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PORTFOLIO SELECTION IN THE ENLARGED MARKOVIAN REGIME-SWITCHING MARKET*

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Abstract. We study a portfolio selection problem in a continuous-time Markovian regime-switching model. The market in this model is, in general, incomplete. We adopt a method to complete the market based on an enlargement of the market using a set of geometric Markovian jump securities. We solve the portfolio selection problem in the enlarged market for a power utility and a logarithmic utility. Closed-form solutions for the optimal portfolio strategies and the value functions are obtained in both cases. We also establish the relationship between the optimal portfolio problems in the enlarged market and the original market.

Key words. portfolio optimization, Markovian regime-switching market, enlargement of market, geometric Markovian jump securities, dynamic programming, Hamilton–Jacobi–Bellman equations

AMS subject classifications. 91B28, 91B16, 93E20

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1. Introduction.

Portfolio selection is an important topic in both the theory and practice of modern banking and finance. Early work in this area can be traced to the work of Markowitz [22], which pioneered the use of mathematical or quantitative methods to formulate and investigate portfolio selection. Markowitz considered a single-period model and formulated the portfolio selection problem as a mean-variance portfolio optimization problem. Merton [23, 24] first considered an intertemporal framework for optimal portfolio selection in a continuous-time economy and explored the use of the stochastic optimal control theory to derive a closed-form solution to the optimal portfolio selection problem. Under the Merton model, the price process of the ordinary share is governed by a geometric Brownian motion. However, numerous empirical studies reveal that this assumption cannot provide a realistic description for the actual behavior of price dynamics.

Regime-switching models seem a good candidate for modeling price processes of risky assets. Hamilton [19] pioneered econometric applications of regime-switching models. In such models, one set of model parameters is in force at a particular time according to the state of an economy at that time. The set of model parameters will change to another set when there is a transition in the state of the economy, which is usually described by a Markov chain. Hence, regime-switching models can describe structural changes in (macro) economic conditions or different stages of business cycles. Regime-switching models have diverse applications in finance. Some works include Elliott, Hunter, and Jamieson [14], Elliott, Malcolm, and Tsoi [16], Elliott,

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Chan, and Siu [13], Guo [18], and Buffington and Elliott [5]. Recently, application of regime-switching models in portfolio selection has received considerable attention, (see, for example, Zhou and Yin [33], Rieder and Bäuerle [29], Yin and Zhou [31], Bäuerle and Rieder [2], and Jang et al. [20]).

In this paper, we introduce a novel approach to investigate the portfolio selection problem in a continuous-time Markovian regime-switching economy based on an enlargement of market. We allow the flexibility that the key market parameters, including the market interest rate of a fixed interest security, the appreciation rate, and the volatility of an ordinary share, are modulated by an observable, continuous-time and finite-state Markov chain whose states represent different states of an economy. Despite the flexibility of the model, the market in the model is incomplete. Here we complete the Markovian regime-switching market by enlarging the market with a set of geometric Markovian jump securities, whose price processes are driven by jump martingales associated with the Markov chain. We consider two utility functions in the portfolio selection problem, namely, a logarithmic utility and a power utility. We are able to obtain closed-form solutions to the optimal portfolio strategies and the value functions in both cases. For the logarithmic utility, a direct differentiation approach is used to derive a closed-form solution to the optimal portfolio strategy. For the power utility, we adopt the dynamic programming approach to derive a closed-form solution to the HJB equation. In both cases, closed-form solutions to the value functions are obtained. We also establish the relationship between the optimal portfolio strategies in the enlarged market and those in the original market. This relationship is consistent with that in some existing literature, such as Karatzas et al. [21].

The geometric Markovian jump securities are a new class of general securities and entail different interpretations from both theoretical and practical perspectives. From a practical perspective, these securities can be interpreted as an alternative class of assets which have different risk and return profiles than some traditional assets, such as a fixed interest security and an ordinary share. For example, these securities can be proxies of some traded assets, such as energy products, commodities and properties, whose price dynamics are closely related to transitions of economic states, or business cycles, (see for example, Culot et al. [8] and Blochlinger [3]). Consequently these securities may be applied to hedge against the uncertainty due to the stochastic variation of investment opportunity set. The idea of using Markov chain model for asset prices was considered by some authors (see, for example, Duffie [9], Norberg [27], and Elliott and Kopp [15]). Indeed, the geometric Markovian jump securities can be related to the securities in the Markov chain market of Norberg [27]. From a theoretical perspective, the geometric Markovian jump securities may be interpreted as a kind of "fictitious" assets, which are used to complete the market. Indeed, the idea of "fictitious" assets to complete markets was considered in Karatzas et al. [21], Guo [18], Corcuera et al. [6], Corcuera, Nualart, and Schoutens [7], and Niu [26]. This idea may have its origin in the Arrow–Debreu securities, which are the foundation of a risk-neutral approach to asset pricing (see, for example, Duffie [9]).

The paper is structured as follows. In the next section, we present the price dynamics in the Markovian regime-switching economy. Then we describe a set of geometric Markovian jump securities and the enlarged market augmented by these securities. In section 3, we show that the enlarged market is arbitrage-free and complete. Section 4 presents the portfolio optimization problem and solves the problem for a power utility and a logarithmic utility. Section 5 gives the relationship between the optimal portfolio strategies in the enlarged market and the original market. The final section summarizes the paper.

2. Enlarged Markovian regime-switching Black–Scholes–Merton market. In this section, we first present a Markovian regime-switching Black–Scholes–Merton market. Then we employ the technique of enlargement of market to complete market.

2.1. Markovian regime-switching Black–Scholes–Merton market. We consider a continuous-time financial market with a fixed interest security and an ordinary share. These primitive securities are tradeable continuously over time on a finite horizon $\mathcal{T} := [0, T]$, where $T < \infty$. To describe uncertainty, we employ a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$, where $\mathcal{P}$ is a real-world probability measure.

First, we describe the evolution of the state of an economy over time by a continuous-time, finite-state, Markov chain $\mathbf{X} := \{\mathbf{X}(t) | t \in \mathcal{T}\}$ defined on $(\Omega, \mathcal{F}, \mathcal{P})$ with a finite state space $\mathcal{S} := \{\mathbf{s}_1, \mathbf{s}_2, \dots, \mathbf{s}_N\}$. We impose the technical conditions that the chain $\mathbf{X}$ is right-continuous and irreducible. Here the chain $\mathbf{X}$ is observable, and its states are interpreted as proxies of the values taken by some observable (macro) economic indicators, such as sovereign credit ratings, the gross domestic product, retail price index, and others.

Following Elliott, Aggoun, and Moore [12], we identify the state space of the chain as a finite set of unit vectors $E := \{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_N\}$ without loss of generality, where $\mathbf{e}_i \in \Re^N$ and the jth component of $\mathbf{e}_i$ is the Kronecker delta δ_{ij} for each $i, j = 1, 2, \dots, N$. The set E is called the canonical state space of the chain. To specify statistical properties of the chain, we define the generator $\mathbf{\Lambda} := [\lambda_{ij}]_{i,j=1,2,\dots,N}$ of the chain $\mathbf{X}$ under $\mathcal{P}$. Here, for each $i, j = 1, 2, \dots, N$, λ_{ij} is the constant intensity of the transition of the chain $\mathbf{X}$ from state $\mathbf{e}_i$ to state $\mathbf{e}_j$ at time t. Note that $\lambda_{ij} \geq 0$, for $i \neq j$ and $\sum_{j=1}^{N} \lambda_{ij} = 0$, so $\lambda_{ii} \leq 0$. Here for each $i, j = 1, 2, \dots, N$ with $i \neq j$, we suppose that $\lambda_{ij} > 0$. Throughout this paper, we denote by $\boldsymbol{y}'$ the transpose of a matrix (or in particular a vector) $\mathbf{y}$.

We now describe the price dynamics of the primitive securities. Let $r(t)$ be the instantaneous interest rate of the fixed interest security at time t for each $t \in \mathcal{T}$. Then the chain determines the interest rate as

$$r(t) = \langle \mathbf{r}, \mathbf{X}(t) \rangle ,$$

where $\mathbf{r} := (r_1, r_2, \dots, r_N)' \in \Re^N$ with $r_i > 0$ for each $i = 1, 2, \dots, N$; $\langle \cdot, \cdot \rangle$ is the scalar product in $\Re^N$. The scalar product is introduced here to select which component of the vector $\mathbf{r}$ of interest rates is in force at time t according to the state of the economy $\mathbf{X}(t)$ at that time. Consequently, the evolution of the price process of the fixed interest security over time follows

$$B(t) = \exp\left(\int_0^t r(u) du \right), \quad t \in \mathcal{T}, \quad B(0) = 1.$$

The chain $\mathbf{X}$ also determines the appreciation rate $\mu_0(t)$ and the volatility $\sigma_0(t)$ of the ordinary share, respectively, as

$$\mu_0(t) = \langle \boldsymbol{\mu}_0, \mathbf{X}(t) \rangle , \quad \sigma_0(t) = \langle \boldsymbol{\sigma}_0, \mathbf{X}(t) \rangle ,$$

where $\boldsymbol{\mu}_0 := (\mu_0^1, \mu_0^2, \dots, \mu_0^N)' \in \Re^N$, $\boldsymbol{\sigma}_0 := (\sigma_0^1, \sigma_0^2, \dots, \sigma_0^N)' \in \Re^N$, $\mu_0^i > r_i$, and $\sigma_0^i > 0$ for each $i = 1, 2, \dots, N$. The condition "$\mu_0^i > r_i$" is required to avoid arbitrage opportunities in the market; μ_0^i and σ_0^i are the appreciation rate and the volatility of the ordinary share, respectively, when the economy is in the ith state. We assume that μ_i's and σ_i's are all distinct.

Let $W_0 := \{W_0(t)|t \in \mathcal{T}\}$ be a standard Brownian motion on $(\Omega, \mathcal{F}, \mathcal{P})$ with respect to the $\mathcal{P}$-augmentation of its natural filtration. To simplify the analysis, we suppose that W_0 and $\mathbf{X}$ are stochastically independent.[1] We assume that the evolution of the price process of the ordinary share over time is governed by the following Markovian, regime-switching, geometric Brownian motion:

$$dS_0(t) = \mu_0(t)S_0(t)dt + \sigma_0(t)S_0(t)dW_0(t), \quad S_0(0) = s > 0.$$

Note that the volatility $\sigma_0(t)$ at time t is completely identified by the (local) quadratic variation of the logarithmic price of the ordinary share $S(t)$ in any small duration prior to time t. So, when σ_i's are all distinct, it is not unreasonable to assume that the chain $\mathbf{X}$ is observable. For detail, interested readers may refer to Guo [18].

2.2. Enlarging the Markovian regime-switching Black–Scholes–Merton market. We employ a technique similar to Zhang, Siu, and Guo [32] to complete the market. The theoretical basis of the technique in Zhang, Siu, and Guo [32] lies on the representation of double martingales developed by Elliott [10]. Indeed, this representation is used in Zhang, Siu, and Guo [32] to justify the completeness of the augmented market after introducing the set of jth Markovian jump securities. One potential problem of the Markovian jump securities in Zhang, Siu, and Guo [32] is that it can take negative values with positive probability. To articulate this problem, we introduce a set of N *geometric Markovian jump securities*, which always take nonnegative values and will be defined precisely later. We use these securities to enlarge the Markovian, regime-switching, Black–Scholes–Merton market.

First, we represent the Markov chain $\mathbf{X}$ as a family of marked point processes, denoted by $\Phi_j := \{\Phi_j(t)|t \in \mathcal{T}\}$, for $j = 1, 2, \dots, N$. Let $(E, \mathcal{E})$ denote a finite marked space, where $E = \{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_N\}$ and $\mathcal{E}$ is the power set of E; that is, $\mathcal{E} := 2^E$. Let $\{T_n|n = 1, 2, \dots\}$ denote the jump epochs of the chain $\mathbf{X}$, where $0 \le T_1 \le T_2 \le \dots, \mathcal{P}$-almost surely. For each $n = 1, 2, \dots$, let $\mathbf{X}_n := \mathbf{X}(T_n)$. Write $\Phi_j(t)$ for the number of jumps into state $\mathbf{e}_j$ up to and including time t; that is,

$$\Phi_j(t) := \Phi([0, t] \times \mathbf{e}_j) = \sum_{n \ge 1} 1_{\{T_n \le t, \mathbf{X}_n = \mathbf{e}_j\}},$$

where $1_{\{A\}}$ is the indicator function of an event A.

Let $\phi_j(t)$ denote the dual predictable projection of $\Phi_j(t)$. Then, it is not difficult to show that $\phi_j(t)$ is unique and that $\phi_j(t) := \int_0^t \lambda_j(s)ds$, where

$$\lambda_j(t) := \sum_{i \ne j} 1_{\{\mathbf{X}(t-) = \mathbf{e}_i\}} \lambda_{ij}. \tag{2.1}$$

Note that $\phi_j := \{\phi_j(t)|t \in \mathcal{T}\}$ is also called the compensator of the jth marked point process Φ_j. So, for each $j = 1, 2, \dots, N$, a compensated version $\bar{\Phi}_j := \{\bar{\Phi}_j(t)|t \in \mathcal{T}\}$ of the jth marked point process Φ_j is defined by

$$\bar{\Phi}_j(t) := \Phi_j(t) - \phi_j(t), \quad t \in \mathcal{T}.$$

[1]The case where W_0 and $\mathbf{X}$ are dependent is an interesting topic for further investigation. We shall pursue this direction in our future work.

For each $j = 1, 2, \ldots, N$, $\bar{\Phi}_j$ is an $(F^{\mathbf{X}}, \mathcal{P})$-martingale, where $F^{\mathbf{X}} := \{\mathcal{F}^{\mathbf{X}}(t) | t \in \mathcal{T}\}$ is the right-continuous, $\mathcal{P}$-completed filtration generated by the chain $\mathbf{X}$. $\bar{\Phi}_j$, $j = 1, 2, \ldots, N$, form the basic martingales associated with the Markov chain $\mathbf{X}$. For each $j = 1, 2, \ldots, N$, we call $\bar{\Phi}_j$ the *jth Markovian jump martingale*.

We now define the price processes of a set of N *geometric Markovian jump securities*. For each $j = 1, 2, \ldots, N$ and each $t \in \mathcal{T}$, we suppose that the appreciation rate $\mu_j(t)$ and the "volatility", or "scale" parameter, $\sigma_j(t)$ of the jth geometric Markovian jump security are determined by the chain $\mathbf{X}$ as

$$\mu_j(t) := \left\langle \boldsymbol{\mu}_j, \mathbf{X}(t) \right\rangle, \quad \sigma_j(t) := \left\langle \boldsymbol{\sigma}_j, \mathbf{X}(t) \right\rangle,$$

where $\boldsymbol{\mu}_j := (\mu_j^1, \mu_j^2, \ldots, \mu_j^N)' \in \Re^N$; $\boldsymbol{\sigma}_j := (\sigma_j^1, \sigma_j^2, \ldots, \sigma_j^N)' \in \Re^N$; μ_j^i and σ_j^i are, respectively, the appreciation rate and the "volatility" parameter of the jth geometric Markovian jump security when the economy is in the ith state for each $i, j = 1, 2, \ldots, N$. We also assume that $\mu_j^i \geq r_i$ for $i = 1, 2, \ldots, N$ and that μ_j^i's and σ_j^i's are all distinct.

For each $j = 1, 2, \ldots, N$, let $S_j := \{S_j(t) | t \in \mathcal{T}\}$ be the price process of the *jth geometric Markovian jump security*. We suppose that the evolution of S_j over time is described by the following Markovian, regime-switching, geometric jump-type process:

$$dS_j(t) = S_j(t-) \left(\mu_j(t-) dt + \sigma_j(t-) d\bar{\Phi}_j(t) \right), \quad S_j(0) = s_j. \tag{2.2}$$

Here $S_j(t) > 0$, $(l \otimes \mathcal{P})$-a.e., where l is a Lebesgue measure on $\mathcal{T}$.

Note that when $\mathbf{X}(t-) = \mathbf{e}_j$, $d\bar{\Phi}_j(t) = 0$. Therefore, $\sigma_j(t-) = \sigma_j^j$ has no impact on the value of the right side of (2.2) when $\mathbf{X}(t-) = \mathbf{e}_j$. In other words, no matter what value σ_j^j takes cannot affect the evolution of the price process $S_j(t)$ for each $j = 1, 2, \ldots, N$. So, without loss of generality, we set $\sigma_j^j = 0$, and, from now on, $\boldsymbol{\sigma}_j$ is the vector with the jth component identical to 0.

Now we summarize the price processes of the securities in the enlarged market as follows:

$$\begin{cases} dB(t) = r(t) B(t) dt, \\ dS_0(t) = S_0(t) \left(\mu_0(t) dt + \sigma_0(t) dW_0(t) \right), \\ dS_j(t) = S_j(t-) \left(\mu_j(t-) dt + \sigma_j(t-) d\bar{\Phi}_j(t) \right), \quad j = 1, 2, \ldots, N. \end{cases} \tag{2.3}$$

3. Arbitrage-free and completeness of the enlarged Markovian regime-switching Black–Scholes–Merton market.

In this section, we first show that the enlarged Markovian regime-switching market is arbitrage-free under the condition that $\mu_j^j = r_j$ for all $j = 1, 2, \ldots, N$. Then, we prove that the enlarged market is complete.

Define an enlarged σ-algebra by $\mathcal{G}(t) := \sigma\{W_0(s), \mathbf{X}(s) | s \in [0, t]\}$ and write $G := \{\mathcal{G}(t) | t \in \mathcal{T}\}$. Consider a square-integrable, $\mathcal{G}$-adapted martingale $L := \{L(t) | t \in \mathcal{T}\}$ on $(\Omega, \mathcal{F}, \mathcal{P})$ such that $L(t) > 0$, $\mathcal{P}$-a.s., for all $t \in \mathcal{T}$, and $L(T) \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathcal{P})$. Suppose $\mathcal{Q}$ is a probability measure equivalent to $\mathcal{P}$ on $\mathcal{G}(T)$ defined by putting

$$\left. \frac{d\mathcal{Q}}{d\mathcal{P}} \right|_{\mathcal{G}(T)} := L(T).$$

So, L is the G-adapted density process for the new measure $\mathcal{Q}$.

The following theorem gives a representation for the density process L.

THEOREM 3.1. *There exist G-predictable processes $\psi_j := \{\psi_j(t) | t \in \mathcal{T}\}$, $j = 0, 1, \ldots, N$, such that*

$$L(t) = 1 + \int_0^t L(s-)\psi_0(s) dW_0(t) + \sum_{j=1}^{N} \int_0^t L(s-)\psi_j(s) d\bar{\Phi}_j(s), \quad t \in \mathcal{T}. \tag{3.1}$$

Proof. The result follows from Theorem 4.15 of Elliott [10] and the fact that $L(t) > 0$, $\mathcal{P}$-a.s. A detailed proof is given in a longer version of the paper, which is available on request. ∎

Suppose $\mathcal{Q}$ is equivalent to $\mathcal{P}$ and the density process is given by (3.1). From Theorem 3.1, we see that to find $\mathcal{Q}$, we first need to determine $\psi_j(s)$, $j = 0, 1, 2, \ldots, N$. By a generalized version of Girsanov's theorem for jump-diffusion processes (see, for example, Elliott [11]), we obtain that under $\mathcal{Q}$ the process

$$W_0^{\mathcal{Q}}(t) := W_0(t) - \int_0^t \psi_0(s) ds, \quad t \in \mathcal{T},$$

is a standard Brownian motion with respect to the filtration G, and for $j = 1, 2, \ldots, N$,

$$\bar{\Phi}_j^{\mathcal{Q}}(t) := \Phi_j(t) - \int_0^t (1 + \psi_j(s))\phi_j(ds), \quad t \in \mathcal{T},$$

are $(G, \mathcal{Q})$-martingales.

Consequently, under $\mathcal{Q}$ the price processes of the securities in the enlarged Markovian regime-switching market can be represented as

$$\begin{aligned}
dB(t) &= r(t)B(t)dt, \\
dS_0(t) &= S_0(t)\left([\mu_0(t) + \sigma_0(t)\psi_0(t)]dt + \sigma_0(t)dW_0^{\mathcal{Q}}(t)\right), \\
dS_j(t) &= S_j(t-)\left([\mu_j(t-) + \sigma_j(t-)\psi_j(t)\lambda_j(t)]dt + \sigma_j(t-)d\bar{\Phi}_j^{\mathcal{Q}}(t)\right), \ j = 1, 2, \ldots, N.
\end{aligned}$$

Thus the necessary and sufficient condition for the discounted price processes of the securities in the enlarged market being martingales reads

$$\mu_0(t) + \sigma_0(t)\psi_0(t) = r(t), \tag{3.2}$$

$$\mu_j(t-) + \sigma_j(t-)\lambda_j(t)\psi_j(t) = r(t-). \tag{3.3}$$

These two equations are called the martingale condition and can be obtained from Proposition 10.1.8 of Musiela and Rutkowski [25]. In fact, by the Itô product rule, the discounted price processes of the risky securities in the enlarged market under $\mathcal{P}$ are governed by the following stochastic differential equations:

$$\begin{aligned}
d\tilde{S}_0(t) &= \tilde{S}_0(t)\left([\mu_0(t) - r(t)]dt + \sigma_0(t)dW_0^{\mathcal{Q}}(t)\right), \\
d\tilde{S}_j(t) &= \tilde{S}_j(t-)\left([\mu_j(t-) - r(t-)]dt + \sigma_j(t-)d\bar{\Phi}_j^{\mathcal{Q}}(t)\right), \ j = 1, 2, \ldots, N,
\end{aligned}$$

where $\tilde{S}_j(t) := B(t)^{-1}S_j(t), \ j = 0, 1, \ldots, N$.

Applying Proposition 10.1.8 of Musiela and Rutkowski [25] gives the following martingale conditions:

$$\int_0^t [\mu_0(s) - r(s) + \sigma_0(s)\psi_0(s)]S_0(s)ds = 0,$$

$$\int_0^t [\mu_j(s-) - r(s-) + \sigma_j(s-)\psi_j(s)\lambda_j(s)]S_j(s-)ds = 0, \ j = 1, \ldots, N, \ \text{for all } t \in [0, T],$$

which are identical to the martingale conditions (3.2) and (3.3), since $S_j(s) > 0, j = 0, 1, \ldots, N$.

Note that $\lambda_j(t)$ is defined by (2.1), and so the necessary and sufficient condition for $\lambda_j(t)$ equal to 0 is $\mathbf{X}(t-) = \mathbf{e}_j$. Thus if $\mu_j^j \neq r_j$, the martingale condition (3.3) would never be satisfied when $\mathbf{X}(t-) = \mathbf{e}_j$, and therefore we cannot find an equivalent martingale measure such that the discounted price processes of all securities in the enlarged market are martingales with respect to the filtration G. This implies that there exist arbitrage opportunities in the enlarged market. To preclude arbitrage opportunities in the enlarged market, we must have $\mu_j^j = r_j$ for all $j = 1, 2, \ldots, N$. Thus, when $\lambda_j(t) \neq 0$ (i.e., $\mathbf{X}(t-) \neq \mathbf{e}_j$), $\psi_j(t), j = 0, 1, \ldots, N$, are determined by

$$\psi_0(t) = \frac{r(t) - \mu_0(t)}{\sigma_0(t)}, \qquad \psi_j(t) = \frac{r(t-) - \mu_j(t-)}{\sigma_j(t-)\lambda_j(t)}, \; j = 1, \ldots, N.$$

Although we can only determine $\psi_j(t)$ when $\mathbf{X}(t-) \neq \mathbf{e}_j$, this is sufficient to determine an equivalent martingale measure $\mathcal{Q}$. Indeed, when $\mathbf{X}(t-) = \mathbf{e}_j$, $d\bar{\Phi}_j(t) = 0$. Therefore $\psi_j(t)$ has no impact on the value of the right side of (3.1) when $\mathbf{X}(t-) = \mathbf{e}_j$.

Now if we set

$$\boldsymbol{\psi}_0 := \left(\frac{r_1 - \mu_0^1}{\sigma_0^1}, \frac{r_2 - \mu_0^2}{\sigma_0^2}, \ldots, \frac{r_N - \mu_0^N}{\sigma_0^N} \right)',$$
$$\boldsymbol{\psi}_j := \left(\frac{r_1 - \mu_j^1}{\sigma_j^1 \lambda_{1j}}, \ldots, \frac{r_{j-1} - \mu_j^{j-1}}{\sigma_j^{j-1} \lambda_{j-1,j}}, 0, \frac{r_{j+1} - \mu_j^{j+1}}{\sigma_j^{j+1} \lambda_{j+1,j}}, \ldots, \frac{r_N - \mu_j^N}{\sigma_j^N \lambda_{Nj}} \right)', \quad j = 1, 2, \ldots, N,$$

we can write $\psi_j(t), j = 0, 1, \ldots, N$, as follows:

$$\psi_0(t) = \langle \boldsymbol{\psi}_0, \mathbf{X}(t) \rangle, \quad \psi_j(t) = \left\langle \boldsymbol{\psi}_j, \mathbf{X}(t-) \right\rangle, \text{ for each } j = 1, 2, \ldots, N. \tag{3.4}$$

The analysis above yields the following theorem.

THEOREM 3.2. *Assume that $\mu_j^j = r_j$, for all $j = 1, 2, \ldots, N$, and that $L(t)$ is given by (3.1), with $\psi_j(t)$, $j = 0, 1, \ldots, N$, given by (3.4). Define a new measure $\mathcal{Q}$ equivalent to $\mathcal{P}$ on $\mathcal{G}(T)$ by*

$$\left.\frac{d\mathcal{Q}}{d\mathcal{P}}\right|_{\mathcal{G}(T)} := L(T).$$

Then under $\mathcal{Q}$ the price processes of the securities in the enlarged market admit the following representations:

$$\begin{aligned} dB(t) &= r(t)B(t)dt, \\ dS_0(t) &= S_0(t)\left(r(t)dt + \sigma_0(t)dW_0^{\mathcal{Q}}(t)\right), \\ dS_j(t) &= S_j(t-)\left(r(t-)dt + \sigma_j(t-)d\bar{\Phi}_j^{\mathcal{Q}}(t)\right), \quad j = 1, 2, \ldots, N, \end{aligned}$$

and therefore the discounted price processes of the securities in the enlarged market are $(\mathcal{G}, \mathcal{Q})$-martingales and the enlarged market is arbitrage-free.

Next we show that the enlarged market is complete.

THEOREM 3.3. *Any nonnegative, square-integrable and $\mathcal{G}(T)$-measurable contingent claim M (i.e., $M \in \mathcal{L}^2(\Omega, \mathcal{G}(T), \mathcal{Q})$) can be perfectly replicated.*

Proof. From the martingale representation theorem in Elliott [10, Theorem 4.15], for a $(G, \mathcal{Q})$-martingale, $\{M(t) | t \in \mathcal{T}\}$, defined by setting

$$M(t) := E_{\mathcal{Q}}\left[\exp\left(-\int_0^T r(s)ds\right) M | \mathcal{G}(t)\right],$$

there exists a unique G-predictable, $\Re^N$-valued process $\mathbf{h}(t) := (h_0(t), h_1(t), h_2(t), \ldots, h_N(t))'$ such that

$$M(t) = M(0) + \int_0^t h_0(s) dW_0^Q(s) + \sum_{j=1}^{N} \int_0^t h_j(s) d\bar{\Phi}_j^Q(s).$$

Now we let $\pi_0(t), \pi_j(t), j = 1, 2, \ldots, N, \pi_r(t)$ be the number of units of the ordinary share, the number of units of the *jth geometric Markovian jump security*, and the number of units of the fixed interest security held in a portfolio, respectively. We construct the portfolio π as follows:

$$\begin{cases} \pi_0(t) := \frac{B(t)h_0(t)}{S_0(t)\sigma_0(t)}, \quad \pi_j(t) := \frac{B(t)h_j(t)}{S_j(t-)\sigma_j(t-)}, \quad j = 1, 2, \ldots, N, \\ \pi_r(t) := M(t) - B^{-1}(t) \sum_{j=0}^{N} \pi_j(t) S_j(t). \end{cases}$$

We claim that π is a self-financing portfolio which replicates M. Indeed, the value $V(t)$ of the portfolio π at time t is

$$V(t) = \pi_r(t)B(t) + \sum_{j=0}^{N} \pi_j(t) S_j(t) = B(t)M(t). \tag{3.5}$$

Consequently, the portfolio π duplicates the claim M. On the other hand, it is not difficult to verify that the gain process $G^\pi(t)$ have the following representation:

$$\begin{aligned} G^\pi(t) &:= \int_0^t r(s)\pi_r(s)B(s)ds + \sum_{j=0}^{N} \int_0^t \pi_j(s) dS_j(s) \\ &= B(t)M(t) - [B, M](t). \end{aligned} \tag{3.6}$$

The last equality follows from substituting $\pi_r(t)$ into the above equation and applying the following stochastic integration by parts:

$$\int_0^t r(s)M(s-)B(s)ds = B(t)M(t) - \int_0^t B(s-)dM(s) - [B, M](t).$$

Based on the theory of semimartingale (see Protter [28]), we can verify that $[B, M](t) = B(0)M(0)$. In fact, from Protter [28, Chapter II, Theorem 26], it can be shown that $\{B(t)|t \in \mathcal{T}\}$ is a pure jump semimartingale. Then observing that $B(t)$ is continuous and from Theorem 28 of Chapter II in Protter [28], we get

$$[B, M](t) = B(0)M(0).$$

Thus, observing that $B(0) = 1$ and combining (3.5) and (3.6) yield

$$G^\pi(t) + M(0) = B(t)M(t).$$

Therefore the result follows. ∎

4. Optimization problems. In this section, we derive closed-form solutions to the portfolio selection problem in the enlarged market.

First, we consider an investor who wishes to invest in the securities whose price processes are governed by (2.3). Due to the fact that for a.a. $\omega \in \Omega$, $\mathbf{X}(s, \omega) =$

$\mathbf{X}(s-,\omega)$, except for countably many s, we use $\mu_0(t-)$ instead of $\mu_0(t)$ to simplify the notation.

The investor selects the amounts invested in the securities in the enlarged market so as to maximize the expected utility on terminal wealth. We restrict ourselves to self-financing portfolio strategies and denote by $\tilde{\pi}_j(t)$ the fraction of wealth invested in $S_j(t)$ at time t for each $j = 0, 1, \ldots, N$. We suppose that $\tilde{\pi}_j := \{\tilde{\pi}_j(t) | t \in \mathcal{T}\}$ is G-predictable.

Remark 4.1. Note that when $\mathbf{X}(t-) = \mathbf{e}_j$ (i.e., $\lambda_j(t) = 0$), $\mu_j(t-) = \mu_j^j = r_j = r(t-)$, and $d\bar{\Phi}_j(t) = 0$, for each $j = 1, 2, \ldots, N$, and so the evolution of the price process of the jth *geometric Markovian jump security* over time is identical to that of the fixed interest security. In other words, when $\mathbf{X}(t-) = \mathbf{e}_j$, investing in the jth *geometric Markovian jump security* is indifferent to investing in the fixed interest security. Therefore when $\mathbf{X}(t-) = \mathbf{e}_j$, we can combine the fixed interest security with the jth *geometric Markovian jump security* as one security, and we need only to determine $\tilde{\pi}_i(t)$, $i \neq j, i = 0, 1, \ldots, N$, since the total fraction of wealth invested in the fixed interest security and $\tilde{\pi}_j(t)$ is equal to $1 - \sum_{i=0, i\neq j} \tilde{\pi}_i(t)$.

The process $\tilde{\pi}(t) := (\tilde{\pi}_0(t), \tilde{\pi}_1(t), \ldots, \tilde{\pi}_N(t))'$ is a portfolio strategy, and its corresponding wealth process, denoted as $R^{\tilde{\pi}} := \{R^{\tilde{\pi}}(t) | t \in \mathcal{T}\}$, is governed by

$$\begin{aligned} \frac{dR^{\tilde{\pi}}(t)}{R^{\tilde{\pi}}(t-)} &= \left[r(t) + \sum_{j=0}^{N} \tilde{\pi}_j(t)(\mu_j(t-) - r(t)) \right] dt \\ &\quad + \tilde{\pi}_0(t)\sigma_0(t) dW_0(t) + \sum_{j=1}^{N} \tilde{\pi}_j(t)\sigma_j(t-) d\bar{\Phi}_j(t). \end{aligned} \tag{4.1}$$

Let $\mathcal{A}$ be the class of admissible portfolio strategies $\tilde{\pi}$ such that

1. $\tilde{\pi}$ is G-predictable;
2. $\int_0^T |\tilde{\pi}(t)|^2 dt < \infty$, $\mathcal{P}$-a.s.;
3. the stochastic differential equation (4.1) has a unique strong solution $R^{\tilde{\pi}}$ associated with $\tilde{\pi}$.

Let $U : \Re^+ \to \Re$ denote a utility function of the investor, which is strictly increasing, strictly concave, and continuously differentiable. We further assume that the utility function satisfies the following Inada conditions, which are technical conditions:

1. $U'(0+) = \lim_{z \to 0+} U'(z) = +\infty$;
2. $U'(+\infty) = \lim_{z \to +\infty} U'(z) = 0$.

For more discussions on properties of utility functions, interested readers may refer to Elliott and Kopp [15].

Define, for each $(t, z) \in \mathcal{T} \times \Re^+$ and each $i = 1, 2, \ldots, N$,

$$V^{\tilde{\pi}}(t, z, \mathbf{e}_i) := E_{t,z,i}[U(R^{\tilde{\pi}}(T))],$$

where $E_{t,z,i}$ is the conditional expectation given $R^{\tilde{\pi}}(t) = z$ and $\mathbf{X}(t) = \mathbf{e}_i$ under $\mathcal{P}$.

We suppose that for each $\tilde{\pi} \in \mathcal{A}$, $(t, z) \in \mathcal{T} \times \Re^+$, and $i = 1, 2, \ldots, N$,

$$E_{t,z,i}[U(R^{\tilde{\pi}}(T))] < \infty.$$

Then, the value function of the investor's portfolio selection problem in the enlarged market is defined by

$$V(t, z, \mathbf{e}_i) = \sup_{\tilde{\pi} \in \mathcal{A}} V^{\tilde{\pi}}(t, z, \mathbf{e}_i).$$

In what follows, we consider two risk-averse utility functions, namely, a logarithmic utility and a power utility.

4.1. Logarithmic utility. In this subsection, we derive the optimal portfolio strategy in the case of a logarithmic utility function of wealth, namely, $U(z) = \log(z)$.

THEOREM 4.2. (i) *Suppose $H^*(s)$ is defined by*

$$H^*(s) := r(s) + \frac{(\mu_0(s-) - r(s))^2}{2\sigma_0(s)^2} - \sum_{j=1}^{N} 1_{\{\mathbf{X}(s-)\neq \mathbf{e}_j\}} \lambda_j(s) \left[\log\left(1 - \frac{\mu_j(s-) - r(s)}{\sigma_j(s-)\lambda_j(s)}\right) - \frac{\mu_j(s-) - r(s)}{\sigma_j(s-)\lambda_j(s)} \right].$$

Then for each $(t, z) \in \mathcal{T} \times \Re^+$ and $i = 1, 2, \ldots, N$,

$$V(t, z, \mathbf{e}_i) = \log(z) + h(t, \mathbf{e}_i),$$

where

$$h(t, \mathbf{e}_i) = E_{t,i}\left\{ \int_t^T H^*(s)ds \right\} \tag{4.2}$$

and $E_{t,i}$ is the conditional expectation given $\mathbf{X}(t) = \mathbf{e}_i$ under $\mathcal{P}$.

(ii) *Let $\tilde{\pi}^*(s) := (\tilde{\pi}_0^*(s), \tilde{\pi}_1^*(s), \ldots, \tilde{\pi}_N^*(s))'$, and $\tilde{\pi}_j^*(s)$ is defined by*

$$\tilde{\pi}_0^*(s) := \frac{\mu_0(s-) - r(s)}{\sigma_0(s)^2}, \tag{4.3}$$

$$\tilde{\pi}_j^*(s) := 1_{\{\mathbf{X}(s-)\neq \mathbf{e}_j\}} \frac{\mu_j(s-) - r(s)}{\sigma_j(s-)^2\lambda_j(s) - \sigma_j(s-)(\mu_j(s-) - r(s))}, \quad j = 1, 2, \ldots, N, \quad s \in [t, T]. \tag{4.4}$$

Then $\tilde{\pi}^* := \{\tilde{\pi}^*(s) | s \in [t, T]\}$ is the optimal portfolio strategy.

Proof. For any portfolio strategy $\tilde{\pi} \in \mathcal{A}$, from the wealth process (4.1) and a generalized version of Itô's differentiation rule for jump-diffusion processes, we have

$$V^{\tilde{\pi}}(t, z, \mathbf{e}_i) = \log(z) + h^{\tilde{\pi}}(t, \mathbf{e}_i),$$

where

$$h^{\tilde{\pi}}(t, \mathbf{e}_i) := E_{t,i}\left[\int_t^T H(\tilde{\pi}_0(s), \tilde{\pi}_1(s), \ldots, \tilde{\pi}_N(s))ds \right]$$

and $H(\tilde{\pi}_0(s), \tilde{\pi}_1(s), \ldots, \tilde{\pi}_N(s))$ is defined by

$$H(\tilde{\pi}_0(s), \tilde{\pi}_1(s), \ldots, \tilde{\pi}_N(s)) := r(s) + \sum_{j=0}^{N} \tilde{\pi}_j(s)(\mu_j(s-) - r(s)) - \frac{\tilde{\pi}_0(s)^2\sigma_0(s)^2}{2} + \sum_{j=1}^{N} \left(\log(1 + \tilde{\pi}_j(s)\sigma_j(s-)) - \tilde{\pi}_j(s)\sigma_j(s-)\right) \lambda_j(s). \tag{4.5}$$

Therefore the value function $V(t, z, \mathbf{e}_i)$ can be written as

$$V(t, z, \mathbf{e}_i) = \log(z) + \sup_{\tilde{\pi}\in\mathcal{A}} h^{\tilde{\pi}}(t, \mathbf{e}_i).$$

Consequently, to determine the optimal portfolio strategy, it suffices to maximize the Hamiltonian $H(\tilde{\pi}_0(s), \tilde{\pi}_1(s), \ldots, \tilde{\pi}_N(s))$ for each $s \in [t, T]$. Recall that $\mu_j^j = r_j$ and $\sigma_j^j = 0$. When $\mathbf{X}(s-) = \mathbf{e}_j$, the Hamiltonian $H(\tilde{\pi}_0(s), \tilde{\pi}_1(s), \ldots, \tilde{\pi}_N(s))$ is independent of $\tilde{\pi}_j$. Thus by direct differentiation with respect to $\tilde{\pi}_j$, $j = 0, 1, \ldots, N$, under the condition that $\mathbf{X}(s-) \neq \mathbf{e}_j$ (i.e., $\lambda_j(s) \neq 0$), we immediately obtain that the portfolio strategies $\tilde{\pi}_j^*(s), j = 0, 1, \ldots, N$, defined by (4.3) and (4.4) are optimal.

Substituting (4.3) and (4.4) into (4.5) yields

$$\sup_{\tilde{\pi} \in \mathcal{A}} H(\tilde{\pi}_0(s), \tilde{\pi}_1(s), \ldots, \tilde{\pi}_N(s)) = H^*(s).$$

Therefore the desired result of the theorem follows. ∎

Remark 4.3. From the proof of the above theorem, we cannot determine the fraction of $\tilde{\pi}_j^*(s)$ when $\mathbf{X}(s-) = \mathbf{e}_j$. In fact, the fraction of $\tilde{\pi}_j^*(s)$ when $\mathbf{X}(s-) = \mathbf{e}_j$ has no impact on the optimal portfolio strategy, since from Remark 4.1, we can combine the jth *geometric Markovian jump security* with the fixed interest as one security when $\mathbf{X}(s-) = \mathbf{e}_j$.

The expression of the value function given in Theorem 4.2 is the Feynman–Kac type representation. We can also give another representation for the value function via a system of linear differential equations.

THEOREM 4.4. *Suppose $h(t, \mathbf{e}_i)$ is as in Theorem 4.2. Then $h(t, \mathbf{e}_i)$, $i = 1, 2, \ldots, N$, satisfy the following system of coupled linear differential equations:*

$$\begin{aligned}&\frac{dh}{dt}(t, \mathbf{e}_i) + r_i + \frac{(\mu_0^i - r_i)^2}{2(\sigma_0^i)^2} \\ &\quad - \sum_{j \neq i, j=1}^{N} \lambda_{ij} \left[\log\left(1 - \frac{\mu_j^i - r_i}{\sigma_j^i \lambda_{ij}}\right) + \frac{\mu_j^i - r_i}{\sigma_j^i \lambda_{ij}} \right] + \sum_{j=1}^{N} \lambda_{ij} h(t, \mathbf{e}_j) = 0,\end{aligned} \tag{4.6}$$

with boundary conditions

$$h(T, \mathbf{e}_i) = 0, \quad i = 1, 2, \ldots, N. \tag{4.7}$$

Proof. We need only to show that if $\boldsymbol{g}(t) := (g(t, \mathbf{e}_1), g(t, \mathbf{e}_2), \ldots, g(t, \mathbf{e}_N))'$ is a solution to the system of coupled linear differential equations (4.6) with boundary conditions (4.7), then $g(t, \mathbf{e}_i)$ must have the same representation as $h(t, \mathbf{e}_i)$ in (4.2).

Note that $g(t, \mathbf{X}(t)) = \langle \boldsymbol{g}(t), \mathbf{X}(t) \rangle$. Thus Lemma 1.5 of Appendix B in Elliott, Aggoun, and Moore [12] implies that the real-valued process $\{M(v) | v \in [t, T]\}$ defined by setting

$$M(v) := g(v, \mathbf{X}(v)) - g(t, \mathbf{X}(t)) - \int_t^v \left(\frac{\partial g}{\partial s}(s, \mathbf{X}(s-)) + \langle \boldsymbol{g}(s), \mathbf{\Lambda}' \mathbf{X}(s-) \rangle \right) ds \tag{4.8}$$

is a $(G, \mathcal{P})$-martingale.

Let

$$G_i = r_i + \frac{(\mu_0^i - r_i)^2}{2(\sigma_0^i)^2} - \sum_{j \neq i, j=1}^{N} \lambda_{ij} \left[\log\left(1 - \frac{\mu_j^i - r_i}{\sigma_j^i \lambda_{ij}}\right) + \frac{\mu_j^i - r_i}{\sigma_j^i \lambda_{ij}} \right], \quad i = 1, 2, \ldots, N, \tag{4.9}$$

and $\mathbf{G} := (G_1, \ldots, G_N)'$.

Note that $\boldsymbol{g}(t) := (g(t, \mathbf{e}_1), g(t, \mathbf{e}_2), \ldots, g(t, \mathbf{e}_N))'$ is the solution to the system of coupled linear differential equations (4.6) with boundary conditions (4.7), and so

$$\frac{d\boldsymbol{g}}{ds}(s) = -\mathbf{G} - \boldsymbol{\Lambda}\boldsymbol{g}(s). \tag{4.10}$$

Therefore,

$$\frac{dg}{ds}(s, \mathbf{X}(s-)) = \langle -\mathbf{G} - \boldsymbol{\Lambda}\boldsymbol{g}(s), \mathbf{X}(s-)\rangle = -\langle \mathbf{G}, \mathbf{X}(s-)\rangle - \langle \boldsymbol{g}(s), \boldsymbol{\Lambda}'\mathbf{X}(s-)\rangle .$$

Substituting this into (4.8) and taking expectation to $M(T)$ yield

$$0 = E_{t,i}[g(T, \mathbf{X}(T))] - g(t, \mathbf{e}_i) + E_{t,i}\left[\int_t^T \langle \mathbf{G}, \mathbf{X}(s-)\rangle\, ds\right].$$

Thus from the boundary conditions $g(T, \mathbf{e}_i) = 0$, for $i = 1, 2, \ldots, N$, the fact that for a.a. $\omega \in \Omega$, $\mathbf{X}(s, \omega) = \mathbf{X}(s-, \omega)$, except for countably many s and the definition of $\mathbf{G}$, we obtain that $g(t, \mathbf{e}_i)$ has the same representation as $h(t, \mathbf{e}_i)$ in (4.2), and so the result follows. ▯

Remark 4.5. Note that the system of differential equations is linear, so we can give a closed-form solution. In fact, from the matrix form of the differential equation system (4.10) and Bronson [4, Chapter 8.4], we have

$$\boldsymbol{g}(t) = \exp(-\boldsymbol{\Lambda} t) \int_t^T \exp(\boldsymbol{\Lambda} s)\mathbf{G} ds,$$

where $\boldsymbol{\Lambda}$ is the rate matrix of the chain $\mathbf{X}$; $\mathbf{G} := (G_1, G_2, \ldots, G_N)' \in \Re^N$ with G_i defined in (4.9).

If we let $\mathbf{D} = (\boldsymbol{\Lambda} - \mathbf{1} \otimes \boldsymbol{\nu})^{-1}$, where $\mathbf{1} = (1, 1, \ldots, 1)'$ and $\boldsymbol{\nu}$ is the stationary distribution of the Markov chain, then from Proposition A4.1 of Asmussen [1], we obtain that

$$\boldsymbol{g}(t) = (T - t) \exp(-\boldsymbol{\Lambda} t)\mathbf{1} \otimes \boldsymbol{\nu}\mathbf{G} + \mathbf{D}[\exp(\boldsymbol{\Lambda}(T - t)) - \mathbf{I}\,]\mathbf{G},$$

where $\mathbf{I}$ is the identity matrix.

Note that $\mathbf{h} = \boldsymbol{g}$. Then we obtain the following closed-form expression for the function V:

$$\begin{aligned} V(t, z, \mathbf{e}_i) = \log(z) &+ \langle (T - t) \exp(-\boldsymbol{\Lambda} t)\mathbf{1} \otimes \boldsymbol{\nu}\mathbf{G}, \mathbf{e}_i\rangle \\ &+ \langle \mathbf{D}[\exp(\boldsymbol{\Lambda}(T - t)) - \mathbf{I}\,]\mathbf{G}, \mathbf{e}_i\rangle , \quad i = 1, 2, \ldots, N. \end{aligned}$$

4.2. Power utility. In this subsection, we deal with a power utility function, namely, $U(z) = z^\alpha$. The direct differentiation approach seems not to be working well in this case. We shall adopt the dynamic programming approach to solve the optimal portfolio selection problem. From some standard arguments in Fleming and Soner [17], we have the following theorem.

THEOREM 4.6. *Suppose the value function V and its partial derivatives V_t, V_z, V_{zz} are continuous on $\mathcal{T} \times \Re^+$, for each $i = 1, 2, \ldots, N$, where V_t, V_z, and V_{zz} represent the first derivative of V with respect to t, the first derivative of V with respect to z, and the second derivative of V with respect to z, respectively. Then, $V(t, z, \mathbf{e}_i)$,*

$i = 1, 2, \dots, N$, *satisfy the following system of HJB equations:*

$$\begin{aligned} V_t(t,z,\mathbf{e}_i) + \sup_{\tilde{\pi}} \Bigg\{ \Bigg[r_i + \sum_{j=0}^{N} \tilde{\pi}_j(\mu_j^i - r_i) \Bigg] z V_z(t,z,\mathbf{e}_i) + \frac{1}{2}\tilde{\pi}_0^2(\sigma_0^i)^2 z^2 V_{zz}(t,z,\mathbf{e}_i) \\ + \sum_{j=1}^{N} \Big[V(t, z(1+\tilde{\pi}_j\sigma_j^i), \mathbf{e}_i) - V(t,z,\mathbf{e}_i) - V_z(t,z,i) z \tilde{\pi}_j \sigma_j^i \Big] \lambda_{ij} \Bigg\} \\ + \sum_{j=1}^{n} \lambda_{ij} V(t,z,\mathbf{e}_j) = 0 \end{aligned} \tag{4.11}$$

for $(t, z, \mathbf{e}_i) \in [0, T) \times \Re^+ \times E$, *with terminal conditions*

$$V(T, z, \mathbf{e}_i) = z^\alpha, \quad i = 1, 2, \dots, N. \tag{4.12}$$

We first give a solution to the system of HJB equations (4.11) with terminal conditions (4.12) and then verify that this solution is indeed the value function V.

THEOREM 4.7. *Let* $\boldsymbol{g}(t) := (g(t,\mathbf{e}_1), g(t,\mathbf{e}_2), \dots, g(t,\mathbf{e}_N))'$ *denote a "classical" solution to the following system of coupled linear differential equations with terminal conditions* $g(T, \mathbf{e}_i) = 1$*:*

$$\begin{aligned} \frac{dg}{dt}(t,\mathbf{e}_i) = -\Bigg[\alpha r_i + \frac{\alpha}{2(1-\alpha)} \left(\frac{\mu_0^i - r_i}{\sigma_0^i} \right)^2 + \sum_{j=1, j\neq i}^{N} \lambda_{ij}(1-\alpha) \left(1 - \frac{\mu_j^i - r_i}{\lambda_{ij}\sigma_j^i} \right)^{\frac{\alpha}{\alpha-1}} \\ + \sum_{j=1, j\neq i}^{N} \lambda_{ij} \left(\alpha \left(1 - \frac{\mu_j^i - r_i}{\lambda_{ij}\sigma_j^i} \right) - 1 \right) \Bigg] g(t,\mathbf{e}_i) - \sum_{j=1}^{N} \lambda_{ij} g(t,\mathbf{e}_j). \end{aligned} \tag{4.13}$$

Then

1. $v(t,z,\mathbf{e}_i) := z^\alpha g(t,\mathbf{e}_i)$ *is a solution to the system of HJB equations* (4.11) *with terminal conditions* (4.12);
2. *The Feynman–Kac formula yields the following representation for* $g(t,\mathbf{e}_i)$*:*

$$\begin{aligned} g(t,\mathbf{e}_i) = E_{t,i} \Bigg\{ \exp\Bigg\{ \int_t^T \Bigg[\alpha r(s) + \frac{\alpha}{2(1-\alpha)} \left(\frac{\mu_0(s-) - r(s)}{\sigma_0(s)} \right)^2 \\ + \sum_{j=1}^{N} 1_{\{\mathbf{X}(s-)\neq \mathbf{e}_j\}} \lambda_j(s) \left[\alpha \left(1 - \frac{\mu_j(s-) - r(s)}{\lambda_j(s)\sigma_j(s-)} \right) - 1 \right] \\ + \sum_{j=1}^{N} 1_{\{\mathbf{X}(s-)\neq \mathbf{e}_j\}} (1-\alpha)\lambda_j(s) \left(1 - \frac{\mu_j(s-) - r(s)}{\lambda_j(s)\sigma_j(s-)} \right)^{\frac{\alpha}{\alpha-1}} \Bigg] ds \Bigg\} \Bigg\}, \end{aligned} \tag{4.14}$$

where $E_{t,i}$ *is the conditional expectation given* $\mathbf{X}(t) = \mathbf{e}_i$ *under* $\mathcal{P}$.

Proof. First, we note that boundary conditions $v(T, z, i) = z^\alpha$, $i = 1, 2, \dots, N$, are satisfied. Since the function v is "sufficiently" smooth on $\mathcal{T} \times \Re^+$, we can compute the derivatives of v as follows:

$$\begin{aligned} v_t(t,z,\mathbf{e}_i) = z^\alpha g_t(t,\mathbf{e}_i),\ v_z(t,z,\mathbf{e}_i) = \alpha z^{\alpha-1} g(t,\mathbf{e}_i), \\ v_{zz}(t,z,\mathbf{e}_i) = \alpha(\alpha-1) z^{\alpha-2} g(t,\mathbf{e}_i). \end{aligned}$$

Substituting these into the system of HJB equations (4.11) yields

$$g_t(t,\mathbf{e}_i)+\sup_{\tilde\pi}\Bigg\{\Bigg[r_i+\sum_{j=0}^{N}\tilde\pi_j(\mu_j^i-r_i)\Bigg]\alpha+\frac{1}{2}\tilde\pi_0^2(\sigma_0^i)^2\alpha(\alpha-1)$$
$$+\sum_{j=1}^{N}[(1+\tilde\pi_j\sigma_j^i)^\alpha-1-\alpha\tilde\pi_j\sigma_j^i]\lambda_{ij}\Bigg\}g(t,\mathbf{e}_i)+\sum_{j=1}^{N}\lambda_{ij}g(t,\mathbf{e}_j)=0. \tag{4.15}$$

For $j\neq i$, differentiating the terms inside the supremum in (4.15) with respect to $\tilde\pi_j$ yields that the maximum points $\tilde\pi_j^*$ are given by

$$\tilde\pi_0^*=\frac{\mu_0^i-r_i}{(1-\alpha)(\sigma_0^i)^2},\qquad \tilde\pi_j^*=\frac{\left(1-\frac{\mu_j^i-r_i}{\lambda_{ij}\sigma_j^i}\right)^{\frac{1}{\alpha-1}}-1}{\sigma_j^i},\quad j\neq i,\ j=1,2,\dots,N.$$

Although we obtain only $\tilde\pi_j^*$ for $j\neq i$, it is sufficient to substitute these into (4.15) to calculate the supremum. Again, the sup term in (4.15) is independent of the value of $\tilde\pi_i$. Thus substituting $\tilde\pi_j^*$, $j=0,1,\dots,N, j\neq i$, into (4.15) and simplifying yield that $g(t,\mathbf{e}_i)$, $i=1,2,\dots,N$, have to satisfy the system of linear differential equations (4.13). Hence, the first result of the theorem follows.

We now prove the Feynman–Kac representation of $g(t,\mathbf{e}_i)$. To simplify the notation, we first rewrite (4.13) in the matrix form. Let

$$H_i=\alpha r_i+\frac{\alpha}{2(1-\alpha)}\left(\frac{\mu_0^i-r_i}{\sigma_0^i}\right)^2+\sum_{j=1,j\neq i}^{N}\lambda_{ij}(1-\alpha)\left(1-\frac{\mu_j^i-r_i}{\lambda_{ij}\sigma_j^i}\right)^{\frac{\alpha}{\alpha-1}}$$
$$+\sum_{j=1,j\neq i}^{N}\lambda_{ij}\left[\alpha\left(1-\frac{\mu_j^i-r_i}{\lambda_{ij}\sigma_j^i}\right)-1\right],\quad i=1,2,\dots,N, \tag{4.16}$$

and $\mathbf{H}=\mathbf{diag}(H_1,\dots,H_N)$ be the diagonal matrix with diagonal elements $(H_1,\dots,H_N)$.

Thus (4.13) can be written in the following matrix form:

$$\frac{d\boldsymbol{g}}{dt}(t)=-[\mathbf{H}+\boldsymbol{\Lambda}]\boldsymbol{g}(t),$$

with terminal conditions

$$\boldsymbol{g}(T)=\mathbf{1}:=(1,1,\dots,1)'\in\Re^N.$$

From Lemma 1.5 of Appendix B in Elliott, Aggoun, and Moore [12], the process $\{M(v)|v\in[t,T]\}$ defined by putting

$$M(v):=g(v,\mathbf{X}(v))-g(t,\mathbf{X}(t))-\int_t^v\left(\frac{\partial g}{\partial s}(s,\mathbf{X}(s-))+\langle\boldsymbol{g}(s),\boldsymbol{\Lambda}'\mathbf{X}(s-)\rangle\right)ds \tag{4.17}$$

is a $(G,\mathcal{P})$-martingale.

Consequently, for the stochastic process $K:=\{K(v)|v\in[t,T]\}$ defined by setting

$$K(v):=\exp\left(\int_t^v\langle\mathbf{H1},\mathbf{X}(s-)\rangle\,ds\right),$$

we have

$$\begin{aligned} d[K(v)g(v,\mathbf{X}(v))] &= K(v)\Big\{dg(v,\mathbf{X}(v)) + \langle \mathbf{H1}, \mathbf{X}(v-)\rangle\, g(v,\mathbf{X}(v))dv\Big\} \\ &= K(v)\Big\{dM(v) + \Big[\frac{\partial g}{\partial v}(v,\mathbf{X}(v-)) + \langle \mathbf{\Lambda}\boldsymbol{g}(v), \mathbf{X}(v-)\rangle\Big]dv \\ &\qquad + \langle \mathbf{H}\boldsymbol{g}(v), \mathbf{X}(v-)\rangle\, dv\Big\}, \end{aligned} \tag{4.18}$$

where the last equality follows from the definition of $M(v)$ in (4.17).

Note that

$$\frac{\partial g}{\partial v}(v,\mathbf{X}(v-)) = \left\langle \frac{d\boldsymbol{g}(v)}{dv}, \mathbf{X}(v-)\right\rangle = -\langle [\mathbf{H}+\mathbf{\Lambda}]\boldsymbol{g}(v), \mathbf{X}(v-)\rangle\,,$$

and substituting this into (4.18) yields that

$$K(v)g(v,\mathbf{X}(v)) - g(t,\mathbf{X}(t)) = \int_t^v K(s)dM(s).$$

Setting $v = T$ and conditioning on $\mathbf{X}(t) = \mathbf{e}_i$ give

$$g(t,\mathbf{e}_i) = E_{t,i}\left[\exp\left(\int_t^T \langle \mathbf{H1}, \mathbf{X}(s-)\rangle\, ds\right)\right]. \tag{4.19}$$

From the fact that for a.a. $\omega \in \Omega$, $\mathbf{X}(s,\omega) = \mathbf{X}(s-,\omega)$, except for countably many s, we can verify that (4.14) is just another representation of (4.19) with the parameters in the enlarged market. This completes the proof of the second result. □

Remark 4.8. From Bronson [4, Chapter 8.4], we can also obtain the closed-form solution to the system of coupled linear differential equations (4.13) as

$$\boldsymbol{g}(t) = \exp[(\mathbf{H}+\mathbf{\Lambda})(T-t)]\,\mathbf{1}, \quad t \in \mathcal{T},$$

where $\boldsymbol{g}(t) = (g(t,\mathbf{e}_1),\ldots,g(t,\mathbf{e}_N))'$, $\mathbf{H} := \mathbf{diag}(H_1,\ldots,H_N)$ with H_i defined in (4.16), and $\mathbf{\Lambda}$ is the rate matrix of the chain $\mathbf{X}$. Then, by the first result of Theorem 4.7, we obtain the following closed-form expression for function v:

$$v(t,z,\mathbf{e}_i) = z^{\alpha}\,\langle \exp[(\mathbf{H}+\mathbf{\Lambda})(T-t)]\,\mathbf{1}, \mathbf{e}_i\rangle\,, \quad i = 1,2,\ldots,N.$$

In the following we shall provide a verification theorem which ensures that the solution v of the system of HJB equations in Theorem 4.7 is indeed the value function of the portfolio selection problem. We first give a generalized version of Itô's lemma for Markov-modulated jump-diffusion processes which will be useful for proving the verification theorem.

LEMMA 4.9. *Suppose that* $\mathbf{X} := \{\mathbf{X}(t) | t \in \mathcal{T}\}$ *is the Markov chain defined in section* 2 *and that* $Z := \{Z(t) | t \in \mathcal{T}\}$ *satisfies the following stochastic differential equation*

$$dZ(t) = \beta(t)dt + \beta_0(t)dW_0(t) + \sum_{j=1}^{N} \beta_j(t-)d\bar{\Phi}_j(t),$$

where $W_0 := \{W_0(t)|t \in \mathcal{T}\}$ *is the standard Brownian motion and* $\bar{\Phi}_j := \{\bar{\Phi}_j(t)|t \in \mathcal{T}\}$ *is the* j*th Markovian jump martingale, for each* $j = 1, 2, \ldots, N$, *both defined on* $(\Omega, \mathcal{F}, \mathcal{P})$. *Let* $g(t, z, \mathbf{e}_i)$ *be a real-valued function defined on* $\Re^+ \times \Re \times E$. *Then,*

$$\begin{aligned} &g(t, Z(t), \mathbf{X}(t)) - g(0, Z(0), \mathbf{X}(0)) \\ &= \int_0^t \langle \mathcal{L}[\boldsymbol{g}(s, Z(s-))], \mathbf{X}(s-)\rangle \, ds + \int_0^t \langle \boldsymbol{g}_z(s, Z(s-)), \mathbf{X}(s-)\rangle \, \beta_0(s-) dW_0(s) \\ &\quad + \sum_{j=1}^N \int_0^t \langle \boldsymbol{g}_z(s, Z(s-)), \mathbf{X}(s-)\rangle \, \beta_j(s-) d\bar{\Phi}_j(s) \\ &\quad + \sum_{j=1}^N \int_0^t \langle \boldsymbol{g}(s, Z(s)) - \boldsymbol{g}(s, Z(s-)) - \boldsymbol{g}_z(s, Z(s-))\beta_j(s-), \mathbf{X}(s-)\rangle \, \lambda_j(s) ds \\ &\quad + \sum_{j=1}^N \int_0^t \Big[\langle \boldsymbol{g}(s, Z(s)), \mathbf{e}_j\rangle - \langle \boldsymbol{g}(s, Z(s-)), \mathbf{X}(s-)\rangle\Big] d\bar{\Phi}_j(s), \end{aligned} \tag{4.20}$$

where the partial differential operator $\mathcal{L}$ *on the vector-valued function* $\boldsymbol{g}(t, z)$ *is defined by*

$$\mathcal{L}[\boldsymbol{g}(t, z)] := \boldsymbol{g}_t(t, z) + \boldsymbol{g}_z(t, z)\beta(t) + \frac{1}{2}\boldsymbol{g}_{zz}(t, z)\beta_0^2(t) + \boldsymbol{\Lambda}\boldsymbol{g}(t, z).$$

Proof. The proof involves the use of Itô's differentiation rule for semimartingales (see Elliott [11]) and a semimartingale representation of the chain $\mathbf{X}$ in (7.2) of Elliott, Aggoun, and Moore [12]. We give the detail of the proof in a longer version of the paper which is available on request. ∎

THEOREM 4.10. *Suppose* $v(t, z, i)$ *is as given in Theorem* 4.7. *Then*

1. $V(t, z, \mathbf{e}_i) = v(t, z, \mathbf{e}_i)$ *for all* $(t, z, \mathbf{e}_i) \in \mathcal{T} \times \Re^+ \times E$;
2. *let* $\tilde{\pi}^*(s) := (\tilde{\pi}_0^*(s), \tilde{\pi}_1^*(s), \ldots, \tilde{\pi}_N^*(s))'$, *and* $\tilde{\pi}_j^*(s)$ *is defined by setting*

$$\tilde{\pi}_0^*(s) := \frac{\mu_0(s-) - r(s)}{(1-\alpha)\sigma_0(s)^2},$$

$$\tilde{\pi}_j^*(s) := 1_{\{\mathbf{X}(s-)\neq \mathbf{e}_j\}} \frac{\left(1 - \frac{\mu_j(s-) - r(s)}{\lambda_j(s)\sigma_j(s-)}\right)^{\frac{1}{\alpha-1}} - 1}{\sigma_j(s-)}, \quad j = 1, 2, \ldots, N,$$

for all $s \in [t, T]$. *Then* $\tilde{\pi}^* := \{\tilde{\pi}^*(s)|s \in [t, T]\}$ *is the optimal portfolio strategy for the portfolio selection problem with power utility.*

Proof. Let $\tilde{\pi}$ be any portfolio strategy and $R^{\tilde{\pi}} := \{R^{\tilde{\pi}}(t)|t \in \mathcal{T}\}$ the corresponding wealth process. Since $v(t, z, \mathbf{e}_i)$ is "sufficiently" smooth in $(t, z) \in \mathcal{T} \times \Re^+$ and is a solution of the HJB equation (4.11), for each $i = 1, 2, \ldots, N$, we can apply Itô's formula to obtain

$$\begin{aligned} &v(T, R^{\tilde{\pi}}(T), \mathbf{X}(T)) - v(t, z, \mathbf{e}_i) \\ &\leq \int_t^T \big\langle \boldsymbol{v}_z(s, R^{\tilde{\pi}}(s-)), \mathbf{X}(s-)\big\rangle R^{\tilde{\pi}}(s-)\tilde{\pi}_0(s)\sigma_0(s-) dW_0(s) \\ &\quad + \sum_{j=1}^N \int_t^T \Big[\big\langle \boldsymbol{v}(s, R^{\tilde{\pi}}(s)), \mathbf{e}_j\big\rangle - \big\langle \boldsymbol{v}(s, R^{\tilde{\pi}}(s-)), \mathbf{X}(s-)\big\rangle\Big] d\bar{\Phi}_j(s). \end{aligned} \tag{4.21}$$

Since $v \geq 0$, the $(G, \mathcal{P})$-(local) martingale $\{M(T)|T \geq t\}$ defined by

$$M(T) := \int_t^T \left\langle \boldsymbol{v}_z(s, R^{\tilde{\pi}}(s-)), \mathbf{X}(s-)\right\rangle R^{\tilde{\pi}}(s-)\tilde{\pi}_0(s)\sigma_0(s-)dW_0(s) + \sum_{j=1}^{N} \int_t^T \left[\left\langle \boldsymbol{v}(s, R^{\tilde{\pi}}(s)), \mathbf{e}_j\right\rangle - \left\langle \boldsymbol{v}(s, R^{\pi}_{s-}), \mathbf{X}(s-)\right\rangle\right] d\bar{\Phi}_j(s)$$

is bounded from below by $-v(t, z, \mathbf{e}_i)$, and so it is a super-martingale. Taking the conditional expectation and using the boundary condition for v, we find that

$$E_{t,z,i}\left[(R^{\tilde{\pi}}(T))^{\alpha}\right] \leq v(t, x, \mathbf{e}_i).$$

Since $\tilde{\pi}$ is arbitrary, we obtain $V(t, z, \mathbf{e}_i) \leq v(t, z, \mathbf{e}_i)$. Now suppose that $\tilde{\pi}^* := \{\tilde{\pi}^*(s)|s \in [t, T]\}$ is given as in the statement of (2) of the theorem. Then we obtain the equality in (4.21) with $\tilde{\pi}^*$. In this case, the $(G, \mathcal{P})$-(local) martingale $\{M(T)\}_{T\geq t}$ is a $(G, \mathcal{P})$-martingale. Taking the expectation, we obtain

$$E_{t,z,i}\left[(R^{\tilde{\pi}^*}(T))^{\alpha}\right] = v(t, z, \mathbf{e}_i),$$

and therefore the result in the theorem follows. ∎

5. Relationship between the optimization problem in the enlarged market and in the original market. Karatzas et al. [21] introduced a set of "fictitious" shares to complete a multidimensional geometric Brownian motion market, where the number of primitive securities in the original market is not large enough to completely hedge against the risk attributed to random shocks. Then they adopted the martingale approach to determine an optimal portfolio that maximizes a general utility function. They regarded the appreciation rates of the fictitious shares as "free" parameters and determined these parameters so that the optimal portfolio in the completed market does not involve these fictitious shares. In the logarithmic utility case, they showed that the appreciation rates of the fictitious stocks are identical to the risk-free interest rate. In this section, as in Karatzas et al. [21], if we consider the appreciation rates $\mu_j(t-)$, $j = 1, \ldots, N$, of the *jth geometric Markovian jump securities* as "free" parameters, we can also give similar results in the Markovian regime-switching market for the logarithmic and power utility. We also give the relationships between the optimization problem in the enlarged market and in the original market in both logarithmic and power utility cases. We first formulate the optimization problems in the original, incomplete market.

Let $\tilde{\pi}_0(t)$ be the fraction of the wealth invested in $S_0(t)$ in the original market. Thus the corresponding wealth process, denoted as $R^{\tilde{\pi}_0} := \{R^{\tilde{\pi}_0}(t)|t \in \mathcal{T}\}$, is given by

$$\frac{dR^{\tilde{\pi}_0}(t)}{R^{\tilde{\pi}_0}(t)} = [r(t) + \tilde{\pi}_0(t)(\mu_0(t) - r(t))]\, dt + \tilde{\pi}_0(t)\sigma_0(t)dW_0(t). \tag{5.1}$$

Let $\mathcal{A}_0$ be the class of admissible portfolio strategies $\tilde{\pi}_0 := \{\tilde{\pi}_0(t)|t \in \mathcal{T}\}$ of the original incomplete market such that

1. $\tilde{\pi}_0$ is G-predictable;
2. $\int_0^T \tilde{\pi}_0^2(t)dt < \infty$, $\mathcal{P}$-a.s.;
3. the stochastic differential equation (5.1) has a unique strong solution $R^{\tilde{\pi}_0}$ associated with $\tilde{\pi}_0$.

Similarly to the definition of the value function in the enlarged market, we define the value function in the original incomplete market as

$$V_0(t,z,\mathbf{e}_i) = \sup_{\tilde{\pi}_0 \in \mathcal{A}_0} E_{t,z,i}[U(R^{\tilde{\pi}_0}(T))].$$

Again we suppose that $E_{t,z,i}[U(R^{\tilde{\pi}_0}(T))] < \infty, \quad i = 1,2,\ldots,N.$

In what follows, we replace the notation of the value function in the enlarged market V by $V_{\boldsymbol{\mu}}$ to emphasize its dependence on the appreciation rates $\boldsymbol{\mu}(t-) := (\mu_1(t-),\ldots,\mu_N(t-))'$ of the N *geometric Markovian jump securities* in the enlarged market. Note that the set of admissible strategies in the original incomplete market $\mathcal{A}_0$ can be regarded as the set of admissible strategies in the enlarged market $\mathcal{A}$ if the following constraints are imposed:

$$\pi_j \equiv 0, \quad j = 1,2,\ldots,N.$$

Consequently, we must have the following inequalities:

$$V_0(t,z,\mathbf{e}_i) \le V_{\boldsymbol{\mu}}(t,z,\mathbf{e}_i), \quad i = 1,2,\ldots,N.$$

Since we assume in this section that the appreciation rates of the "fictitious" shares are "free" parameters,

$$V_0(t,z,\mathbf{e}_i) \le \inf_{\boldsymbol{\mu}} V_{\boldsymbol{\mu}}(t,z,\mathbf{e}_i), \quad i = 1,2,\ldots,N. \tag{5.2}$$

In what follows, we shall show that the above inequality is indeed an equality for both logarithmic and power utility functions.

THEOREM 5.1. *Let $V_r(t,z,\mathbf{e}_i)$ be the value function in the enlarged market with the appreciation rates of the N geometric Markovian jump securities equal to the risk-free interest rate. Then*

1. *for the logarithmic utility,*

$$\begin{aligned} V_0(t,z,\mathbf{e}_i) &= V_r(t,z,\mathbf{e}_i) = \inf_{\boldsymbol{\mu}} V_{\boldsymbol{\mu}}(t,z,\mathbf{e}_i) \\ &= \log(z) + E_{t,i}\left[\int_t^T \left(r(s) + \frac{(\mu_0(s-) - r(s))^2}{2\sigma_0(s)^2}\right) ds\right], \end{aligned}$$

and the corresponding optimal portfolio strategies are

$$\tilde{\pi}_0^*(s) = \frac{\mu_0(s-) - r(s)}{\sigma_0(s)^2}, \quad \tilde{\pi}_j^*(s) = 0, \quad j = 1,2,\ldots,N. \tag{5.3}$$

2. *for the power utility,*

$$\begin{aligned} V_0(t,z,\mathbf{e}_i) &= V_r(t,z,\mathbf{e}_i) = \inf_{\boldsymbol{\mu}} V_{\boldsymbol{\mu}}(t,z,\mathbf{e}_i) \\ &= z^{\alpha} E_{t,i}\left\{\exp\left\{\int_t^T \left[\alpha r(s) + \frac{\alpha}{2(1-\alpha)}\left(\frac{\mu_0(s-) - r(s)}{\sigma_0(s)}\right)^2\right] ds\right\}\right\}, \end{aligned}$$

and the corresponding optimal portfolio strategies are

$$\tilde{\pi}_0^*(s) := \frac{\mu_0(s-) - r(s)}{(1-\alpha)\sigma_0(s)^2}, \quad \tilde{\pi}_j^*(s) := 0, \ j = 1,2,\ldots,N.$$

Proof. We first consider the logarithmic utility case. From Theorem 4.2,

$$V_{\boldsymbol{\mu}}(t, z, \mathbf{e}_i) = \log z + h(t, \mathbf{e}_i),$$

where $h(t, \mathbf{e}_i)$ is given by (4.2).

Using a similar argument as in Remark 3.1 of Varaiya [30],

$$1 + \psi_j(t) \geq 0, \ \mathcal{P} - a.s., \quad j = 1, 2, \ldots, N,$$

where $\psi_j(t), j = 1, 2, \ldots, N$, are defined by (3.4). Therefore from the martingale conditions, we have

$$0 \leq \frac{\mu_j(s-) - r(s)}{\sigma_j(s-)\lambda_j(s)} = -\psi_j(s) \leq 1, \quad j = 1, 2, \ldots, N.$$

Consequently, by noting that $f_j(x_j) := x_j - \log(1 - x_j), j = 1, 2, \ldots, N$, are increasing functions on the interval $[0, 1]$, it is easy to verify that

$$\sum_{j=1}^{N} 1_{\{\mathbf{X}(s-) \neq \mathbf{e}_j\}} \lambda_j(s) \left[-\log\left(1 - \frac{\mu_j(s-) - r(s)}{\sigma_j(s-)\lambda_j(s)}\right) + \frac{\mu_j(s-) - r(s)}{\sigma_j(s-)\lambda_j(s)} \right]$$

attains its minimum value at

$$\frac{\mu_j(s-) - r(s)}{\sigma_j(s-)\lambda_j(s)} = 0, \quad j = 1, 2, \ldots, N.$$

That is,

$$\mu_j(s-) = r(s), \quad j = 1, 2, \ldots, N.$$

Thus

$$\begin{aligned}
\inf_{\boldsymbol{\mu}} V_{\boldsymbol{\mu}}(t, z, \mathbf{e}_i) &= \log z + \inf_{\boldsymbol{\mu}} h(t, \mathbf{e}_i) \\
&= \log(z) + E_{t,i}\left[\int_t^T \left(r(s) + \frac{(\mu_0(s-) - r(s))^2}{2\sigma_0(s)^2}\right) ds\right] \\
&= V_r(t, z, \mathbf{e}_i),
\end{aligned}$$

where the last equality follows from Theorem 4.2 with the appreciation rates of the N *geometric Markovian jump securities* equal to the risk-free interest rate. From Theorem 4.2, we also obtain that $\tilde{\pi}_j^*(s)$, $j = 1, 2, \ldots, N$, defined by (5.3) are optimal portfolio strategies. The optimal portfolio strategies defined by (5.3) belong to $\mathcal{A}_0$, since

$$\tilde{\pi}_j^*(s) = 0, \quad j = 1, 2, \ldots, N.$$

Therefore, we must have

$$V_r(t, z, \mathbf{e}_i) \leq V_0(t, z, \mathbf{e}_i) \leq \inf_{\boldsymbol{\mu}} V_{\boldsymbol{\mu}}(t, z, \mathbf{e}_i),$$

where the second inequality follows from (5.2). Thus the first result of the theorem is proved.

From Theorems 4.7 and 4.10, we can also obtain the second result of the theorem using similar arguments as in the logarithmic utility. The main difference is that we need to set the first derivative equal to 0 to determine the minimum point of $V_{\boldsymbol{\mu}}$ over $\boldsymbol{\mu}$ in the power utility case. ∎

6. Summary. We introduced a novel approach to solve the portfolio selection problem in a continuous-time Markovian regime-switching market. By augmenting the market with a set of geometric Markovian jump securities, we made the market complete and considered the portfolio selection problem in the enlarged market. We obtained closed-form solutions to the optimal portfolio strategies and the value functions in both the cases of logarithmic utility and power utility. We also established the relationship between the optimization problem in the enlarged market and in the original market in both logarithmic and power utility cases.

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