

## EXPONENTIAL SUMS OVER POINTS OF ELLIPTIC CURVES WITH RECIPROCAL OF PRIMES

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*Abstract.* We consider exponential sums with  $x$ -coordinates of points  $qG$  and  $q^{-1}G$  where  $G$  is a point of order  $T$  on an elliptic curve modulo a prime  $p$  and  $q$  runs through all primes up to  $N$  (with  $\gcd(q, T) = 1$  in the case of the points  $q^{-1}G$ ). We obtain a new bound on exponential sums with  $q^{-1}G$  and correct an imprecision in the work of W. D. Banks, J. B. Friedlander, M. Z. Garaev and I. E. Shparlinski on exponential sums with  $qG$ . We also note that similar sums with  $g^{1/q}$  for an integer  $g$  with  $\gcd(g, p) = 1$  have been estimated by J. Bourgain and I. E. Shparlinski.

§1. *Introduction.* Let  $p \geq 5$  be a prime and  $\mathcal{E}$  be an elliptic curve defined over a finite field  $\mathbb{F}_p$  of  $p$  elements given by an affine Weierstraß equation

$$\mathcal{E} : Y^2 = X^3 + AX + B$$

with some  $A, B \in \mathbb{F}_p$ , see [1, 3, 21].

We recall that the set of all points on  $\mathcal{E}$  forms an abelian group, with the “point at infinity”  $\mathcal{O}$  as the neutral element, and we use  $\oplus$  to denote the group operation. As usual, we write every point  $P \neq \mathcal{O}$  on  $\mathcal{E}$  as  $P = (\mathbf{x}(P), \mathbf{y}(P))$ .

Let  $\mathcal{E}(\mathbb{F}_p)$  denote the set of  $\mathbb{F}_p$ -rational points on  $\mathcal{E}$ . We recall that the celebrated result of Bombieri [4] implies in particular an estimate of order  $p^{1/2}$  for exponential sums with functions from the function field of  $\mathcal{E}$  taken over all points of  $\mathcal{E}(\mathbb{F}_p)$ . More recently, various character sums over points of elliptic curves have been considered in a number of papers, see [2, 7, 9, 10, 14–16, 18, 20] and references therein; many of these estimates are motivated by applications to pseudorandom number generators on elliptic curves [19].

Let  $G \in \mathcal{E}(\mathbb{F}_p)$  be a point of order  $T$ , in other words,  $T$  is the cardinality of the cyclic group  $\langle G \rangle$  generated by  $G$  in  $\mathcal{E}(\mathbb{F}_p)$ .

We also denote

$$\mathbf{e}(z) = \exp(2\pi iz) \quad \text{and} \quad \mathbf{e}_m(z) = \mathbf{e}(z/m),$$

and consider the sums

$$S_a(N) = \sum_{\substack{q \leq N \\ q \text{ prime} \\ \gcd(q, T) = 1}} \mathbf{e}_p(ax(q^{-1}G)), \quad a \in \mathbb{F}_p, \quad (1)$$

where the parameter  $q$  varies over prime numbers. We estimate these sums (uniformly over  $a \not\equiv 0 \pmod{p}$ ), provided that  $N$  and  $T$  are sufficiently large compared to  $p$ . Since our results are based on those of [17], they apply only to ordinary curves, see [1, 3, 21] for a definition of ordinary elliptic curves.

We note that the sums  $S_a(N)$  are elliptic curve analogues of the exponential sums with reciprocals of primes  $1/q$  that have been considered in [5, 11, 12] and with  $g^{1/q}$  for  $g \in \mathbb{F}_p$  that have been considered in [6].

In particular, in the most interesting case of  $T = p^{1+o(1)}$  (that is, when  $G$  generates a large subgroup of  $\mathcal{E}(\mathbb{F}_p)$ ) we obtain the bound

$$|S_a(N)| \leq (Np^{-1/256} + N^{5/6} p^{5/12}) N^{o(1)} \quad (2)$$

which is non-trivial if  $p^C \geq N \geq p^{5/2+\varepsilon}$  for some fixed  $C$  and  $\varepsilon > 0$ . Furthermore, for  $N \geq p^{323/128}$  the bound (2) simplifies as

$$|S_a(N)| \leq N^{1+o(1)} p^{-1/256}. \quad (3)$$

One can use our bounds in a standard fashion to obtain an asymptotic formula for the number of solutions to the congruence

$$\mathbf{x}(q_1^{-1}G) + \cdots + \mathbf{x}(q_k^{-1}G) \equiv c \pmod{p} \quad (4)$$

in primes  $q_1, \dots, q_k \leq N$ , which is an analogue of the congruence

$$q_1^{-1} + \cdots + q_k^{-1} \equiv c \pmod{p}$$

studied in [11]. We do not derive all possible results of this kind but simply give one example which relies on the bound (3).

We remark that the sums

$$T_a(N) = \sum_{\substack{q \leq N \\ q \text{ prime}}} \mathbf{e}_p(a\mathbf{x}(qG)), \quad a \in \mathbb{F}_p, \quad (5)$$

have been considered in [2]. However, the proof of [2, Theorem 6] unfortunately contains an imprecision. Here we present an estimate on  $T_a(N)$  which can be obtained by the same method as our bound on  $S_a(N)$ .

## §2. Preparations.

2.1. *Notation.* We use  $\mathbb{Z}_M^*$  to denote the unit group of the residue ring  $\mathbb{Z}_M$  modulo a positive integer  $M$ .

As usual, let  $\mu$  be the Möbius function. Let  $\Lambda$  denote the von Mangoldt function which we recall to be defined for positive integers  $n$  by

$$\Lambda(n) = \begin{cases} \log q & \text{if } n > 1 \text{ is a power of a prime } q, \\ 0 & \text{otherwise} \end{cases}$$

with  $\log$  being the natural logarithm.

Throughout the paper, the implied constants in symbols “ $O$ ” and “ $\ll$ ” may occasionally depend on the integer parameters  $r$  and  $s$ , and are absolute otherwise (we recall that  $U \ll V$  and  $U = O(V)$  are both equivalent to the inequality  $|U| \leq cV$  with some constant  $c > 0$ ).

2.2. *Vaughan identity.* We decompose  $\Lambda$  by means of the Vaughan identity, given for example in [8, Ch. 24], which we use in the following form.

LEMMA 1. *For any complex-valued function  $f(n)$  and any real numbers  $U, V > 1$  with  $UV \leq N$ , we have*

$$\sum_{n \leq N} \Lambda(n) f(n) \ll \Sigma_1 + \Sigma_2 + \Sigma_3 + \Sigma_4,$$

where

$$\begin{aligned} \Sigma_1 &= \left| \sum_{n \leq U} \Lambda(n) f(n) \right|, \\ \Sigma_2 &= (\log UV) \sum_{k \leq UV} \left| \sum_{\ell \leq N/k} f(k\ell) \right|, \\ \Sigma_3 &= (\log N) \sum_{k \leq V} \max_{w \geq 1} \left| \sum_{w \leq \ell \leq N/k} f(k\ell) \right|, \\ \Sigma_4 &= \left| \sum_{\substack{k\ell \leq N \\ k > V, \ell > U}} \Lambda(\ell) \sum_{d|k, d \leq V} \mu(d) f(k\ell) \right|. \end{aligned}$$

2.3. *Single sums.* We also need the following result which is proved in [17, Theorem 6].

LEMMA 2. *Let  $\mathcal{E}$  be an ordinary curve defined over  $\mathbb{F}_p$  and let  $G \in \mathcal{E}$  of order  $T$ . Then for any  $d \geq 1$  fixed pairwise distinct integers  $e_1, \dots, e_d$  and positive integers  $r, s \geq 2$ , uniformly over  $a \in \mathbb{F}_p^*$  and  $b_1, \dots, b_d \in \mathbb{Z}$ , we have the bound*

$$\sum_{n \in \mathbb{Z}_T^*} \mathbf{e}_p(ax(nP)) \mathbf{e}_T(H(n)) \ll T^{1-2\eta_{r,s} + \kappa_{d,r,s}} p^{\eta_{r,s} + o(1)},$$

where the rational function  $H$  is given by

$$H(X) = b_1 X^{e_1} + \dots + b_d X^{e_d},$$

and

$$\eta_{r,s} = \frac{1}{4(4r+s)} \quad \text{and} \quad \kappa_{d,r,s} = \frac{2(d-1)s+1}{4rs}.$$

Taking  $d = 1$  and  $e_1 = -1$  in Lemma 2 and using the standard reduction between complete and incomplete sums, see [13, §12.2], we obtain the following estimate.

COROLLARY 3. *Let  $\mathcal{E}$  be an ordinary elliptic curve defined over  $\mathbb{F}_p$  and let  $G \in \mathcal{E}(\mathbb{F}_p)$  be a point of order  $T$ . Then for any integers  $r, s \geq 2$ , real numbers  $H_1 < H_2$  and integer  $a$  not divisible by  $p$ , the following estimate holds:*

$$\sum_{\substack{H_1 < n \leq H_2 \\ \gcd(n, T) = 1}} \mathbf{e}_p(a\mathbf{x}(n^{-1}G)) \ll \left( \frac{H_2 - H_1}{T} + 1 \right) T^{1-1/2(4r+s)+1/4rs} p^{1/4(4r+s)+o(1)}.$$

2.4. *Double sums.* We need an estimate of certain double exponential sums that follows directly from [2, Theorem 3].

LEMMA 4. *Let  $\mathcal{E}$  be an ordinary elliptic curve defined over  $\mathbb{F}_p$ , and let  $G \in \mathcal{E}(\mathbb{F}_p)$  be a point of order  $T$ . Then, for all subsets  $\mathcal{K}, \mathcal{L} \subset \mathbb{Z}_T^*$ , sequences  $\alpha_k$  and  $\beta_\ell$  of arbitrary complex numbers, supported on the sets  $\mathcal{K}$  and  $\mathcal{L}$ , respectively, and all  $a \in \mathbb{F}_p^*$ , the following bound holds:*

$$\sum_{k \in \mathcal{K}} \sum_{\ell \in \mathcal{L}} \alpha_k \beta_\ell \mathbf{e}_p(a\mathbf{x}(k^{-1}\ell^{-1}G)) \ll AB T^{5/6} (\#\mathcal{K}\#\mathcal{L})^{1/2} p^{1/12+o(1)},$$

where

$$A = \max_{k \in \mathcal{K}} |\alpha_k| \quad \text{and} \quad B = \max_{\ell \in \mathcal{L}} |\beta_\ell|.$$

*Proof.* We define the subsets of  $\mathbb{Z}_T^*$

$$\mathcal{K}^* = \{k^{-1} : k \in \mathcal{K}\}, \quad \mathcal{L}^* = \{\ell^{-1} : \ell \in \mathcal{L}\}.$$

Using these notations, we obtain

$$\sum_{k \in \mathcal{K}} \sum_{\ell \in \mathcal{L}} \alpha_k \beta_\ell \mathbf{e}_p(a\mathbf{x}(k^{-1}\ell^{-1}G)) = \sum_{k \in \mathcal{K}^*} \sum_{\ell \in \mathcal{L}^*} \alpha_{k^{-1}} \beta_{\ell^{-1}} \mathbf{e}_p(a\mathbf{x}(k\ell G)).$$

Now applying [2, Theorem 3], we obtain the desired result.  $\square$

We now immediately derive the following corollary.

COROLLARY 5. *Let  $\mathcal{E}$  be an ordinary elliptic curve defined over  $\mathbb{F}_p$ , and let  $G \in \mathcal{E}(\mathbb{F}_p)$  be a point of order  $T$ . Then, for arbitrary positive integers  $K, L$ , sequences  $\alpha_k$  and  $\beta_\ell$  of arbitrary complex numbers, supported on the intervals  $[1, K]$  and  $[1, L]$ , respectively, and all  $a \in \mathbb{F}_p^*$ , the following bound holds:*

$$\begin{aligned} & \sum_{\substack{k \leq K \\ \gcd(k, T) = 1}} \sum_{\substack{\ell \leq L \\ \gcd(\ell, T) = 1}} \alpha_k \beta_\ell \mathbf{e}_p(a\mathbf{x}(k^{-1}\ell^{-1}G)) \\ & \ll AB T^{5/6} (KT^{-1/2} + K^{1/2})(LT^{-1/2} + L^{1/2}) p^{1/12+o(1)}, \end{aligned}$$

where

$$A = \max_{1 \leq k \leq K} |\alpha_k| \quad \text{and} \quad B = \max_{1 \leq \ell \leq L} |\beta_\ell|.$$

*Proof.* We split the sum into at most  $(K/T + 1)(L/T + 1)$  double sums with at most  $\min\{K, T\} \min\{L, T\}$  terms obtaining from Lemma 4 that

$$\begin{aligned} & \sum_{\substack{k \leq K \\ \gcd(k, T) = 1}} \sum_{\substack{\ell \leq L \\ \gcd(\ell, T) = 1}} \alpha_k \beta_\ell \mathbf{e}_p(a\mathbf{x}(k^{-1}\ell^{-1}G)) \\ & \ll AB T^{5/6} \left(\frac{K}{T} + 1\right) \left(\frac{L}{T} + 1\right) (\min\{K, T\} \min\{L, T\})^{1/2} p^{1/12+o(1)}. \end{aligned}$$

Since for any  $R > 0$

$$\max\{R, T\} (\min\{R, T\})^{1/2} = R^{1/2} T^{1/2} (\max\{R, T\})^{1/2}$$

we derive

$$\begin{aligned}
 \left(\frac{R}{T} + 1\right) \max\{R^{1/2}, T^{1/2}\} &\ll T^{-1} \max\{R, T\} \max\{R^{1/2}, T^{1/2}\} \\
 &= R^{1/2} T^{-1/2} \max\{R^{1/2}, T^{1/2}\} \\
 &\leq R^{1/2} T^{-1/2} (R^{1/2} + T^{1/2}).
 \end{aligned}$$

The result now follows.  $\square$

We now use the idea of [12] which allows us to vary the limit of summation for  $\ell$ .

LEMMA 6. *Let  $\mathcal{E}$  be an ordinary elliptic curve defined over  $\mathbb{F}_p$ , and let  $G \in \mathcal{E}(\mathbb{F}_p)$  be a point of order  $T$ . Then, for arbitrary positive integers  $K, L$ , a sequence of positive integers  $L_k$  with  $L_k \leq L$ ,  $1 \leq k \leq K$ , sequences  $\alpha_k$  and  $\beta_\ell$  of arbitrary complex numbers, supported on the intervals  $[1, K]$  and  $[1, L]$ , and all  $a \in \mathbb{F}_p^*$ , the following bound holds:*

$$\begin{aligned}
 &\sum_{\substack{k \leq K \\ \gcd(k, T) = 1}} \sum_{\substack{\ell \leq L_k \\ \gcd(\ell, T) = 1}} \alpha_k \beta_\ell \mathbf{e}_p(a\mathbf{x}(k^{-1}\ell^{-1}G)) \\
 &\ll AB T^{5/6} (KT^{-1/2} + K^{1/2})(LT^{-1/2} + L^{1/2}) p^{1/12+o(1)} \log L,
 \end{aligned}$$

where

$$A = \max_{1 \leq k \leq K} |\alpha_k| \quad \text{and} \quad B = \max_{1 \leq \ell \leq L} |\beta_\ell|.$$

*Proof.* We have

$$\begin{aligned}
 &\sum_{\ell \leq L_k} \alpha_k \beta_\ell \mathbf{e}_p(a\mathbf{x}(k^{-1}\ell^{-1}G)) \\
 &= \sum_{\substack{\ell \leq L \\ \gcd(\ell, T) = 1}} \alpha_k \beta_\ell \mathbf{e}_p(a\mathbf{x}(k^{-1}\ell^{-1}G)) \frac{1}{L} \sum_{-(L-1)/2 \leq s \leq L/2} \sum_{w \leq L_k} \mathbf{e}_L(s(\ell - w)) \\
 &= \frac{1}{L} \sum_{-(L-1)/2 \leq s \leq L/2} \sum_{w \leq L_k} \mathbf{e}_L(-sw) \\
 &\quad \times \sum_{\substack{\ell \leq L \\ \gcd(\ell, T) = 1}} \alpha_k \beta_\ell \mathbf{e}_L(s\ell) \mathbf{e}_p(a\mathbf{x}(k^{-1}\ell^{-1}G)).
 \end{aligned}$$

Since for  $|s| \leq L/2$  we have

$$\sum_{w \leq L_k} \mathbf{e}_L(sw) = \eta_{k,s} \frac{L}{|s| + 1},$$

for some complex numbers  $\eta_{k,s} \ll 1$ , see [13, Bound (8.6)], we conclude that for  $|s| \leq L/2$  and  $k \leq K$  there are some complex numbers  $\gamma'_{k,s} = \eta_{k,s} \alpha_k$

such that

$$\begin{aligned}
& \sum_{\substack{k \leq K \\ \gcd(k, T) = 1}} \sum_{\substack{\ell \leq L/k \\ \gcd(\ell, T) = 1}} \alpha_k \beta_\ell \mathbf{e}_p(a \mathbf{x}(k^{-1} \ell^{-1} G)) \\
&= \sum_{-(L-1)/2 \leq s \leq L/2} \frac{1}{|s| + 1} \\
&\quad \times \sum_{\substack{k \leq K \\ \gcd(k, T) = 1}} \sum_{\substack{\ell \leq L \\ \gcd(\ell, T) = 1}} \gamma_{k, s} \beta_\ell \mathbf{e}_L(s \ell) \mathbf{e}_p(a \mathbf{x}(k^{-1} \ell^{-1} G)).
\end{aligned}$$

Using Corollary 5, we derive the desired result.  $\square$

Finally, we are ready to derive our main technical tool of this section.

LEMMA 7. *Let  $\mathcal{E}$  be an ordinary elliptic curve defined over  $\mathbb{F}_p$ , and let  $G \in \mathcal{E}(\mathbb{F}_p)$  be a point of order  $T$ . Then, for arbitrary positive integers  $H, K, L$ , sequences  $\alpha_k$  and  $\beta_\ell$  of arbitrary complex numbers, supported on the intervals  $[1, K]$  and  $[1, L]$ , and all  $a \in \mathbb{F}_p^*$ , the following bound holds:*

$$\begin{aligned}
& \sum_{\substack{H \leq k \leq K \\ \gcd(k, T) = 1}} \sum_{\substack{\ell \leq L/k \\ \gcd(\ell, T) = 1}} \alpha_k \beta_\ell \mathbf{e}_p(a \mathbf{x}(k^{-1} \ell^{-1} G)) \\
&\ll AB T^{5/6} \left( \frac{L}{T} + \frac{K^{1/2} L^{1/2}}{T^{1/2}} + \frac{L}{H^{1/2} T^{1/2}} + L^{1/2} \right) p^{1/12 + o(1)} (KL)^{o(1)},
\end{aligned}$$

where

$$A = \max_{1 \leq k \leq K} |\alpha_k| \quad \text{and} \quad B = \max_{1 \leq \ell \leq L} |\beta_\ell|.$$

*Proof.* Defining some values of  $\alpha_k$  as zeros we write

$$\begin{aligned}
& \sum_{\substack{H \leq k \leq K \\ \gcd(k, T) = 1}} \sum_{\substack{\ell \leq L/k \\ \gcd(\ell, T) = 1}} \alpha_k \beta_\ell \mathbf{e}_p(a \mathbf{x}(k^{-1} \ell^{-1} G)) \\
&= \sum_{j=I}^J \sum_{e^j \leq k \leq e^{j+1}} \sum_{\substack{\ell \leq L/k \\ \gcd(\ell, T) = 1}} \alpha_k \beta_\ell \mathbf{e}_p(a \mathbf{x}(k^{-1} \ell^{-1} G)),
\end{aligned}$$

where  $I = \lfloor \log H \rfloor$  and  $J = \lfloor \log K \rfloor$ . So, by Lemma 6, we get

$$\begin{aligned}
& \sum_{\substack{H \leq k \leq K \\ \gcd(k, T) = 1}} \sum_{\substack{\ell \leq L/k \\ \gcd(\ell, T) = 1}} \alpha_k \beta_\ell \mathbf{e}_p(a \mathbf{x}(k^{-1} \ell^{-1} G)) \\
&\ll AB T^{5/6} p^{1/12 + o(1)} \log L \sum_{j=I}^J (LT^{-1} + L^{1/2} e^{j/2} T^{-1/2} \\
&\quad + Le^{-j/2} T^{-1/2} + L^{1/2}) \\
&\leq AB T^{5/6} p^{1/12 + o(1)} \log L (JLT^{-1} + e^{J/2} L^{1/2} T^{-1/2} \\
&\quad + Le^{-I/2} T^{-1/2} + JL^{1/2}).
\end{aligned}$$

Since  $H \ll e^I \leq e^J \ll K$ , we immediately obtain the desired result.  $\square$

### §3. Main results.

3.1. *Sums over primes.* In this subsection, we combine Lemma 1 with the bounds of Corollary 3 and Lemma 4 to estimate the sums  $S_a(N)$  defined by (1).

**THEOREM 8.** *Let  $\mathcal{E}$  be an ordinary elliptic curve defined over  $\mathbb{F}_p$ , and let  $G \in \mathcal{E}(\mathbb{F}_p)$  be a point of order  $T$ . Then, for every  $a \in \mathbb{F}_p^*$  and integers  $r, s \geq 2$ , we have*

$$\left| \sum_{\substack{n \leq N \\ \gcd(n, T) = 1}} \Lambda(n) \mathbf{e}_p(ax(n^{-1}G)) \right| \leq (N\Delta + N^{5/6}T^{1/3}p^{1/12})N^{o(1)},$$

where

$$\Delta = T^{-1/2(4r+s)+1/4rs} p^{1/4(4r+s)}.$$

*Proof.* We remark that the result is trivial if  $T \leq p^{1/2}$  or  $N \leq T^{7/3}$ . Hence we can always assume that

$$T > p^{1/2} \quad \text{and} \quad N \geq T^{7/3} \geq p. \quad (6)$$

Let  $U, V > 1$  with  $UV \leq N$  and apply Lemma 1 with the function  $f(n) = \mathbf{e}_p(ax(n^{-1}G))$ . By the prime number theorem, we have

$$\Sigma_1 = \left| \sum_{\substack{n \leq U \\ \gcd(n, T) = 1}} \Lambda(n) f(n) \right| \leq \sum_{n \leq U} \Lambda(n) \ll U. \quad (7)$$

We now write

$$\Sigma_2 = \Sigma_{2,1} + \Sigma_{2,2}$$

where

$$\begin{aligned} \Sigma_{2,1} &= (\log UV) \sum_{\substack{k \leq N/T \\ \gcd(k, T) = 1}} \left| \sum_{\substack{\ell \leq N/k \\ \gcd(\ell, T) = 1}} \mathbf{e}_p(ax(k^{-1}\ell^{-1}G)) \right| \\ \Sigma_{2,2} &= (\log UV) \sum_{\substack{N/T \leq k \leq UV \\ \gcd(k, T) = 1}} \left| \sum_{\substack{\ell \leq N/k \\ \gcd(\ell, T) = 1}} \mathbf{e}_p(ax(k^{-1}\ell^{-1}G)) \right|. \end{aligned}$$

Next, for any  $k \geq 1$  with  $\gcd(k, T) = 1$  the point  $kG$  has also order  $T$  in  $\mathcal{E}(\mathbb{F}_p)$ ; thus Corollary 3 provides the bound

$$\begin{aligned} \Sigma_{2,1} &= (\log UV) \sum_{\substack{k \leq N/T \\ \gcd(k, T) = 1}} \left| \sum_{\substack{\ell \leq N/k \\ \gcd(\ell, T) = 1}} \mathbf{e}_p(ax(k^{-1}\ell^{-1}G)) \right| \\ &\leq N^{o(1)} \sum_{k \leq N/T} \left( \frac{N}{kT} + 1 \right) T\Delta \\ &\leq N^{1+o(1)} \Delta \sum_{k \leq N/T} \frac{1}{k} \\ &= N^{1+o(1)} \Delta. \end{aligned}$$

To estimate  $\Sigma_{2,2}$  we use Lemma 7:

$$\Sigma_{2,2} \leq T^{5/6}(NT^{-1} + N^{1/2}U^{1/2}V^{1/2}T^{-1/2} + N^{1/2})p^{1/12}N^{o(1)}.$$

Since under the conditions (6)

$$\Delta \geq T^{-1/2(4r+s)}p^{1/4(4r+s)} \geq T^{-1/6}p^{1/12} \quad \text{and} \quad NT^{-1} \geq N^{1/2}$$

for  $T \geq p^{1/2}$  and  $r, s \geq 2$ , we derive

$$\Sigma_2 \leq N^{1+o(1)}\Delta + N^{1/2+o(1)}U^{1/2}V^{1/2}T^{1/3}p^{1/12}. \quad (8)$$

Similar to the estimate of  $\Sigma_{2,1}$ , we also have

$$\Sigma_3 \leq N^{o(1)} \sum_{k \leq V} \left( \frac{N}{kT} + 1 \right) T \Delta.$$

Thus

$$\Sigma_3 \leq (N + VT)\Delta N^{o(1)}. \quad (9)$$

We now turn to the estimate of  $\Sigma_4$ . For every positive integer  $k$  let

$$A(k) = \left| \sum_{d|k, d \leq V} \mu(d) \right|.$$

Since  $k, \ell \leq N$ , we have

$$A(k) \leq \tau(k) \ll N^{o(1)} \quad \text{and} \quad \Lambda(\ell) \leq \log \ell \leq N^{o(1)},$$

where  $\tau(k)$  is the number of integer positive divisors of  $k$ .

Then,

$$\begin{aligned} \Sigma_4 &= \left| \sum_{\substack{k\ell \leq N, \gcd(k\ell, T)=1 \\ k > V, \ell > U}} A(k)\Lambda(\ell)\mathbf{e}_p(\mathbf{a}\mathbf{x}(k^{-1}\ell^{-1}G)) \right| \\ &= \left| \sum_{\substack{V \leq k \leq N/U \\ \gcd(k, T)=1}} \sum_{\substack{U \leq \ell \leq N/k \\ \gcd(\ell, T)=1}} A(k)\Lambda(\ell)\mathbf{e}_p(\mathbf{a}\mathbf{x}(k^{-1}\ell^{-1}G)) \right|. \end{aligned}$$

Applying Lemma 7 we derive

$$\begin{aligned} \Sigma_4 &\leq T^{5/6}(NT^{-1} + NT^{-1/2}U^{-1/2} + NT^{-1/2}V^{-1/2} + N^{1/2})p^{1/12}N^{o(1)} \\ &\leq T^{5/6}(NT^{-1} + NT^{-1/2}U^{-1/2} + NT^{-1/2}V^{-1/2})p^{1/12}N^{o(1)} \quad (10) \end{aligned}$$

since, as we have noticed,  $NT^{-1} \geq N^{1/2}$  under the conditions (6).

Combining (7), (8), (9) and (10), and recalling that  $\Delta \geq T^{-1/6}p^{1/12}$ , we find that

$$\left| \sum_{n \leq N} \Lambda(n)\mathbf{e}_p(\mathbf{a}\mathbf{x}(nG)) \right| \leq U + (N + VT)\Delta N^{o(1)} + (\Psi_1 + \Psi_2 + \Psi_3)N^{o(1)},$$



where

$$\begin{aligned}\Psi_1 &= N^{1/2}T^{1/3}U^{1/2}V^{1/2}p^{1/12}, \\ \Psi_2 &= NT^{1/3}U^{-1/2}p^{1/12}, \\ \Psi_3 &= NT^{1/3}V^{-1/2}p^{1/12}.\end{aligned}$$

Choosing  $U = V = N^{1/3}$ , we obtain

$$\left| \sum_{n \leq N} \Lambda(n) \mathbf{e}_p(a\mathbf{x}(nG)) \right| \leq (N + N^{1/3}T)\Delta + N^{5/6+o(1)}T^{1/2}p^{1/12}.$$

Since under the conditions (6) we have

$$N \geq N^{1/3}T \quad \text{and} \quad N^{5/6}T^{-1/2} \geq N^{1/2},$$

the desired result follows.  $\square$

Using partial summation we immediately derive the following corollary from Theorem 8.

**COROLLARY 9.** *Let  $\mathcal{E}$  be an ordinary elliptic curve defined over  $\mathbb{F}_p$ , and let  $G \in \mathcal{E}(\mathbb{F}_p)$  be a point of order  $T$ . Then, for every  $a \in \mathbb{F}_p^*$  and integers  $r, s \geq 2$ , we have*

$$|S_a(N)| \leq (N\Delta + N^{5/6}T^{1/3}p^{1/12})N^{o(1)},$$

where

$$\Delta = T^{-1/2(4r+s)+1/4rs} p^{1/4(4r+s)}.$$

We see that if  $T = p^{1+o(1)}$  then taking  $r = 4$  and  $s = 16$  in Corollary 9 we derive the bound (2).

Furthermore, if  $N \geq T^2 p^{1/2+\varepsilon}$  and  $T \geq p^{1/2+\varepsilon}$  for some fixed  $\varepsilon > 0$  then taking sufficiently large  $r = s$  we obtain

$$|S_a(N)| \leq N^{1+o(1)} p^{-\delta}$$

where  $\delta > 0$  depends only on  $\varepsilon$ .

**3.2. Congruences with primes.** Here we study the congruence (4). As we have mentioned, we only consider the case in which the bound (3) applies.

Let  $\pi(N)$  be the number of primes  $q \leq N$  as usual.

**THEOREM 10.** *Let  $\mathcal{E}$  be an ordinary elliptic curve defined over  $\mathbb{F}_p$ , and let  $G \in \mathcal{E}(\mathbb{F}_p)$  be a point of order  $T = p^{1+o(1)}$ . Then, for  $N \geq p^{323/128}$  and any fixed integer  $k \geq 3$  the congruence (4) has*

$$R_k(N, c) = \frac{1}{p} \pi(N)^k + O(N^{k+o(1)} p^{-1-(k-2)/256})$$

solutions.

*Proof.* We recall that for any integer  $m \geq 1$  we have the identity

$$\frac{1}{m} \sum_{\lambda \in \mathbb{Z}_m} \mathbf{e}_m(\lambda v) = \begin{cases} 1 & \text{if } v \equiv 0 \pmod{m}, \\ 0 & \text{if } v \not\equiv 0 \pmod{m}. \end{cases}$$

Therefore, for any integer  $h$ , the number of solutions to the congruence (4) can be written as

$$\begin{aligned} R_k(N, c) &= \frac{1}{p} \sum_{\substack{q_1, \dots, q_k \leq N \\ q_i \text{ prime} \\ \gcd(q_i, T)=1}} \sum_{\lambda \in \mathbb{F}_p} \mathbf{e}_p \left( \lambda \left( \sum_{j=1}^k x(q_j^{-1}G) - c \right) \right) \\ &= \frac{1}{p} \sum_{\lambda \in \mathbb{F}_p} \mathbf{e}_p(-\lambda c) \prod_{j=1}^k \sum_{\substack{q_j \leq N \\ q_j \text{ prime} \\ \gcd(q_j, T)=1}} \mathbf{e}_p(\lambda x(q_j^{-1}G)) \\ &= \frac{1}{p} \sum_{\lambda \in \mathbb{F}_p} \mathbf{e}_p(-\lambda c) S_\lambda(N)^k. \end{aligned}$$

Separating the term  $\pi(N)^k/p$  corresponding to  $\lambda = 0$  we obtain

$$\left| R_k(N, c) - \frac{1}{p} \pi(N)^k \right| \leq \frac{1}{p} \sum_{\lambda \in \mathbb{F}_p^*} |S_\lambda(N)|^k. \quad (11)$$

Since under the conditions of the theorem the bound (3) holds, we derive

$$\begin{aligned} \sum_{\lambda \in \mathbb{F}_p^*} |S_\lambda(N)|^k &\leq (N^{1+o(1)} p^{-1/256})^{k-2} \sum_{\lambda \in \mathbb{F}_p^*} |S_\lambda(N)|^2 \\ &\leq (N^{1+o(1)} p^{-1/256})^{k-2} \sum_{\lambda \in \mathbb{F}_p} |S_\lambda(N)|^2 = (N^{1+o(1)} p^{-1/256})^{k-2} pW, \end{aligned}$$

where  $W$  is the number of solutions to the congruence

$$x(q_1^{-1}G) \equiv x(q_2^{-1}G) \pmod{p}.$$

For every prime  $q_1 \leq N$  we have at most  $2(N/T + 1) = Np^{-1+o(1)}$  possibilities for  $q_2$ , thus  $W \ll N^2 p^{-1+o(1)}$ , from where we obtain

$$\sum_{\lambda \in \mathbb{F}_p^*} |S_\lambda(N)|^k \leq N^{k+o(1)} p^{-(k-2)/256} \quad (12)$$

and after the substitution in (11) the desired result follows.  $\square$

One can easily see that under the conditions of Theorem 10 we have the asymptotic formula  $R_k(N, c) = (1 + o(1))\pi(N)^k/p$  for any  $k \geq 3$ . We now consider the moments

$$M_{k,v}(N) = \sum_{c \in \mathbb{F}_p} \left| R_k(N, c) - \frac{1}{p} \pi(N)^k \right|^{2v},$$

for which we obtain a non-trivial estimate starting with  $k = 2$ .

**THEOREM 11.** *Let  $\mathcal{E}$  be an ordinary elliptic curve defined over  $\mathbb{F}_p$ , and let  $G \in \mathcal{E}(\mathbb{F}_p)$  be a point of order  $T = p^{1+o(1)}$ . Then, for  $N \geq p^{323/128}$  and any fixed integers  $k \geq 2$ ,  $v \geq 1$  we have*

$$M_{k,v}(N) \leq N^{2kv+o(1)} p^{-2v+1-(kv-2v+1)/128}.$$

*Proof.* As in the proof of Theorem 10, we have

$$R_k(N, c) - \frac{1}{p} \pi(N)^k = \frac{1}{p} \sum_{\lambda \in \mathbb{F}_p^*} \mathbf{e}_p(-\lambda c) S_\lambda(N)^k.$$

Therefore,

$$\begin{aligned} M_{k,v}(N) &= \frac{1}{p^{2v}} \sum_{c \in \mathbb{F}_p} \sum_{\lambda_1, \dots, \lambda_{2v} \in \mathbb{F}_p^*} \mathbf{e}_p(-(\lambda_1 + \dots + \lambda_{2v})c) \\ &\quad \times S_{\lambda_1}(N)^k \dots S_{\lambda_{2v}}(N)^k. \end{aligned}$$

Thus we obtain

$$M_{k,v}(N) = \frac{1}{p^{2v-1}} \sum_{\substack{\lambda_1, \dots, \lambda_{2v} \in \mathbb{F}_p^* \\ \lambda_1 + \dots + \lambda_{2v} = 0}} S_{\lambda_1}(N)^k \dots S_{\lambda_{2v-1}}(N)^k S_{\lambda_{2v}}(N)^k.$$

Since under the conditions of the theorem the bound (3) holds for every sum  $S_{\lambda_j}(N)$  above, which we apply to  $S_{\lambda_{2v}}(N)$ , we derive

$$\begin{aligned} M_{k,v}(N) &\leq \frac{1}{p^{2v-1}} (N^{1+o(1)} p^{-1/256})^k \sum_{\substack{\lambda_1, \dots, \lambda_{2v-1} \in \mathbb{F}_p^* \\ \lambda_1 + \dots + \lambda_{2v-1} \neq 0}} |S_{\lambda_1}(N)|^k \dots |S_{\lambda_{2v-1}}(N)|^k \\ &\leq \frac{1}{p^{2v-1}} (N^{1+o(1)} p^{-1/256})^k \sum_{\lambda_1, \dots, \lambda_{2v-1} \in \mathbb{F}_p^*} |S_{\lambda_1}(N)|^k \dots |S_{\lambda_{2v-1}}(N)|^k \\ &= \frac{1}{p^{2v-1}} (N^{1+o(1)} p^{-1/256})^k \left( \sum_{\lambda \in \mathbb{F}_p^*} |S_\lambda(N)|^k \right)^{2v-1}. \end{aligned}$$

Using (12) we conclude the proof.  $\square$

Theorem 11 (taken with  $k = 2$  and, say,  $v = 1$ ) immediately implies that under the same conditions we have  $R_2(N, c) > 0$  for all but at most  $p^{127/128+o(1)}$  elements  $c \in \mathbb{F}_p$ .

§4. *Comments.* As we have noticed, the proof of [2, Theorem 6] contains a gap as the double sums which appear in the proof are sometimes over sets which are not subsets of  $\mathbb{Z}_T$  and thus [2, Theorem 6] does not apply. However using the estimate

$$\left| \sum_{H_1 < n \leq H_2} \mathbf{e}_p(ax(nG)) \right| \leq \left( \frac{H_2 - H_1}{T} + 1 \right) p^{1/2+o(1)}, \quad (13)$$

see [2, Lemma 5], instead of Corollary 3, and also a full analogue of Lemma 7 with  $k\ell$  instead of  $k^{-1}\ell^{-1}$ , one can easily derive the following analogue of the estimate of Corollary 9 for the sums  $T_a(N)$  given by (5): for every  $a \in \mathbb{F}_p^*$  we have

$$|T_a(N)| \leq N^{1+o(1)} T^{-1} p^{1/2} + N^{5/6+o(1)} T^{1/3} p^{1/12}.$$

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### References

1. R. Avanzi, H. Cohen, C. Doche, G. Frey, T. Lange, K. Nguyen and F. Vercauteren, *Elliptic and Hyperelliptic Curve Cryptography: Theory and Practice*, CRC Press (Boca Raton, FL, 2005).
2. W. D. Banks, J. B. Friedlander, M. Z. Garaev and I. E. Shparlinski, Double character sums over elliptic curves and finite fields. *Pure Appl. Math. Q.* **2** (2006), 179–197.
3. I. Blake, G. Seroussi and N. Smart, *Elliptic Curves in Cryptography (London Mathematical Society Lecture Note Series 265)*, Cambridge University Press (Cambridge, MA, 1999).
4. E. Bombieri, On exponential sums in finite fields. *Amer. J. Math.* **88** (1966), 71–105.
5. J. Bourgain, More on the sum–product phenomenon in prime fields and its applications. *Int. J. Number Theory* **1** (2005), 1–32.
6. J. Bourgain and I. E. Shparlinski, Distribution of consecutive modular roots of an integer. *Acta Arith.* **134** (2008), 83–91.
7. Z. Chen, Elliptic curve analogue of Legendre sequences. *Monatsh. Math.* **154** (2008), 1–10.
8. H. Davenport, *Multiplicative Number Theory*, 2nd edn., Springer (New York, 1980).
9. E. El Mahassni and I. E. Shparlinski, On the distribution of the elliptic curve power generator. In *Proceedings of the 8th International Conference on Finite Fields and Applications (Contemporary Mathematics 461)*, American Mathematical Society (Providence, RI, 2008), 111–119.
10. R. R. Farashahi and I. E. Shparlinski, Pseudorandom bits from points on elliptic curves. *Preprint*, 2009.
11. E. Fouvry and P. Michel, Sur certaines sommes d'exponentielles sur les nombres premiers. *Ann. Sci. Éc. Norm. Supér.* (4) **31** (1998), 93–130.
12. M. Z. Garaev, An estimate of Kloosterman sums with prime numbers and an application. *Mat. Zametki* **88**(3) (2010), 365–373 (in Russian).
13. H. Iwaniec and E. Kowalski, *Analytic Number Theory*, American Mathematical Society (Providence, RI, 2004).
14. D. R. Kohel and I. E. Shparlinski, Exponential sums and group generators for elliptic curves over finite fields. In *Proceedings of the 4th Algorithmic Number Theory Symposium (Lecture Notes in Computer Science 1838)*, Springer (Berlin, 2000), 395–404.
15. T. Lange and I. E. Shparlinski, Certain exponential sums and random walks on elliptic curves. *Canad. J. Math.* **57** (2005), 338–350.
16. T. Lange and I. E. Shparlinski, Distribution of some sequences of points on elliptic curves. *J. Math. Cryptol.* **1** (2007), 1–11.
17. A. Ostafe and I. E. Shparlinski, Twisted exponential sums over points of elliptic curves. *Acta Arith.* **148** (2011), 77–92.
18. I. E. Shparlinski, Bilinear character sums over elliptic curves. *Finite Fields Appl.* **14** (2008), 132–141.
19. I. E. Shparlinski, Pseudorandom number generators from elliptic curves. In *Recent Trends in Cryptography (Contemporary Mathematics 477)*, American Mathematical Society (Providence, RI, 2009), 121–141.
20. I. E. Shparlinski, Some special character sums over elliptic curves. *Bol. Soc. Mat. Mexicana* (3) **15** (2009), 37–40.
21. J. H. Silverman, *The Arithmetic of Elliptic Curves*, Springer (Berlin, 1995).

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