$-category corresponds not to a preorder internal to sheaves, but to a stack of preorders.

The classical MacNeille completion [17] of a poset P is the smallest complete poset into which P embeds, and may be constructed by Dedekind cuts: its elements are pairs (L, U) of subsets of P wherein L is the set of all lower bounds of U, and U the set of all upper bounds of L. In [8] is described a more general "MacNeille completion", which turns a faithful functor into $\mathcal{B}$ to a topological one, and includes the classical MacNeille completion as the special case $\mathcal{B} = 1$. We unify these constructions by showing that they are an instance of the general notion of "MacNeille completion" for enriched categories first described by Isbell [12].

2. Topological functors

Since questions of size are relevant in what follows, let us first make clear our conventions. A set will be called *small* if it lies within some fixed Grothendieck universe κ, and *large* otherwise. All categories in this paper will be assumed to have small hom-sets and a (possibly large) set of objects[2].

We begin by recalling the definition of topological functor [1, 19].

2.1. Definition. Let $p\colon \mathcal{E} \to \mathcal{B}$ be a faithful functor, and I a (possibly large) set. Given objects $(x_i \in \mathcal{E})_{i \in I}$ and morphisms $(g_i\colon px_i \to z \in \mathcal{B})_{i \in I}$, a *final lifting* is an object $\bar{z} \in \mathcal{E}$ with $p\bar{z} = z$ such that, for any $\theta \in \mathcal{B}(z, pe)$,

$$z \xrightarrow{\theta} pe \text{ lifts to a map } \bar{z} \to e \text{ iff each } px_i \xrightarrow{\theta . g_i} pe \text{ lifts to a map } x_i \to e. \qquad (*)$$

The functor p is called *topological* if it admits all final liftings.

2.2. Examples. The basic example of a topological functor is the forgetful functor $\mathbf{Top} \to \mathbf{Set}$: given spaces $(X_i \mid i \in I)$ and functions $(g_i\colon UX_i \to Z)$, we obtain a final lifting by equipping Z with the topology in which $V \in \mathcal{O}(Z)$ just when $g_i^{-1}(V) \in \mathcal{O}(X_i)$ for each i. Similarly, the categories of quasitopological spaces, limit spaces, filter spaces, subsequential spaces and so on, all admit topological forgetful functors to $\mathbf{Set}$; see [6]. Other interesting examples of categories topological over $\mathbf{Set}$ include the category of bornological spaces; the category $\mathcal{F}$ of sets equipped with a filter of subsets [5]; and the categories of diffeological and Chen spaces [2].

An easy way of obtaining topological functors whose codomain is not $\mathbf{Set}$ is using the result that, if $\mathbb{T}$ is a small category with finite limits, and $p\colon \mathcal{E} \to \mathcal{B}$ is topological, then so too is $\mathbf{Lex}(\mathbb{T}, \mathcal{E}) \to \mathbf{Lex}(\mathbb{T}, \mathcal{B})$. So, for example, the forgetful functors from topological groups or topological vector spaces to groups or vector spaces are topological. Another class of examples to bear in mind are those with $\mathcal{B} = 1$; as in the introduction, a faithful functor into 1 is just a (large) preorder, and such a functor is topological just when the preorder admits all joins.

[2]Our conventions deviate here from those commonly used in the categorical topology literature, where a small category is defined as one with a set of objects, and a large category may have a proper class of them. This avoids problems such as the failure of the collection of subclasses of a fixed class to form a class.

2.3. Remark. Dually, an *initial lifting* of a family $(g_i \colon z \to px_i \in \mathcal{B})_{i \in I}$ is a final lifting with respect to p^{op}; and clearly, a functor p admits all initial liftings just when p^{op} is topological. One might be tempted to call such a p *optopological*, but it turns out that this is unnecessary: a functor is topological if and only if its opposite is so. This is the *topological duality theorem* [1, 19, 10]; in the case $\mathcal{B} = 1$, it reduces to the result that a preorder admits all meets if and only if it admits all joins. We return to this point in Section 6 below.

2.4. Remark. The definition of a topological functor is sometimes taken to include extra side-conditions. One is that it should be *amnestic*—meaning that the fibres are posets, not preorders. This requirement is inessential for the basic theory. Another common side-condition is that the fibres should be small categories; again, this is unnecessary for the basic theory.

3. Faithful functors as enriched categories

Towards our characterisation of topological functors, we now describe how faithful functors into a fixed base category $\mathcal{B}$ may be seen as categories enriched in an associated quantaloid. As in the introduction, a *quantaloid* [20] is a bicategory $\mathcal{Q}$ whose hom-categories $\mathcal{Q}(A, B)$ are complete (small) lattices, and whose whiskering functors $(–) \circ f \colon \mathcal{Q}(B, C) \to \mathcal{Q}(A, C)$ and $g \circ (–) \colon \mathcal{Q}(A, B) \to \mathcal{Q}(A, C)$ preserve all joins. It follows that these whiskering functors have right adjoints, denoted by

$$[f, –] \colon \mathcal{Q}(A, C) \to \mathcal{Q}(B, C) \qquad \text{and} \qquad \{g, –\} \colon \mathcal{Q}(A, C) \to \mathcal{Q}(A, B)$$

respectively. Existence of these right adjoints is the bicategorical counterpart to a monoidal category's being left and right closed, and ensures that there is a workable theory of enriched categories over such a base.

3.1. Examples. A one-object quantaloid is a (unital) *quantale* $(Q, \&, 1)$ in the sense of [18]: a complete lattice Q with an associative unital multiplication $\& \colon Q \times Q \to Q$ that preserves joins in each variable. In particular, any locale L yields a quantale $(L, \wedge, \top)$. Another important example, due to Lawvere [15], is the quantale $(\mathbb{R}_+, +, 0)$ of non-negative real numbers with the reverse of the usual ordering.

Quantaloids with more than one object arise, amongst other places, in the work of Walters [27, 26]: given a topological space (or locale) X, he considers the quantaloid $\mathcal{Q}_X$ whose objects are open sets of X, and for which $\mathcal{Q}_X(U, V)$ is the complete lattice of open sets of X contained in $U \cap V$. Analogously, if $(\mathcal{C}, J)$ is a (standard) site, then there is a quantaloid $\mathcal{Q}_\mathcal{C}$ whose objects are those of $\mathcal{C}$, and for which $\mathcal{Q}_\mathcal{C}(X, Y)$ is the complete lattice of subsheaves of $\mathcal{C}(–, X \times Y)$. See [9] for an abstract characterisation of such quantaloids.

The notions of category, functor and transformation enriched in a quantaloid are particular cases of the bicategorical ones of [22], though rather easier to state in this special case due to the partially ordered homs, which ensure that all 2-cell axioms are automatically satisfied. For more on quantaloid-enriched category theory, see [24, 25].

3.2. Definition. If $\mathcal{Q}$ is a quantaloid, a *$\mathcal{Q}$-enriched category*, or *$\mathcal{Q}$-category* $\mathcal{C}$, is given by:

- A (possibly large) set of objects $\mathrm{ob}\,\mathcal{C}$;
- For each $x \in \mathrm{ob}\,\mathcal{C}$, an object $|x| \in \mathcal{Q}$, called the *extent* of x; and
- For all $x, y \in \mathrm{ob}\,\mathcal{C}$, a hom-object $\mathcal{C}(x, y) \in \mathcal{Q}(|x|, |y|)$;

all such that

- For all $x \in \mathrm{ob}\,\mathcal{C}$, we have $1_{|x|} \leqslant \mathcal{C}(x, x)$ in $\mathcal{Q}(|x|, |x|)$;
- For all $x, y, z \in \mathrm{ob}\,\mathcal{C}$, we have $\mathcal{C}(y, z) \circ \mathcal{C}(x, y) \leqslant \mathcal{C}(x, z)$ in $\mathcal{Q}(|x|, |z|)$.

If $\mathcal{C}$ and $\mathcal{D}$ are $\mathcal{Q}$-categories, a *$\mathcal{Q}$-functor* $F \colon \mathcal{C} \to \mathcal{D}$ is an extent-preserving function $F\colon \mathrm{ob}\,\mathcal{C} \to \mathrm{ob}\,\mathcal{D}$ such that $\mathcal{C}(x, y) \leqslant \mathcal{D}(Fx, Fy)$ in $\mathcal{Q}(|x|, |y|)$ for all $x, y \in \mathrm{ob}\,\mathcal{C}$. Between $\mathcal{Q}$-functors $F, G \colon \mathcal{C} \to \mathcal{D}$ there exists at most one $\mathcal{Q}$-transformation, which exists just when $1_{|x|} \leqslant \mathcal{D}(Fx, Gx)$ for all $x \in \mathrm{ob}\,\mathcal{C}$, and will then be notated as $F \leqslant G$. We write $\mathcal{Q}$-$\mathbf{CAT}$ for the locally posetal 2-category of (possibly large) $\mathcal{Q}$-categories.

3.3. Example. When $\mathcal{Q}$ is a quantale seen as a one-object quantaloid, $\mathcal{Q}$-enriched categories are categories enriched in the quantale seen as a monoidal poset. For example, categories enriched in the one-object quantaloid $(\mathbf{2}, \wedge, \top)$ are preordered sets; whilst Lawvere showed in [15] that categories enriched in the one-object quantaloid $(\mathbb{R}_+, +, 0)$ are (generalised) metric spaces. For the quantaloid $\mathcal{Q}_X$ associated to a topological space X, Cauchy-complete skeletal $\mathcal{Q}_X$-enriched categories are, as in the introduction, posets internal to $\mathbf{Sh}(X)$, and correspondingly for the quantaloid associated to a site $(\mathcal{C}, J)$.

We now describe the quantaloid-enrichments that will be relevant in this paper.

3.4. Definition. [20] The *free quantaloid* $\mathcal{Q}_\mathcal{B}$ on an ordinary category $\mathcal{B}$ has the same objects as $\mathcal{B}$; morphisms $U \colon X \twoheadrightarrow Y$ in $\mathcal{Q}_\mathcal{B}$ are subsets $U \subseteq \mathcal{B}(X, Y)$, ordered by inclusion; the composition of $U \colon X \twoheadrightarrow Y$ with $V \colon Y \twoheadrightarrow Z$ is given by $V \circ U = \{v \circ u \mid v \in V, u \in U\} \subseteq \mathcal{B}(X, Z)$; while the identity map at X is $\{1_X\} \colon X \twoheadrightarrow X$. The right adjoints to whiskering are constructed as follows for each $U \colon X \twoheadrightarrow Y$, $V \colon Y \twoheadrightarrow Z$ and $W \colon X \twoheadrightarrow Z$ in $\mathcal{Q}_\mathcal{B}$:

$$\begin{aligned} [U, W] &= \{v \in \mathcal{B}(Y, Z) \mid vu \in W \text{ for all } u \in U\} \\ \text{and} \quad \{V, W\} &= \{u \in \mathcal{B}(X, Y) \mid vu \in W \text{ for all } v \in V\} \,. \end{aligned}$$

Given an ordinary category $\mathcal{B}$, consider now what it is to give a category enriched in the free quantaloid $\mathcal{Q}_\mathcal{B}$. We have a set $\mathrm{ob}\,\mathcal{E}$ of objects; for each $x \in \mathrm{ob}\,\mathcal{E}$ an object $|x| \in \mathrm{ob}\,\mathcal{B}$; and for each $x, y \in \mathrm{ob}\,\mathcal{E}$ a subset $\mathcal{E}(x, y) \subseteq \mathcal{B}(|x|, |y|)$, in such a way that $1_{|x|} \in \mathcal{E}(x, x)$ and whenever $f \in \mathcal{E}(x, y)$ and $g \in \mathcal{E}(y, z)$ we have also $g \circ f \in \mathcal{E}(x, z)$. These data are clearly those of an ordinary category $\mathcal{E}$ together with a faithful functor $p \colon \mathcal{E} \to \mathcal{B}$ sending x to $|x|$. Similar calculations with respect to functors and transformations now show that:

3.5. PROPOSITION. *If $\mathcal{Q}_\mathcal{B}$ is the free quantaloid on the ordinary category $\mathcal{B}$, then the 2-category $\mathcal{Q}_\mathcal{B}$-$\mathbf{CAT}$ is 2-equivalent to the full sub-2-category of the strict slice $\mathbf{CAT}/\mathcal{B}$ on the faithful functors.*

4. Totality for quantaloid-enriched categories

In this section, we discuss totality in the context of quantaloid-enriched categories. The notion of *totality* of a category was introduced in [21, 23] in an abstract context broad enough to encompass ordinary categories but also enriched and internal ones. An ordinary (possibly large) category $\mathcal{C}$ is called *total* if its Yoneda embedding $\mathcal{C} \to [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ admits a left adjoint[3]. Totality implies cocompleteness, the existence of all small colimits, but also the existence of certain large ones, which can be used, for example, to avoid solution-set conditions in the adjoint functor theorem; see [21, Theorem 18]. Most categories arising in mathematical practice are total: for instance, any locally presentable category, in particular, any Grothendieck topos; any category monadic over $\mathbf{Set}$; any category admitting a topological functor to $\mathbf{Set}$; and so on.

As explained in [14], defining totality in the enriched context requires some delicacy: the naïve definition—involving a left adjoint to the Yoneda embedding into the enriched presheaf category—is complicated by the fact that the presheaves on a large $\mathcal{W}$-category in general only form a $\mathcal{W}'$-category for some universe-enlargement $\mathcal{W}'$ of $\mathcal{W}$. One can avoid this issue by restating the definition of totality purely in terms of the large colimits required to exist, but when enriching in a quantaloid $\mathcal{Q}$ this is unnecessary. Because the hom-categories of $\mathcal{Q}$ are complete *posets*, they admit all of the large limits necessary for the presheaves on a large $\mathcal{Q}$-category to exist as a $\mathcal{Q}$-category; and so the naïve definition of totality is, in this context, valid.

4.1. DEFINITION. Let $\mathcal{Q}$ be a quantaloid, and $\mathcal{C}$ a $\mathcal{Q}$-enriched category. A *presheaf* φ on $\mathcal{C}$ is given by:

- An object $|\varphi| \in \mathcal{Q}$, the *extent* of φ; and
- For each $x \in \mathrm{ob}\,\mathcal{C}$, an arrow $\varphi(x)\colon |x| \to |\varphi|$ in $\mathcal{Q}$,

such that we have $\varphi(y) \circ \mathcal{C}(x,y) \leqslant \varphi(x)\colon |x| \to |\varphi|$ for all $x, y \in \mathrm{ob}\,\mathcal{C}$. The *presheaf $\mathcal{Q}$-category* $\mathcal{PC}$ has these presheaves as objects, with the specified extents, and hom-arrows given by

$$\mathcal{PC}(\varphi,\psi) = \textstyle\bigwedge_{x\in\mathcal{C}}[\varphi(x),\psi(x)]\colon |\varphi| \to |\psi|\ .$$

The *Yoneda embedding* $Y\colon \mathcal{C} \to \mathcal{PC}$ sends x to the representable presheaf $\mathcal{C}(-,x)$ with extent $|x|$ and with components $\mathcal{C}(-,x)(y) = \mathcal{C}(y,x)$.

[3]Our standing assumptions, that every category be locally small, ensure that the Yoneda embedding $\mathcal{C} \to [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ exists; yet if $\mathcal{C}$ is not small, then $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ may itself fail to satisfy this same standing assumption. This is, in fact, an instance of the delicacy described in the following paragraph: the category of presheaves on the large $\mathbf{Set}$-category $\mathcal{C}$ need not form a $\mathbf{Set}$-category.

4.2. Definition. A $\mathcal{Q}$-category $\mathcal{C}$ is called *total* if the Yoneda embedding $Y : \mathcal{C} \to \mathcal{PC}$ admits a left adjoint in $\mathcal{Q}$-$\mathbf{CAT}$.

We prefer to use *total* rather than *cocomplete* for this notion, reserving the latter to mean, as is standard, "having all small colimits". Note that in [24], Stubbe uses *cocomplete* for what appears to be our *total*; but he in fact works under the restriction that $\mathcal{Q}$ should be a *small* quantaloid, and $\mathcal{C}$ a *small* $\mathcal{Q}$-category, so that his nomenclature is compatible with ours.

4.3. Example. For any $\mathcal{Q}$-category $\mathcal{C}$, $\mathcal{PC}$ is total; the left adjoint $\mu : \mathcal{PPC} \to \mathcal{PC}$ of the Yoneda embedding is defined at Φ by $\mu(\Phi)(x) = \bigvee_{\varphi \in \mathcal{PC}} \Phi(\varphi) \circ \varphi(x)$.

As explained above, totality is equivalent to the existence of certain large colimits. In the quantaloid-enriched case, it is actually equivalent to the existence of *all* large colimits.

4.4. Definition. Let $\mathcal{Q}$ be a quantaloid and $F : \mathcal{I} \to \mathcal{C}$ a $\mathcal{Q}$-functor.

- The *singular $\mathcal{Q}$-functor* $\mathcal{C}(F, \mathord{-}) : \mathcal{C} \to \mathcal{PI}$ sends $c \in \mathcal{C}$ to $\mathcal{C}(F, c)$ with extent $|c|$ and components $\mathcal{C}(F, c)(x) = \mathcal{C}(Fx, d) : |x| \to |c|$.
- Given $\varphi \in \mathcal{PI}$, a *weighted colimit of F by φ* is a left adjoint to $\mathcal{C}(F, \mathord{-})$ at φ, given by an object $\varphi \star F$ of $\mathcal{C}$ with extent $|\varphi|$ such that $\mathcal{C}(\varphi \star F, c) = \mathcal{PI}(\varphi, \mathcal{C}(F, c))$ in $\mathcal{Q}(|\varphi|, |c|)$ for all $c \in \mathcal{C}$.

4.5. Proposition. *[24, Corollary 5.4] A $\mathcal{Q}$-category $\mathcal{C}$ is total if and only if it admits all (possibly large) colimits.*

Proof. All colimits exist in $\mathcal{C}$ just when every singular functor $\mathcal{C}(F, \mathord{-}) : \mathcal{C} \to \mathcal{PI}$ admits a left adjoint. Now $\mathcal{C}(F, \mathord{-})$ is the composite of $Y : \mathcal{C} \to \mathcal{PC}$ with the $\mathcal{Q}$-functor $F^* : \mathcal{PC} \to \mathcal{PI}$ defined by $(F^*\varphi)(x) = \varphi(Fx)$, and F^* always has a left adjoint $F_! : \mathcal{PI} \to \mathcal{PC}$, defined by $(F_!\psi)(c) = \bigvee_{x \in \mathcal{I}} \psi(x) \circ \mathcal{C}(c, Fx)$; whence every $\mathcal{C}(F, \mathord{-})$ will have a left adjoint just when Y does, that is, just when $\mathcal{C}$ is total. ∎

5. Topological functors as total categories

We are now ready to prove our main result: that total categories enriched in the free quantaloid $\mathcal{Q}_\mathcal{B}$ correspond to topological functors into $\mathcal{B}$. Given a faithful functor $p : \mathcal{E} \to \mathcal{B}$, corresponding via Proposition 3.5 to a $\mathcal{Q}_\mathcal{B}$-category $\bar{\mathcal{E}}$, we will notate by

$$\begin{tikzcd} \mathcal{E} \arrow[rr, "Y"] \arrow[dr, "p"'] & & \mathcal{PE} \arrow[dl, "\mathcal{P}p"] \\ & \mathcal{B} & \end{tikzcd} \tag{1}$$

the arrow of $\mathbf{CAT}/\mathcal{B}$ corresponding to the Yoneda $\mathcal{Q}_\mathcal{B}$-functor $\bar{\mathcal{E}} \to \mathcal{P}\bar{\mathcal{E}}$. Unfolding the definitions, an object of $\mathcal{PE}$ over $z \in \mathcal{B}$ is an object of $\mathcal{P}\bar{\mathcal{E}}$ with extent z; thus given by subsets $\varphi(x) \subseteq \mathcal{B}(px, z)$ for each $x \in \mathcal{E}$ such that

$$f \in \varphi(y) \text{ and } g \in \mathcal{E}(x, y) \quad \Longrightarrow \quad f \circ p(g) \in \varphi(x) \ ;$$

in other words, by a subfunctor $\varphi \subseteq \mathcal{B}(p\text{–}, z)$, which we call a *$p$-sieve on z*. Given another p-sieve $\psi \subseteq \mathcal{B}(p\text{–}, w)$, a morphism $\varphi \to \psi$ in $\mathcal{PE}$ is a map $\theta\colon z \to w$ in $\mathcal{B}$ such that $\theta \circ f \in \psi(x)$ whenever $f \in \varphi(x)$. The functor $Y\colon \mathcal{E} \to \mathcal{PE}$ sends $e \in \mathcal{E}$ to the p-sieve $\mathcal{E}(\text{–}, e) \subseteq \mathcal{B}(p\text{–}, pe)$. Note that $\mathcal{P}p$ is topological by Example 4.3.

5.1. Remark. The construction of $\mathcal{P}p$ from p is well-known in the categorical topology literature; in the notation of [8], it is the completion $(\mathcal{E}^{-2}, p^{-2})$ of $(\mathcal{E}, p)$.

We now give our main result; in (ii), we allow ourselves to identify a p-sieve $\varphi \subseteq \mathcal{B}(p\text{–}, z)$ with the family of maps $(g\colon px \to z)_{x\in\mathcal{E}, g\in\varphi(x)}$.

5.2. Theorem. *Let $p\colon \mathcal{E} \to \mathcal{B}$ be a faithful functor and $\bar{\mathcal{E}}$ the corresponding $\mathcal{Q}_{\mathcal{B}}$-category. The following are equivalent:*

(a) p is topological;
(b) p admits all final liftings of p-sieves;
(c) $Y\colon \mathcal{E} \to \mathcal{PE}$ in (1) admits a left adjoint over $\mathcal{B}$;
(d) $\bar{\mathcal{E}}$ is total.

Proof. Clearly (a) implies (b); conversely, suppose that p admits final liftings of p-sieves; we will show that any family $\mathbf{g} = (g_i\colon px_i \to z)_{i\in I}$ admits a final lifting. Given such a $\mathbf{g}$, form the p-sieve $\varphi \subseteq \mathcal{B}(p\text{–}, z)$ with

$$\varphi(x) = \{f\colon px \to z \mid f = g_i \circ pk \text{ for some } i \in I \text{ and } k\colon x \to x_i \text{ in } \mathcal{E}\}\ , \qquad (2)$$

and let $\bar{z}$ be a final lifting of φ. We claim that $\bar{z}$ is also a final lifting of $\mathbf{g}$; which will follow so long as for all $\theta \in \mathcal{B}(z, pe)$,

$$\begin{aligned} &\theta \circ g_i\colon px_i \to pe \text{ lifts to a map } x_i \to e \text{ for all } i \in I \\ \iff\ &\theta \circ g\colon px \to pe \text{ lifts to a map } x \to e \text{ for all } g \in \varphi(x)\ . \end{aligned}$$

The leftward implication follows since each $g_i \in \varphi(x_i)$; the rightward since each $g \in \varphi(x)$ factors as $g_i \circ pk$. Thus (b) implies (a). We now show that (b) $\Leftrightarrow$ (c). To say that Y has a left adjoint at an object $\varphi \subseteq \mathcal{B}(p\text{–}, z)$ of $\mathcal{PE}$ is to say that there is an object $\bar{\varphi} \in \mathcal{E}$ with $p\bar{\varphi} = z$ such that, for all $\theta \in \mathcal{B}(z, pe)$,

$$z \xrightarrow{\theta} pe \text{ lifts to a map } \bar{\varphi} \to e \text{ in } \mathcal{E} \text{ iff it lifts to a map } \varphi \to Ye \text{ in } \mathcal{PE}. \qquad (\dagger)$$

But to say that θ lifts to a map $\varphi \to Ye$ is to say that $\theta \circ g\colon px \to pe$ lifts to a map $x \to e$ for every $g \in \varphi(x)$, and so condition (†) says precisely that $\bar{\varphi}$ is a final lifting of the sieve φ. Finally, (c) $\Leftrightarrow$ (d) by Proposition 3.5. ∎

6. Duality

In Remark 2.3 we mentioned the *topological duality theorem*, which states that a functor $p\colon \mathcal{E} \to \mathcal{B}$ is topological if and only if $p^{\mathrm{op}}\colon \mathcal{E}^{\mathrm{op}} \to \mathcal{B}^{\mathrm{op}}$ is so. In this section, we explain this result in terms of a general result of quantaloid-enriched category theory: namely, that the notion of total $\mathcal{Q}$-category is self-dual. The starting point is the adjoint functor theorem for $\mathcal{Q}$-categories.

6.1. Proposition. *[24, Proposition 4.6] If $F\colon \mathcal{C} \to \mathcal{D}$ is a $\mathcal{Q}$-functor and $d \in \mathcal{D}$, then a right adjoint for F at d is given by $Gd = \mathcal{D}(F, d) \star 1_{\mathcal{C}}$ whenever this colimit exists in $\mathcal{C}$ and is preserved by F.*

Proof. Under the stated hypotheses, we must verify that $\mathcal{C}(c, Gd) = \mathcal{D}(Fc, d)$. On the one hand, we have $1_{|d|} \leqslant \mathcal{C}(Gd, Gd) = \mathcal{PC}(\mathcal{D}(F, d), \mathcal{C}(1, Gd))$, whence $\mathcal{D}(Fc, d) \leqslant \mathcal{C}(c, Gd)$ as required; conversely, since FGd has the universal property of $\mathcal{D}(F, d) \star F$, we have $1_{|d|} \leqslant \mathcal{PC}(\mathcal{D}(F, d), \mathcal{D}(F, d)) = \mathcal{D}(FGd, d)$ whence $\mathcal{C}(c, Gd) \leqslant \mathcal{D}(FGd, d) \circ \mathcal{D}(Fc, FGd) \leqslant \mathcal{D}(Fc, d)$ as required. ∎

6.2. Corollary. [Adjoint functor theorem] *If $\mathcal{C}$ is a total $\mathcal{Q}$-category, and $F\colon \mathcal{C} \to \mathcal{D}$ preserves all (large) colimits, then F has a right adjoint in $\mathcal{Q}$-**CAT**.*

We now apply this result to show that the notion of totality for $\mathcal{Q}$-categories is self-dual. First we make clear the sense of that duality.

6.3. Definition. The *copresheaf category* $\mathcal{P}^\dagger\mathcal{C}$ on a $\mathcal{Q}$-category $\mathcal{C}$ has as objects φ, families of maps $\varphi(x)\colon |\varphi| \to |x|$ satisfying $\mathcal{C}(x, y) \circ \varphi(x) \leqslant \varphi(y)$, and hom-objects defined by $\mathcal{P}^\dagger\mathcal{C}(\varphi, \psi) = \bigwedge_{x \in C}\{\psi(x), \varphi(x)\}$ (note the reversal of order!). The dual Yoneda embedding $Y^\dagger\colon \mathcal{C} \to \mathcal{P}^\dagger\mathcal{C}$ sends x to $\mathcal{C}(x, \text{–})$. We say that $\mathcal{C}$ is *cototal* if $Y^\dagger\colon \mathcal{C} \to \mathcal{P}^\dagger\mathcal{C}$ admits a right adjoint.

6.4. Remark. We can go on to define a *weighted limit* $\{\psi, F\}$ of a $\mathcal{Q}$-functor $F\colon \mathcal{I} \to \mathcal{C}$ by a weight $\psi \in \mathcal{P}^\dagger\mathcal{I}$ as a right adjoint at ψ to the dual singular functor $\mathcal{C}(1, F)\colon \mathcal{C} \to \mathcal{P}^\dagger\mathcal{I}$; then as in Proposition 4.5, we may conclude that a $\mathcal{Q}$-category is cototal just when it admits all large limits.

6.5. Remark. Any $\mathcal{Q}$-category $\mathcal{C}$ has an opposite $\mathcal{C}^{\mathrm{op}}$ which is a $\mathcal{Q}^{\mathrm{op}}$-category; now $\mathcal{P}^\dagger$ as defined above is equally $(\mathcal{P}(\mathcal{C}^{\mathrm{op}}))^{\mathrm{op}}$, and $Y^\dagger$ the opposite of $Y\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{P}(\mathcal{C}^{\mathrm{op}})$. In these terms, the $\mathcal{Q}$-category $\mathcal{C}$ is cototal just when $\mathcal{C}^{\mathrm{op}}$ is a total $\mathcal{Q}^{\mathrm{op}}$-category.

For a faithful functor $p\colon \mathcal{E} \to \mathcal{B}$, seen as a category enriched over the free quantaloid $\mathcal{Q}_{\mathcal{B}}$, applying the copresheaf construction yields the topological functor $\mathcal{P}^\dagger p\colon \mathcal{P}^\dagger\mathcal{E} \to \mathcal{B}$ for which objects over z are cosieves $\varphi \subseteq \mathcal{B}(z, p\text{–})$ and morphisms from φ to $\psi \subseteq \mathcal{B}(w, p\text{–})$ are morphisms $\theta\colon z \to w$ of $\mathcal{B}$ such that $f \in \psi(x)$ implies $f \circ \theta \in \varphi(x)$; in the notation of [8], this is the completion $(\mathcal{E}^2, p^2)$ of $(\mathcal{E}, p)$.

6.6. Theorem. *[24, Proposition 5.10] A $\mathcal{Q}$-category $\mathcal{C}$ is total just when it is cototal.*

PROOF. A straightforward calculation shows that the dual Yoneda embedding $Y^\dagger \colon \mathcal{C} \to \mathcal{P}^\dagger\mathcal{C}$ preserves all colimits; thus, if $\mathcal{C}$ is total, then $Y^\dagger$ has a right adjoint, whence $\mathcal{C}$ is cototal. The converse implication is dual. ∎

6.7. REMARK. More generally, Proposition 6.1 and its dual provide formulas for computing limits in $\mathcal{Q}$-categories in terms of colimits and vice versa; c.f. [24, Proposition 5.8]. The situation is encapsulated in terms of the *Isbell adjunction* [12]

$$\begin{tikzcd} \mathcal{P}\mathcal{C} \arrow[r, shift left, "\uparrow = \mathcal{P}\mathcal{C}(1,Y)"] \arrow[r, phantom, "\perp"] & \mathcal{P}^\dagger\mathcal{C} \arrow[l, shift left, "\downarrow = \mathcal{P}^\dagger\mathcal{C}(Y^\dagger,1)"] \end{tikzcd} \tag{3}$$

between the singular functor of $Y^\dagger$ and the dual singular functor of Y. Given $\psi \in \mathcal{P}^\dagger\mathcal{C}$, the weighted limit $\{\psi, 1_\mathcal{C}\}$ can be computed as $(\downarrow\psi) \star 1_\mathcal{C}$ whenever the latter colimit exists. Dually, $\varphi \star 1_\mathcal{C}$ can be computed as the limit $\{\uparrow\varphi, 1_\mathcal{C}\}$.

In the context of enrichment over a free quantaloid $\mathcal{Q}_\mathcal{B}$, taking Theorem 6.6 together with Remark 6.5 and Theorem 5.2 recovers the topological duality theorem of Remark 2.3. From Remark 6.7, we obtain the explicit formula for computing final liftings along p in terms of initial ones: the Isbell adjunction for $p \colon \mathcal{E} \to B$ sends a sieve $\varphi \subseteq \mathcal{B}(p\text{–}, z)$ to the cosieve $\uparrow\varphi$, and a cosieve $\psi \subseteq \mathcal{B}(z, p\text{–})$ to the sieve $\downarrow\psi$ defined by the following formulae:

$$\begin{aligned} \uparrow\varphi &= \{g \colon z \to px \mid g \circ h \colon py \to px \text{ lifts to } y \to x \ \forall h \colon py \to z \text{ in } \varphi\} \\ \downarrow\psi &= \{h \colon py \to z \mid g \circ h \colon py \to px \text{ lifts to } y \to x \ \forall g \colon z \to px \text{ in } \psi\}\ . \end{aligned} \tag{4}$$

Thus given a family of maps $\mathbf{g} = (g_i \colon px_i \to z)_{i \in I}$, on forming the sieve $\varphi \subseteq \mathcal{B}(p\text{–}, z)$ they generate, and the conjugate cosieve $\uparrow\varphi \subseteq \mathcal{B}(p\text{–}, z)$, Remark 6.7 asserts that an initial lifting for $\mathbf{g}$ can be obtained as a final lifting for $\uparrow\varphi$; and dually.

7. MacNeille completions

In this final section, we give an application of our main result to MacNeille completions. As in the introduction, the *MacNeille completion* of a preorder X is the poset $\mathcal{R}X$ whose elements are pairs (L, U) of subsets of X with $L = {\downarrow}U$ and $U = {\uparrow}L$, ordered by $(L, U) \leqslant (L', U')$ iff $L \subseteq L'$ (equivalently, $U \supseteq U'$); here we write

$$\downarrow U = \{\ell \in X \mid \ell \leqslant u \ \forall u \in U\} \quad \text{and} \quad \uparrow L = \{u \in X \mid \ell \leqslant u \ \forall \ell \in L\}\ .$$

$\mathcal{R}X$ is a complete lattice, with meets $\bigwedge_i (L_i, U_i) = (\bigwedge_i L_i, \uparrow(\bigwedge_i L_i))$ and joins defined dually, and there is an order-embedding $J \colon X \to \mathcal{R}X$ sending x to $(\downarrow\{x\}, \uparrow\{x\})$ that preserves all meets and joins that exist in X and is join- and meet-dense. These properties in fact serve to characterise $\mathcal{R}X$ up to equivalence of preorders; it follows that $\mathcal{R}$ is idempotent, in the sense that J is an equivalence whenever X is a complete preorder. There is an alternate characterisation in terms of cut-continuous maps [4]. A *lower cut* in a poset is a subset of the form $\downarrow U$, and a map of preorders $f \colon X \to Y$ is called *lower* *cut-continuous* when f^{-1} preserves lower cuts. Now the assignation $X \mapsto \mathcal{R}X$ provides a bireflection of the 2-category of preorders and lower cut-continuous maps to its full sub-2-category on the complete preorders.

In [8], Herrlich introduced the notion of *MacNeille completion* of a faithful functor to a topological one; this completion has the same good properties of the MacNeille completion of a preorder, and indeed reduces to it in the case where $\mathcal{B} = 1$. In this section, we will justify the nomenclature by exhibiting the topological MacNeille completion as an instance of the general process of "MacNeille completion" for categories enriched over an arbitrary base; the construction here was first described (for unenriched categories) by Isbell [12]. It is rather less well-behaved for general enriched categories than for preorders, but when the enrichment base happens to be a quantaloid, all of the good properties of the order-theoretic case are retained; it is this situation that we shall now describe.

We motivate the construction with some remarks about the MacNeille completion of a preorder X. Note that if $(L, U) \in \mathcal{R}X$, then L is a downset and U an upset in X; and on viewing X as a category enriched in the one-object quantaloid $(\mathbf{2}, \wedge, \top)$, such downsets and upsets in X are the respective objects of the presheaf and copresheaf categories $\mathcal{P}X$ and $\mathcal{P}^\dagger X$. The operations $\uparrow$ and $\downarrow$ described above are now precisely those of the Isbell adjunction (3) for X. This adjunction is a Galois connection between posets; and $\mathcal{R}X$ is one of the several equivalent descriptions of the poset of fixpoints of this Galois connection.

Motivated by this, we now describe the MacNeille completion of a general quantaloid-enriched category. First a preparatory result on adjunctions in the $\mathcal{Q}$-enriched context; the proof is entirely straightforward and so omitted.

7.1. Proposition. *For any adjunction $F \colon \mathcal{C} \rightleftarrows \mathcal{D} \colon G$ of $\mathcal{Q}$-categories, there are canonical isomorphisms induced by F and G between the following $\mathcal{Q}$-categories:*

- *(a) The full replete image of G in $\mathcal{C}$;*
- *(b) The full subcategory of $\mathcal{C}$ on those objects X with $X \cong GFX$;*
- *(c) The $\mathcal{Q}$-category of GF-algebras in $\mathcal{C}$;*
- *(d)–(f) The duals of (a)–(c);*
- *(g) The $\mathcal{Q}$-category whose objects are pairs (c, d) with $Fc \cong d$ and $c \cong Gd$ and whose hom-object from (c, d) to (c', d') is $\mathcal{C}(c, c') = \mathcal{D}(d, d')$.*

The $\mathcal{Q}$-categories (a)–(c) are reflective in $\mathcal{C}$, while (d)–(f) are coreflective in $\mathcal{D}$.

We write $\mathbf{Fix}(F, G)$ for any of the isomorphic $\mathcal{Q}$-categories (a)–(g).

7.2. Definition. The *MacNeille completion* $\mathcal{R}\mathcal{C}$ of a $\mathcal{Q}$-category $\mathcal{C}$ is the $\mathcal{Q}$-category $\mathbf{Fix}(\uparrow, \downarrow)$ associated to the Isbell adjunction (3); for concreteness, we take the representation in (b), so that $\mathcal{R}\mathcal{C}$ comprises those $\varphi \in \mathcal{P}\mathcal{C}$ with $\varphi = {\downarrow}{\uparrow}\varphi$. Since $\uparrow$ and $\downarrow$ send representables to representables, the Yoneda embedding factors through $\mathcal{R}\mathcal{C}$ as $J \colon \mathcal{C} \to \mathcal{R}\mathcal{C}$, say.

7.3. Remark. Since ${\downarrow} \colon \mathcal{P}^\dagger\mathcal{C} \to \mathcal{P}\mathcal{C}$ is right adjoint to the dual singular functor $\mathcal{P}\mathcal{C}(1, Y)$, it must send ψ to $\{\psi, Y\}$. Thus every $\varphi \in \mathcal{R}\mathcal{C}$ is a limit $\{{\uparrow}\varphi, Y\}$ of representables in $\mathcal{P}\mathcal{C}$;

on the other hand, as $\mathcal{RC}$ is reflective in $\mathcal{PC}$, it is limit-closed, and so $\mathcal{RC}$ is in fact the closure of the representables in $\mathcal{PC}$ under (large) limits.

When $\mathcal{Q}$ is the one-object quantaloid $(\mathbf{2}, \wedge, \top)$, the MacNeille completion of a $\mathcal{Q}$-category (=preorder) X is the classical MacNeille completion $\mathcal{R}X$. When $\mathcal{Q}$ is the one-object quantaloid $(\mathbb{R}_+, +, 0)$, the MacNeille completion of $\mathcal{Q}$-category (=generalised metric space) was identified in [28] with its *directed tight span* completion. When $\mathcal{Q}$ is the quantaloid associated to a space X or a site $(\mathcal{C}, J)$, the MacNeille completion of a (Cauchy-complete, skeletal) $\mathcal{Q}$-category is its MacNeille completion internal to the topos of sheaves. Finally, for the free quantaloids which are our main concern in this paper, we have that:

7.4. Proposition. *Let $\mathcal{Q}_\mathcal{B}$ be the free quantaloid on a category $\mathcal{B}$. If $p\colon \mathcal{E} \to \mathcal{B}$ is a faithful functor corresponding to a $\mathcal{Q}_\mathcal{B}$-category $\bar{\mathcal{E}}$, then the MacNeille completion of* p qua *faithful functor corresponds to the MacNeille completion of $\bar{\mathcal{E}}$* qua *quantaloid-enriched category.*

Proof. The faithful functor $\mathcal{R}p\colon \mathcal{RE} \to \mathcal{B}$ corresponding to $\mathcal{R}\overline{\mathcal{E}}$ has as domain the full subcategory of $\mathcal{PE}$ on those sieves $\varphi \subseteq \mathcal{P}(p\text{–}, b)$ with $\varphi = {\downarrow}{\uparrow}\varphi$, for $\downarrow$ and $\uparrow$ defined as in (4). This is precisely the construction of the MacNeille completion of p given in [8]. ∎

Our remaining results give characterisations of the MacNeille completion of a $\mathcal{Q}$-category that extend those for the classical case; having identified the scope of the notions in particular examples, we shall henceforth feel free to work only in the general situation. We begin with a preparatory result concerning density.

7.5. Proposition. *For a $\mathcal{Q}$-functor $F\colon \mathcal{C} \to \mathcal{D}$, the following are equivalent:*

(a) $\mathcal{D}(F, 1)\colon \mathcal{D} \to \mathcal{PC}$ is fully faithful;
(b) Each $d \in \mathcal{D}$ is the colimit $\mathcal{D}(F, d) \star F$;
(c) Each $d \in \mathcal{D}$ is a colimit $\varphi \star F$ for some $\varphi \in \mathcal{PC}$.

We call $F\colon \mathcal{C} \to \mathcal{D}$ with any of these equivalent properties *dense*. The equivalence of (a) and (b) is standard enriched category theory [13, Theorem 5.1]; that of (b) and (c) is peculiar to the quantaloid-enriched case.

Proof. (a) and (b) both say that $\mathcal{D}(d, d') = \mathcal{PC}(\mathcal{D}(F, d), \mathcal{D}(F, d'))$ for all $d, d' \in \mathcal{D}$, and (b) clearly implies (c). It remains to show (c) $\Rightarrow$ (b); thus fixing $d \in \mathcal{D}$, we must show that $\mathcal{PC}(\mathcal{D}(F, d), \mathcal{D}(F, d')) \leqslant \mathcal{D}(d, d')$ for all $d' \in \mathcal{D}$ (the converse inequality is automatic by functoriality of $\mathcal{D}(F, 1)$). By assumption, d is $\varphi \star F$ for some φ; and so $1_{|d|} \leqslant \mathcal{D}(d, d) = \mathcal{PC}(\varphi, \mathcal{D}(F, d))$, whence $\mathcal{PC}(\mathcal{B}(F, d), \mathcal{B}(F, d')) \leqslant \mathcal{PC}(\varphi, \mathcal{B}(F, d')) = \mathcal{D}(d, d')$ as required. ∎

7.6. Proposition. *The MacNeille completion $J\colon \mathcal{C} \to \mathcal{RC}$ has the properties that:*

(a) $\mathcal{RC}$ is total;
(b) J is fully faithful;
(c) J is dense and codense.

These properties imply that:

(d) J preserves all limits and colimits that exist in $\mathcal{C}$;
(e) Any full embedding of $\mathcal{C}$ into a total $\mathcal{Q}$-category $\mathcal{D}$ extends to a full embedding of $\mathcal{RC}$ into $\mathcal{D}$.

and moreover characterise $\mathcal{RC}$ and J up to equivalence; in particular, $\mathcal{R}$ is idempotent in the sense that J is an equivalence whenever $\mathcal{C}$ is total.

Proof. $\mathcal{RC}$ is reflective in the total $\mathcal{PC}$, and hence itself total; while J is a factorisation of the fully faithful $Y : \mathcal{C} \to \mathcal{PC}$ and so itself fully faithful. Codensity follows from Remark 7.3 and density from its dual. For (d), a fully faithful dense functor $F \colon \mathcal{A} \to \mathcal{B}$ must preserve limits, since it factorises the limit-preserving Yoneda embedding; dually, codensity implies colimit-preservation. For (e), given $F \colon \mathcal{C} \to \mathcal{D}$ a full embedding with $\mathcal{D}$ total, let $\mathcal{D}'$ be the closure of $\mathcal{C}$ in $\mathcal{D}$ under colimits; then $\mathcal{D}'$ is again total, and $\mathcal{C} \to \mathcal{D}'$ is dense, so that $\mathcal{D}'$ can be identified with a limit-closed subcategory of $\mathcal{PC}$ containing the representables. Now let $\mathcal{D}''$ be the closure of the representables in $\mathcal{D}'$ under limits; then by construction and Remark 7.3, $\mathcal{D}'' \simeq \mathcal{RC}$, as required. Finally, for the uniqueness, observe that in the construction just given, we have $\mathcal{D}' = \mathcal{D}$ if J is dense, and then $\mathcal{D} = \mathcal{D}' = \mathcal{D}''$ if in addition J is codense. The final clause follows since, if $\mathcal{C}$ is total, then $1_{\mathcal{C}} \colon \mathcal{C} \to \mathcal{C}$ satisfies properties (a)–(c). ∎

We conclude by describing a universal property of the MacNeille completion of a $\mathcal{Q}$-category which generalises a universal property [4] of the classical case. In the following definition, recall that for any $\mathcal{Q}$-functor $F \colon \mathcal{C} \to \mathcal{D}$, the $\mathcal{Q}$-functor $F^* \colon \mathcal{PD} \to \mathcal{PC}$ is defined at φ by $(F^*\varphi)(x) = \varphi(Fx)$.

7.7. Definition. A $\mathcal{Q}$-functor $F \colon \mathcal{C} \to \mathcal{D}$ is *cut-cocontinuous* if $F^* \colon \mathcal{PD} \to \mathcal{PC}$ maps $\mathcal{RD}$ into $\mathcal{RC}$.

7.8. Proposition. *$F \colon \mathcal{C} \to \mathcal{D}$ is cut-cocontinuous if and only if the singular functor $\mathcal{D}(F, 1) \colon \mathcal{D} \to \mathcal{PC}$ lands inside $\mathcal{RC}$.*

Proof. We must show that F^* maps $\mathcal{RD}$ into $\mathcal{RC}$ if and only if it maps the representables into $\mathcal{RC}$. For the non-trivial direction, note that F^* preserves all limits, each $\varphi \in \mathcal{RD}$ is a limit of representables in $\mathcal{PD}$, and $\mathcal{RC}$ is closed under limits in $\mathcal{PC}$. ∎

7.9. Proposition. *For any $\mathcal{Q}$-functor $F \colon \mathcal{C} \to \mathcal{D}$, we have*

$$\begin{aligned} & F \text{ is a left adjoint} \\ \Longrightarrow\ & F \text{ is cut-cocontinuous} \\ \Longrightarrow\ & F \text{ preserves all (large) colimits.} \end{aligned}$$

If $\mathcal{C}$ is total, then all three conditions are equivalent.

Proof. For the first implication, if F has a right adjoint G, then $\mathcal{D}(F,1) = YG\colon \mathcal{D} \to \mathcal{PC}$ clearly lands in $\mathcal{RC}$. For the second implication, it suffices by Proposition 4.5 to show that a cut-cocontinuous F preserves any colimit $v = \varphi \star 1_{\mathcal{C}}$ existing in $\mathcal{C}$. Now $\uparrow\varphi = \mathcal{PC}(\varphi, Y) = \mathcal{C}(v,1) = Y^\dagger v$; and since $\mathcal{D}(F,1)\colon \mathcal{D} \to \mathcal{PC}$ factors through $\mathcal{RC}$, we have that $\mathcal{PC}(\varphi, \mathcal{D}(F,1)) = \mathcal{P}^\dagger\mathcal{C}(\uparrow\varphi, \uparrow\mathcal{D}(F,1)) = \mathcal{P}^\dagger\mathcal{C}(Y^\dagger v, \uparrow\mathcal{D}(F,1)) = \mathcal{D}(Fv,1)$, so that Fv is $\varphi \star F$ as claimed. The final clause follows from the adjoint functor theorem. ■

Now let **CCOCTS** be the 2-category of $\mathcal{Q}$-categories, cut-cocontinuous $\mathcal{Q}$-functors, and $\mathcal{Q}$-transformations, and let **TOT** denote the full sub-2-category on the total $\mathcal{Q}$-categories; by the preceding proposition, the morphisms of **TOT** are the functors preserving all colimits.

7.10. Proposition. *The MacNeille completion $J\colon \mathcal{C} \to \mathcal{RC}$ is the value at $\mathcal{C}$ of a left biadjoint to the inclusion 2-functor* **TOT** $\to$ **CCOCTS**.

Proof. Let $\mathcal{C}$ be a $\mathcal{Q}$-category. Since $\mathcal{RC}$ is reflective in the total $\mathcal{PC}$, it is itself total; moreover, $J\colon \mathcal{C} \to \mathcal{RC}$ is cut-cocontinuous by Proposition 7.8, as its singular functor $\mathcal{RC}(J,1)\colon \mathcal{RC} \to \mathcal{PC}$ is the inclusion. It remains to show that, for any total $\mathcal{Q}$-category $\mathcal{D}$, the restriction $\mathcal{Q}$-functor

$$\mathbf{CCOCTS}(\mathcal{RC}, \mathcal{D}) \xrightarrow{(-)\circ J} \mathbf{CCOCTS}(\mathcal{C}, \mathcal{D}) \tag{5}$$

is an equivalence of preorders; we do so by exhibiting an explicit pseudoinverse. Given a cut-cocontinuous $F\colon \mathcal{C} \to \mathcal{D}$, define $F^\#\colon \mathcal{RC} \to \mathcal{D}$ by $F^\#(\varphi) = \varphi \star F$. We must show that $F^\#$ is cut-cocontinuous; equivalently, by Proposition 7.9, that it is a left adjoint. By Proposition 7.8 and cut-cocontinuity of F, the singular functor $\mathcal{D}(F,1)\colon \mathcal{D} \to \mathcal{PC}$ factors through $\mathcal{RC}$, as $H\colon \mathcal{D} \to \mathcal{RC}$, say; then as $(-) \star F\colon \mathcal{PC} \to \mathcal{D}$ is left adjoint to $\mathcal{D}(F,1)$, it follows that $F^\#$ is left adjoint to H, as required. It remains to show that $(-)^\#$ is pseudoinverse to $(-) \circ J$, for which we need two things:

(a) For any cut-cocontinuous $F\colon \mathcal{C} \to \mathcal{D}$, we have $F^\# J \cong F$; but for each $x \in \mathcal{C}$ we have $F^\#(Jx) = \mathcal{C}(-,x) \star F \cong Fx$ by the Yoneda lemma.
(b) For any (cut-)cocontinuous $G\colon \mathcal{RC} \to \mathcal{D}$, we have $(GJ)^\# \cong G$. But $(GJ)^\#(\varphi) = \varphi \star GJ \cong G(\varphi \star J) \cong G(\varphi)$ using cocontinuity of G and density of J.

■

7.11. Remark. The notion of MacNeille completion is self-dual in that $\mathcal{RC} = (\mathcal{R}(\mathcal{C}^{\mathrm{op}}))^{\mathrm{op}}$. It follows that $\mathcal{RC}$ is also characterised by a dual universal property: it provides a bireflection of $\mathcal{C}$, seen as an object of the 2-category **CCONTS** of $\mathcal{Q}$-categories and cut-continuous $\mathcal{Q}$-functors, onto the full sub-2-category **COTOT** whose objects are the cototal (=total) $\mathcal{Q}$-categories.

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