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Option Valuation Under a Multivariate Markov Chain Model

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Abstract

In this paper, we develop an option valuation model in the context of a discrete-time multivariate Markov chain model using the Esscher transform. The multivariate Markov chain provides a flexible way to incorporate the dependency of the underlying asset price processes and price multi-state options written on several dependent underlying assets. In our model, the price of an individual asset can take finitely many values. The market described by our model is incomplete in general, hence there are more than one equivalent martingale pricing measures. We adopt conditional Esscher transform to determine an equivalent martingale measure for option valuation. We also document consequences for option prices of the dependency of the underlying asset prices described by the multivariate Markov chain model.

1. Introduction

Option valuation has long been a very important topic in financial economics. Since the seminal work of Black and Scholes [2] and Merton [20], there has been an explosive growth in the amount of literature on the theory and practice of option pricing models. The Black-Scholes-Merton option pricing formula is preference-free; that is, the formula does not depend on the real-world expected return of the underlying asset, which is replaced by the risk-free interest rate.Pricing is done in a risk-neutral world in which the expected return on each asset is the same as the risk-free interest rate. Harrison and Kreps [14], Harrison and Pliska ([15], [16]) showed that the absence of arbitrage is equivalent to the existence of an equivalent martingale measure. If the securities market is complete, there is a unique martingale measure, hence the unique price of any contingent claim is given by its expected discounted payoff at expiry under the martingale measure. However, in an incomplete market, there are infinitely many equivalent martingale measures and therefore, a range of no-arbitrage prices for a contingent claim. The Cox-Ross-Rubinstein (CRR) model introduced in the seminal paper by Cox, Ross and Rubinstein [10] describes the price dynamics of an underlying asset as a binomial lattice in which the price of the asset at a particular time period can take one of the two possible values, namely, "up" and "down". Boyle [3] developed a trinomial model in complete market for option valuation,which assumed that the price of the underlying asset at a particular time period can take three possible values, "up", "middle" and "down". He [17] further extended the lattice models and proposed a discrete-time multinomial model for the valuation of the option written on several correlated risky assets. In the finance literature, there are other important discrete-time models for approximating multivariate diffusion processes, for instances, Boyle [3], Cheyette [8], Boyle, Evnine and Gibbs [4], Madan, Milne an Shefrin [19] and Ho, Stapleton and Subrahmanyam [18]. See Boyle [5] for a comprehensive account on various discrete-time models.

In this paper, we develop an option valuation model in the context of a discrete-time multivariate Markov chain model using a well-known tool, namely, the Esscher transform. The multivariate Markov chain model provides a flexible way to incorporate the dependency of the underlying asset price processes in a discrete framework. In our model, the price of an individual asset can take finitely many values. The market described by our model is incomplete. We adopt the conditional Esscher transform in Buhlmann et al. ([6], [7]) to determine an equivalent martingale measure for option valuation. Our model can incorporate dependency of the price dynamics for individual assets described by a Markovian version of the multinomial model. We also document consequences for option prices of dependency of the underlying asset prices described by multivariate Markov chain model. In particular, we investigate whether misspecification of the level of the dependency of the underlying asset prices can have significant impact on the option prices.

The rest of the paper is organized as follows. In Section 2, we introduce the multivariate Markov chain model for

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modelling the dependency of the price dynamics of the underlying asset. Section 3 presents the Esscher transform for determining an equivalent martingale pricing measure. Section 4 documents consequences for option prices of the dependency of the underlying asset prices described by the multivariate Markov chain model. Finally, concluding remarks are given in Section 5.

2. Asset Price Dynamics

We consider a discrete-time financial model with one primary risk-free asset and n underlying risky assets. Suppose $\mathcal{T}$ represents the time index set $\{0, 1, 2, \ldots, \}$. Let r_t denote the risk-free interest rate over the time interval $[t-1, t]$. We assume that the price dynamics of the risk-free asset B is governed by:

$$B_t = B_{t-1} e^{r_t} \ , t \in \mathcal{T} \backslash \{0\}.$$

Fix a complete probability space $(\Omega, F, \mathcal{P})$, where $\mathcal{P}$ is a real-world physical probability measure. For each $j = 1, 2, \ldots, n$, let $\{S_{jt}\}_{t \in \mathcal{T}}$ denote the price dynamics of the j^{th} risky asset at time t. Let $R_t^{(j)}$ denote the rate of return of the j^{th} risky asset from time $t-1$ to time t; i.e.,

$$S_{jt} = S_{j,t-1} \exp(R_t^{(j)}) \ , t \in \mathcal{T} \backslash \{0\}.$$

We assume that the return processes

$$R := (R^{(1)}, R^{(2)}, \ldots, R^{(n)})$$

of the n risky assets are governed by a multivariate Markov Chain model by Ching, et al. [9]. For each $j = 1, 2, \ldots, n$, we suppose that $R^{(j)} := \{R_t^{(j)}\}_{t \in \mathcal{T} \backslash \{0\}}$ denotes a stochastic process on $(\Omega, F, \mathcal{P})$ with a common state space $L = (L_0, L_1, \ldots, L_{m-1})$. Note that L_i $(i = 0, 1, \ldots, m-1)$ represents one possible state of the return of a risky asset.

On $(\Omega, F, \mathcal{P})$, we define n categorical time series $Y^{(1)}, \ldots, Y^{(n)}$ with common time index set $\mathcal{T}$. Let $\mathcal{S}$ denote a set of unit basis vectors $\{e_0, e_1, \ldots, e_{m-1}\}$ in R^m, where

$$e_i = (0, \ldots, 0, \overbrace{1}^{i^{th} entry}, 0, \ldots, 0)^T \in R^m.$$

For each $j = 1, 2, \ldots, n$, $Y^{(j)} := \{Y_t^{(j)}\}_{t \in \mathcal{T}}$ represents a discrete-time and finite-state stochastic process with state space $\mathcal{S}$. Here $\{Y_t^{(j)}\}_{t \in \mathcal{T}}$ represents the underlying state process for the return dynamics of the j^{th} risky asset. Define the space $\hat{\mathcal{S}}$ as

$$\{s \in R^m | s = \sum_{i=0}^{m-1} \alpha_i e_i, \quad 0 \le \alpha_i \le 1, \quad \sum_{i=0}^{m-1} \alpha_i = 1\}.$$

For each $t \in \mathcal{T}$, we assume that

$$R_t^{(j)} = < L, Y_t^{(j)} >,$$

$< X, Y >$ represents the inner product of two vectors X and Y in $\mathcal{R}^m$.

For each $j = 1, 2, \ldots, n$, we denote the dynamics of the discrete probability distributions for $Y^{(j)}$ as $\{X_t^{(j)}\}_{t \in \mathcal{T}}$, where $X_t^{(j)} \in \hat{\mathcal{S}}$. In particular, for each $t \in \mathcal{T}$, the i^{th} entity of the probability vector $X_t^{(j)}$ represents the probability that the return of the j^{th} asset is in the i^{th} state at time t.

Let $P^{(jk)}$ be a transition probability matrix from the states in the return dynamics of the k^{th} risky asset to the states in the return dynamics of the j^{th} risky asset. Write $X_t^{(k)}$ for the state probability distribution of the return of the k^{th} risky asset at time t. Then, we assume that the dynamics of the probability distributions of the return dynamics for the j^{th} risky asset are governed by the following equation:

$$X_{t+1}^{(j)} = \sum_{k=1}^{n} \lambda_{jk} P^{(jk)} X_t^{(k)}, \quad \text{for} \quad j = 1, 2, \cdots, n \quad (1)$$

$$\lambda_{jk} \ge 0, \quad 1 \le j, k \le n \quad \text{and} \quad \sum_{k=1}^{n} \lambda_{jk} = 1.$$

λ_{ii} $(i = 1, 2)$ describes the intra-dependency of the price dynamics of the i^{th} underlying asset while λ_{ij} describes the inter-dependency of the price dynamics of the i^{th} asset on the price dynamics of the j^{th} asset.

3. Option Valuation by the conditional Esscher transform

We employ a multivariate version of the conditional Esscher transform proposed by Bühlmann et al. ([6], [7]) to determine an equivalent martingale measure in the context of the multivariate Markov chain model. First, suppose $\mathcal{F} := \{\mathcal{F}_t\}_{t \in \mathcal{T}}$ denote the $\mathcal{P}$-augmentation of the natural filtration $\mathcal{F}^Y := \{\mathcal{F}_t^Y\}_{t \in \mathcal{T}}$ generated by the n-dimensional process $Y := (Y^{(1)}, Y^{(2)}, \ldots, Y^{(n)})$ where $\mathcal{F}_t^Y := \sigma\{Y_t^{(1)}, Y_t^{(2)}, \ldots, Y_t^{(n)}\}$ for each $t \in \mathcal{T}$. For each $t \in \mathcal{T} \backslash \{0\}$ and $j = 1, 2, \ldots, n$, the joint conditional distribution $P_{t|t-1}^{(j)}$ of $Y_t^{(j)}$ given $\mathcal{F}_{t-1}$ under $\mathcal{P}$ is given by

$$P_{t|t-1}^{(j)} := (p_{t|t-1}^{(j0)}, p_{t|t-1}^{(j1)}, \ldots, p_{t|t-1}^{(j,m-1)})$$

where

$$p_{t|t-1}^{(ji)} := \mathcal{P}(\{Y_t^{(j)} = e_i\} | \mathcal{F}_{t-1}).$$

Lemma 1. *Let $[V]^i$ denote the i^{th} element of the column vector V. Then, for each $j = 1, 2, \ldots, n$ and $i = 0, 1, \ldots, m-1$,*

$$p_{t|t-1}^{(ji)} = [\sum_{k=1}^{n} \lambda_{jk} P^{(jk)} X_{t-1}^{(k)}]^i |_{X_{t-1} = (e_{i_1}, \ldots, e_{i_n})}. \quad (2)$$

Let $M_{S_t|\mathcal{F}_{t-1}}(Z)$ denote the moment generating function of the joint conditional distribution of the random vector $S_t := (S_{1t}, S_{2t}, \ldots, S_{nt})^*$ given $\mathcal{F}_{t-1}$ under $\mathcal{P}$, where $Z := (Z_1, Z_2, \ldots, Z_n)^* \in \mathcal{R}^n$; i.e.,

$$M_{S_t|\mathcal{F}_{t-1}}(Z) := E(e^{Z^* S_t} | \mathcal{F}_{t-1})$$

where Z^* represents the transpose of the vector Z.

Since S_t can take values in a finite state space, $M_{S_t|\mathcal{F}_{t-1}}(Z) < \infty$, for some $Z \in \mathcal{R}^n$, for each $t \in \mathcal{T}$. Let $\{\Theta_t\}_{t\in\mathcal{T}\backslash\{0\}}$ denote an n-dimensional stochastic process which is predictable with respect to $\mathcal{F}$; i.e., Θ_t is measurable with respect to $\mathcal{F}_{t-1}$, for each $t \in \mathcal{T}\backslash\{0\}$. Then, we define a sequence $\{\Lambda_t\}_{t\in\mathcal{T}}$ with $\Lambda_0 = 1$ and

$$\Lambda_t = \prod_{k=1}^{t} \frac{e^{\Theta_k^* S_k}}{M_{S_k|\mathcal{F}_{k-1}}(\Theta_k)} \ , \ t \in \mathcal{T}\backslash\{0\}.$$

Lemma 2. *$\{\Lambda_t\}_{t\in\mathcal{T}}$ is a $(\mathcal{F}, \mathcal{P})$-martingale.*

Define a probability measure $\mathcal{P}_\Theta \sim \mathcal{P}$ on (Ω, F) by the following multivariate conditional Esscher transform:

$$\left.\frac{d\mathcal{P}_\Theta}{d\mathcal{P}}\right|_{\mathcal{F}_t} = \Lambda_t.$$

We note that Θ_t is the vector of conditional Esscher parameters given $\mathcal{F}_{t-1}$, for each $t \in \mathcal{T}\backslash\{0\}$. Let $P_{t|t-1}^{(i_1,i_2,\ldots,i_n)}$ denote the joint conditional distribution of $Y_t := (Y_t^{(1)}, Y_t^{(2)}, \ldots, Y_t^{(n)})$ given $\mathcal{F}_{t-1}$ under $\mathcal{P}$; i.e.,

$$\begin{aligned} P_{t|t-1}^{(i_1,i_2,\ldots,i_n)} &:= \mathcal{P}(Y_t^{(1)} = e_{i_1}, \ldots, Y_t^{(n)} = e_{i_n}|\mathcal{F}_{t-1}) \\ &= \prod_{j=1}^{n} p_{t|t-1}^{(ji_j)}. \end{aligned} \quad (3)$$

Suppose $P_{t|t-1}^{(i_1,i_2,\ldots,i_n)}(\Theta)$ denotes the joint conditional distribution of $Y_t := (Y_t^{(1)}, Y_t^{(2)}, \ldots, Y_t^{(n)})$ given $\mathcal{F}_{t-1}$ under $\mathcal{P}_\Theta$; i.e.,

$$\begin{aligned} & P_{t|t-1}^{(i_1,i_2,\ldots,i_n)}(\Theta) \\ := \ & \mathcal{P}_\Theta(Y_t^{(1)} = e_{i_1}, \ldots, Y_t^{(n)} = e_{i_n}|\mathcal{F}_{t-1}) \\ = \ & \frac{P_{t|t-1}^{(i_1,i_2,\ldots,i_n)} \exp(\sum_{j=1}^n S_{j,t-1}\Theta_t^{(j)} e^{L_{i_j}})}{E(e^{\Theta_t^* S_t}|\mathcal{F}_{t-1})} \\ = \ & \frac{\prod_{j=1}^n p_{t|t-1}^{(ji_j)} \exp(\sum_{j=1}^n S_{j,t-1}\Theta_t^{(j)} e^{L_{i_j}})}{\sum_{i_1=0}^{m-1} \cdots \sum_{i_n=0}^{m-1} \prod_{j=1}^n p_{t|t-1}^{(ji_j)} \exp(\sum_{j=1}^n S_{j,t-1}\Theta_t^{(j)} e^{L_{i_j}})}, \end{aligned}$$

where $\Theta_t := (\Theta_t^{(1)}, \Theta_t^{(2)}, \ldots, \Theta_t^{(n)})$.

Harrison and Kreps [14] established the relationship between the absence of arbitrage opportunities and the existence of an equivalent martingale measure under which all discounted asset price processes are martingale. This result is known as the fundamental theorem of asset pricing and further extended by Harrison and Pliska ([15], [16]), Back and Pliska [1], Schachermayer [21] and Delbaen and Schachermayer [11]. Back and Pliska [1] showed that the absence of arbitrage opportunities is equivalent to the existence of an equivalent martingale measure in a discrete-time and infinite-state-space setting.

When there is no arbitrage and the market is complete, there exists a unique equivalent martingale measure.However, when the market is incomplete, there are infinitely many equivalent martingale measures, and, hence a range of no-arbitrage prices for the claim. We employ the conditional Esscher transform by Bühlmann et al. (1996) [6] to determine an equivalent martingale measure in the sequel.

First, we present an expression for the moment generating function $M_S(t, U; \Theta)$ of the joint conditional distribution of the random vector S_t given $\mathcal{F}_{t-1}$ under $\mathcal{P}_\Theta$

Lemma 3.

$$M_S(t, U; \Theta) := E^\Theta(e^{U^* S_t}|\mathcal{F}_{t-1}) = \frac{M_{S_t|\mathcal{F}_{t-1}}(\Theta_t + U)}{M_{S_t|\mathcal{F}_{t-1}}(\Theta_t)}, \quad (4)$$

where $E^\Theta(\cdot)$ represents the expectation operator with respect to the probability measure $\mathcal{P}_\Theta$ and U^* is the transpose of the vector $U := (U_1, U_2, \ldots, U_n) \in \mathcal{R}^n$.

By employing the multivariate conditional Esscher transform, we determine an equivalent martingale pricing measure under which the discounted price processes of the n risky assets at the risk-free interest rate $\{\frac{S_{jt}}{B_t}\}$, for $i = 1, 2, \ldots, n$, are martingale with respect to $\{\mathcal{F}_t\}_{t\in\mathcal{T}}$. The following proposition presents a sufficient condition on the sequence Θ for $\mathcal{P}_\Theta$ to be an equivalent martingale measure.

Proposition 1. *Let $I_j := (0, 0, \ldots, \overbrace{1}^{j^{th} entry}, \ldots, 0, 0) \in \mathcal{R}^n$, for each $t \in \mathcal{T}, j = 1, 2, \ldots, n$. Suppose Θ_t satisfies the following system of n coupled non-linear equations:*

$$E[e^{\Theta_t^* S_t}(e^{I_j^* S_t} - e^{r_t})|\mathcal{F}_{t-1}] = 0 \ . \quad (5)$$

Then, the discounted price processes $\{\frac{S_{jt}}{B_t}\}$ are martingales with respect to $\{\mathcal{F}_t\}_{t\in\mathcal{T}}$ under $\mathcal{P}_\Theta$, for each $j = 1, 2, \ldots, n$.

Proposition 2. *Suppose for each $t \in \mathcal{T}, \Theta_t^*(S_t e^{-r_t} - S_{t-1})$ either equals zero with probability one or has both signs with positive probability. Then, there exist a Θ_t satisfying the following system of n coupled non-linear equations:*

$$E[e^{\Theta_t^* S_t}(e^{I_j^* S_t} - e^{r_t})|\mathcal{F}_{t-1}] = 0 \ .$$

It can also be shown that the martingale conditions can be written as follows:

$$\sum_{i_1=0}^{m-1} \cdots \sum_{i_n=0}^{m-1} \left\{ \prod_{j=1}^{n} p_{t|t-1}^{(ji_j)} \exp\left(\sum_{j=1}^{n} S_{j,t-1}\Theta_t^{(j)} e^{L_{i_j}} \right) X \right\} = 0, \quad (6)$$

where $X = \exp(S_{k,t-1}e^{L_{i_k}}) - \exp(r_t)$ for $k = 1, 2, \ldots, n$.

Now, we consider a European-style contingent claim V written on the n risky assets with maturity at time T and payoff function $V(S_{1T}, S_{2T}, \ldots, S_{nT}, T)$ at time T. Then, a price of the contingent claim V_t at time t is given by:

$$V_t = E^\Theta\left[\exp\left(- \sum_{k=t+1}^{T} r_k \right) V(S_{1T}, \ldots, S_{nT}, T) \middle| \mathcal{F}_t \right], \quad (7)$$

which is the conditional expectation of the discounted payoff of the claim V given the $\mathcal{F}_t$ under $\mathcal{P}_\Theta$. We note that there are other possible ways to determine a price of the option.

4. Consequences for option prices of the dependency of asset prices

In this section, we document consequences for option prices of the dependency of the underlying asset prices described by the multivariate Markov chain model. We investigate whether misspecification of the level of the dependency of the underlying asset prices can have significant impact on option prices. We shall consider a financial model with one risk-free asset B and two risky assets S_1 and S_2. The state space of the return process $R^{(j)}$ of the j^{th} risky asset ($j = 1, 2$) is given by $\{L_0, L_1, L_2\}$. In practice, the returns of a risky asset take real values instead of the categorical values $\{L_0, L_1, L_2\}$. The discrete state space can only serve as a proxy for the "actual" state space of the returns. Similar procedure has been adopted in Elliott and Rishell [12] for approximating the state space of a short rate process. We assume that $R_t^{(j)} = L_i = (i-1)/20$, for $i = 0, 1, 2$.

First, we investigate the situation that the "true" model is described by the multivariate Markov chain model with a "Strong" level of the inter-dependency of the price dynamics of the two underlying assets. While the "assumed" model used for the evaluation of option prices is described by the multivariate Markov chain model with the level of inter-dependency of the price dynamics of the two underlying assets ranged from "Strong" to "Weak". We suppose that in the "true" model, the model parameters are given as follow:

$$P^{(11)} = \begin{bmatrix} 0.4069 & 0.3995 & 0.5642 \\ 0.3536 & 0.5588 & 0.0470 \\ 0.2395 & 0.0416 & 0.3887 \end{bmatrix},$$

$$P^{(12)} = \begin{bmatrix} 0.2016 & 0.2737 & 0.2056 \\ 0.2970 & 0.1303 & 0.4917 \\ 0.5014 & 0.5959 & 0.3027 \end{bmatrix},$$

$$P^{(21)} = \begin{bmatrix} 0.2554 & 0.2814 & 0.4571 \\ 0.7321 & 0.3558 & 0.2542 \\ 0.0126 & 0.3628 & 0.2887 \end{bmatrix},$$

$$P^{(22)} = \begin{bmatrix} 0.5102 & 0.5239 & 0.1434 \\ 0.3736 & 0.3925 & 0.4204 \\ 0.1162 & 0.0835 & 0.4361 \end{bmatrix},$$

and

$$\Lambda = \begin{bmatrix} 0.0000 & 1.0000 \\ 0.5000 & 0.5000 \end{bmatrix}.$$

In this case, 100% is allocated to the inter-transition probability matrix $P^{(12)}$ under the "true" model. This represents a "Strong" inter-dependency effect.

For the "assumed" model, we assume that the intra-transition probability matrices, inter-transition probability matrices and the parameters λ_{21}, λ_{22} are the same as those in the "true" model. The only difference is that λ_{11} increases from 0.0000 to 1.0000; in other words, λ_{12} decreases from 1.0000 to 0.0000.

We consider a European-style exchange option written on the two risky assets, which provides a buyer with the right, but not the obligation, to exchange the first risky asset for the second risky asset at the maturity of the option. We have that $V(S_{1T}, S_{2T}, T) := \max(S_{2T} - S_{1T}, 0)$ at the maturity time $T = 3$. We then adopt the option pricing formula in Equation (7) to determine the price of the exchange option. We consider three cases that the exchange option is at-the-money, out-of-the-money and in-the-money respectively. We suppose that the compound risk-free interest rate is constant and equal to 2.5%. We first assume that $S_{10} = 100$ and $S_{20} = 100$. We note that the price of the exchange option obtained from the "true" model is 0.4776. Now, we consider the case that $S_{10} = 120$ and $S_{20} = 100$. In this case, the price of the exchange option obtained from the "true" model is $5.6903e-005$. Finally, we consider the case that $S_{10} = 80$ and $S_{20} = 100$. The price of the exchange option obtained from the "true" model is 17.9539. Figures 1-3 display the plots of the prices for the exchange option obtained from the "assumed" models against different levels of inter-dependence λ_{12} for the three cases respectively.

From Figures 1-3, we can see that the option prices change significantly as the level of dependency λ_{12} varies for all cases, namely, at-the-money, in-the-money and out-of-the-money. Given the configuration for the transition probability matrices, the option prices implied by the "assumed" model increase as the "assumed" level of the dependency does for all cases. In other words, if the "true" model has a "Strong" level of the dependency, say $\lambda_{12} = 1$, the underpricing of the "assumed" model becomes more pronounced when the "assumed" level of the dependency decreases from $\lambda_{12} = 1$ to $\lambda_{12} = 0$.

5. Conclusion

We have developed an option valuation model in the context of a discrete-time multivariate Markov chain model using the conditional Esscher transform introduced by Bühlmann et al. [6]. This model can provide market practitioners with a flexible way to incorporate the dependency of the underlying asset price processes in a discrete framework. It also allows the price of an individual risky asset taking finitely many values. We have documented consequences for option prices of the dependency of the underlying asset prices described by the multivariate Markov chain model.

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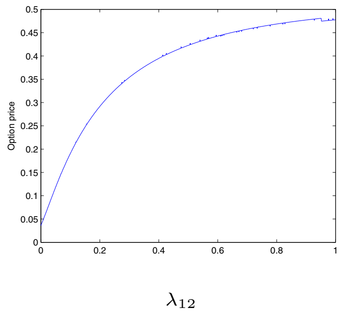

Figure 1. At-the-money with various value of λ_{12}

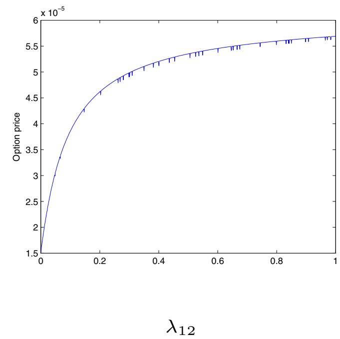

Figure 2. Out-of-money with various value of λ_{12}

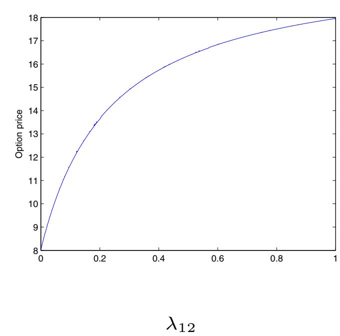

Figure 3. In-the-money with various value of λ_{12}

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