Decoherence-free quantum information in the presence of dynamical evolution

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We analyze decoherence-free (DF) quantum information in the presence of an arbitrary non-nearest-neighbor bath-induced system Hamiltonian using a Markovian master equation. We show that the most appropriate encoding for N qubits is probably contained within the $\sim \frac{2}{9}N$ excitation subspace. We give a time scale over which one would expect to apply other methods to correct for the system Hamiltonian. In order to remain applicable to experiment, we then focus on small systems, and present examples of DF quantum information for three and four qubits. We give an encoding for four qubits that, while quantum information remains in the two-excitation subspace, protects against an arbitrary bath-induced system Hamiltonian. Although our results are general to any system of qubits that satisfies our assumptions, throughout the paper we use dipole-coupled qubits as an example physical system.

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I. INTRODUCTION

Storing quantum information for long periods is difficult: excited quantum states decohere. One method of counteracting the effects of decoherence is to encode logical information into decoherence-free subspaces and subsystems (DFSs) [1–8]. These are groups of states that have robust symmetry properties, and that in ideal cases serve as perfect quantum memory. See Ref. [9] for a comprehensive review of DFS theory. As well as theoretical developments, there has been much success experimentally. For example, DFSs have been prepared in optical systems, in NMR, and in ion traps [10–13], and there has been a proposal for decoherence-free (DF) quantum-information processing in nitrogen-vacancy (NV) centers in diamond [14].

Here, we focus on the semigroup formulation of DFSs for a number of reasons. First, the unitary and nonunitary evolutions of the system are naturally separated. Second, when the DF condition is satisfied, there is only one nonunitary Lindblad operator [6,7]. Finally, the scalability properties of various encodings can be quantified [7,9]. The separation of unitary and nonunitary evolution enables the detrimental effect of any environment-induced unitary evolution to be isolated. We refer to "environment-induced" or equivalently, "bath-induced" evolution because both the unitary and nonunitary parts of the master equation result directly from the system-environment interaction Hamiltonian. For a single atom, the unitary part gives rise to the Lamb shift and the nonunitary part gives rise to spontaneous decay. For many qubits, decoherence can result indirectly from the action of Lamb-shift-type terms because these can cause quantum information to leak into states that decay. A full analysis of the effect of unitary evolution on DF quantum information in Markovian systems is the purpose of this paper. Throughout

the paper, when we refer to a system Hamiltonian causing leakage of DF quantum information, we are referring explicitly to the action of a Lamb-shift-type Hamiltonian on DF states.

It is well known that the condition for DF dynamics derived in Ref. [7] does not rule out the possibility of an environment-induced unitary operator evolving information encoded in a DFS into non-DFS states. The results in Ref. [7] were extended in Ref. [15], where the authors gave conditions for DF quantum information that accounted for a system Hamiltonian. Here, we focus on quantum information that is encoded within states that satisfy the DF condition [7], but that could evolve into non-DF states due to a system Hamiltonian. We emphasize that for Markovian systems, the property of the bath operators is such that generally, the presence of a non-nearest-neighbor system Hamiltonian is unavoidable. Physical justification for the regime studied in this paper is given in Ref. [16]. Here, the authors showed that for closely spaced dipole-coupled qubits, the nonunitary evolution can be approximated as being the same as that for colocated qubits (and so satisfy the DF requirements derived in Ref. [7]), but the unitary evolution depends on both the position and orientation of the dipoles. Our results are applicable to any physical system that can be appropriately described by a Markovian master equation and, in light of recent experimental progress, are directly applicable to NV centers in diamond [17].

The paper is structured as follows. In Sec. II, we state the conditions on the bath operators that lead to a bath-induced unitary evolution. In Sec. III, we give a condition for subspaces which are immune to nonunitary evolution. This condition is weaker than that derived in Ref. [15], but stronger than that derived in Ref. [7]. Then, in Sec. IV we analyze the scalability—which we define as the encoding efficiency multiplied by the proportion of the Hilbert space that is DF—for DF encoding in the presence of a system Hamiltonian. An interesting consequence of our results is that, when the effect of a system Hamiltonian is included, the most suitable sub-

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space for quantum information storage in N qubits is the subspace with $\sim \frac{2}{9}N$ excitations. Then, we give a time scale over which other methods will have to be applied to counter the effect of the system Hamiltonian. In Sec. V, we concentrate on three and four qubit systems. We give conditions on the system Hamiltonian for DF quantum information, and we give an explicit encoding for four qubits that, while quantum information remains in the two-excitation subspace, protects against errors induced by an arbitrary system Hamiltonian. We hope that this encoding is directly relevant to experimental quantum information processing.

II. MASTER EQUATION

For clarity and to establish our notation, we briefly summarize a derivation of the Lindblad master equation. A system A coupled to a bath B can be described by the Hamiltonian $\mathbf{H} = \mathbf{H}_A \otimes \mathbf{I}_B + \mathbf{I}_A \otimes \mathbf{H}_B + \mathbf{H}_I$, where $\mathbf{H}_A$ ($\mathbf{H}_B$) the system (bath) Hamiltonian acts on the system (bath) Hilbert space, $\mathbf{I}_A$ ($\mathbf{I}_B$) is the identity operator on the system (bath) Hilbert space, and $\mathbf{H}_I$ is the interaction Hamiltonian that contains all nontrivial couplings between the system and the bath and is written as

$$\mathbf{H}_I = \sum_\alpha \mathbf{S}_\alpha \otimes \mathbf{B}_\alpha, \tag{1}$$

for $\mathbf{S}_\alpha$ ($\mathbf{B}_\alpha$) the system (bath) operators. Treating the interaction Hamiltonian as a perturbation, the equation of motion for the density matrix χ of the system and bath in the interaction picture is ($\hbar = 1$)

$$\dot{\chi}(t) = -i[\mathbf{H}_I(t), \chi(t)], \tag{2}$$

which gives

$$\dot{\rho}(t) = -\int_0^t ds \, \mathrm{Tr}_B[\mathbf{H}_I(t), [\mathbf{H}_I(s), \chi(s)]], \tag{3}$$

where Tr_B denotes the trace over the bath and ρ is the (reduced) system density matrix. So, for the rest of this paper, we focus on the Hilbert space of the system. Making the Born-Markov approximations, Eq. (3) becomes

$$\dot{\rho}(t) = -\int_0^\infty ds \, \mathrm{Tr}_B[\mathbf{H}_I(t), [\mathbf{H}_I(s), \rho(t) \otimes R_0]] \tag{4}$$

for R_0 the initial density operator of the (stationary) bath. Introducing the correlation function

$$C_{\alpha\beta}(s) \equiv \mathrm{Tr}_B[\mathbf{B}_\alpha^\dagger(s)\mathbf{B}_\beta(0)R_0], \tag{5}$$

which we note has real and imaginary parts because, in general, $\mathbf{B}_\alpha^\dagger(s)$ and $\mathbf{B}_\beta(0)$ do not commute, the master equation in the rotating-wave approximation is written [18]

$$\dot{\rho} = L[\rho] = -i[\mathbf{H}_S, \rho] + L_D[\rho],$$

$$L_D[\rho] = \sum_{\alpha,\beta} a_{\alpha\beta} L_{\mathbf{S}_\alpha, \mathbf{S}_\beta}[\rho],$$

$$L_{\mathbf{S}_\alpha, \mathbf{S}_\beta}[\rho] = [\mathbf{S}_\beta, \rho \mathbf{S}_\alpha^\dagger] + [\mathbf{S}_\beta \rho, \mathbf{S}_\alpha^\dagger], \tag{6}$$

for the system Hamiltonian

$$\mathbf{H}_S = \sum_{\alpha,\beta} b_{\alpha\beta} \mathbf{S}_\alpha^\dagger \mathbf{S}_\beta, \tag{7}$$

and where $a_{\alpha\beta} = \Gamma_{\alpha\beta} + \Gamma_{\beta\alpha}^*$ and $b_{\alpha\beta} = \frac{1}{2i}[\Gamma_{\alpha\beta} - \Gamma_{\beta\alpha}^*]$ for

$$\Gamma_{\alpha\beta} \equiv \int_0^\infty ds \, e^{i\omega_0 s} C_{\alpha\beta}(s). \tag{8}$$

The correlation function $C_{\alpha\beta}(s)$ contains all the information about the physical system, and satisfies the Kramers-Kronig relations. For a single atom, $\mathbf{H}_S$ describes the Lamb shift, and $L_D[\rho]$ describes the spontaneous emission. For two or more dipole-coupled qubits, the off-diagonal terms in the Hermitian matrix ($a_{\alpha\beta}$) describe the rate of spontaneous emission between separate physical qubits, and the off-diagonal terms in the matrix ($b_{\alpha\beta}$) describe the coherent dipole-dipole interaction between separate qubits. Generally, the terms in ($b_{\alpha\beta}$) diverge and require renormalization [16].

A. Decoherence-free subspaces

A subspace $\tilde{\rho}$ of the system Hilbert space ρ is a DFS if it satisfies $L_D[\tilde{\rho}] = 0$ [7]. The condition that leads to $L_D[\tilde{\rho}] = 0$ being satisfied is $a_{\alpha\beta} \equiv a$ [19]. For dipole-coupled qubits, this implies that the spontaneous emission rate between separated qubits is the same as that for individual qubits—all qubits experience the same nonunitary couplings. Here, we focus on the effect of the system Hamiltonian on $\tilde{\rho}$. Note that when we refer to the system Hamiltonian, we are not referring to $\mathbf{H}_A$, but $\mathbf{H}_S$ which results from $\mathbf{H}_I$. We consider the situation in which the Hamiltonian $\mathbf{H}_S$ evolves information encoded in a DFS into non-DFS states [7,20]. We are interested in the physically applicable perturbative regime $(a_{\alpha\beta}) = a\mathbf{X} + \epsilon\mathbf{A}$ and $(b_{\alpha\beta}) = b\mathbf{X} + \epsilon\mathbf{B}$ for $\mathbf{X}$ a matrix of size $(a_{\alpha\beta})$ with all entries equal to one, $\mathbf{A}$ and $\mathbf{B}$ matrices also of size $(a_{\alpha\beta})$, but with arbitrary entries, and expansion parameter $\epsilon \ll 1$.

Specifically, we concentrate on the regime $(a_{\alpha\beta}) = a\mathbf{X}$ and $(b_{\alpha\beta}) = b\mathbf{X} + \epsilon\mathbf{B}$. We do this for two reasons. First, it has been shown that DFSs are stable to first order even in the presence of a nonunitary symmetry breaking perturbation [20]. All else being equal, small changes in the form of $L_D[\rho]$ do not prevent infinite-lifetime quantum information storage. Second, the strict requirement that all qubits experience the same environment is unlikely to ever be met in practice. If one perturbs the system-environment couplings, the perturbation applies to both the nonunitary and unitary parts of the master equation.

We illustrate the regime of interest to this paper using dipole-coupled qubits. For these qubits the appropriate expansion parameter is physical separation r. So, we concentrate on collections of qubits that satisfy

$$a_{\alpha\beta} = a + O(r^2), \tag{9}$$

$$b_{\alpha\beta} = b + b_{\alpha\beta}'(r) + O(r^2), \tag{10}$$

where the system of qubits is such that any $O(r)$ contribution to $a_{\alpha\beta}$ can be neglected. Equations (9) and (10) are satisfied by closely spaced dipole-coupled qubits—explicit forms for $a_{\alpha\beta}$ and $b_{\alpha\beta}$ are given in Ref. [16]. Note that $b_{\alpha\beta}'(r)$ can be many orders of magnitude larger than a.

We emphasize that the results presented here are applicable to any Markovian system for which $\mathbf{A}=\mathbf{0}$ and $\mathbf{B}\neq\mathbf{0}$. This is the most general case for Markovian systems that satisfy the DF condition, and is more likely to be realized in the laboratory than $\mathbf{A}=\mathbf{0}$ and $\mathbf{B}=\mathbf{0}$. Note that if $\mathbf{A}=\mathbf{0}$ and $\mathbf{B}=\mathbf{0}$, then $\Gamma_{\alpha\beta}$ is strictly qubit independent, and $\mathbf{H}_S$ does not evolve information encoded in $\tilde{\rho}$ into non-DF states [4,6].

For non-Markovian systems, leakage resulting from Lamb-shift-type terms in the master equation can be accounted for using dynamical decoupling methods, or "bang-bang" pulses [21–24]. Here, we focus on *passive* error correction that does not require fast and strong pulses and so is more amenable to experimental implementations. Also, bang-bang pulses are inherently non-Markovian, and applying similar techniques in a Markovian environment is difficult.

III. COMPLETELY DECOHERENCE-FREE SUBSPACES

The condition $a_{\alpha\beta}\equiv a$ means that the dissipator can be written

$$L_D[\rho] = \lambda(2\mathbf{J}\rho\mathbf{J}^\dagger - \mathbf{J}^\dagger\mathbf{J}\rho - \rho\mathbf{J}^\dagger\mathbf{J}), \quad (11)$$

where λ is the only nonzero eigenvalue of $a_{\alpha\beta}$ and $\mathbf{J}=\Sigma_\alpha^N \mathbf{S}_\alpha$ for N the number of qubits. The jump operator can be written

$$\mathbf{J} = \sum_{i=1}^N \hat{\sigma}_{i-} = \sum_{i=1}^N I\otimes\cdots I\otimes\underbrace{\hat{\sigma}_-}_{i\text{th qubit}}\otimes I\cdots, \quad (12)$$

where I is the 2×2 identity operator. So, $\hat{\sigma}_{ia}$, for $a=+,-,z$, acts on the ith qubit, and satisfies $[\hat{\sigma}_{iz},\hat{\sigma}_{j\pm}]=\pm 2\delta_{ij}\hat{\sigma}_{i\pm}$, and $[\hat{\sigma}_{i+},\hat{\sigma}_{j-}]=\delta_{ij}\hat{\sigma}_{iz}$. The operators $\mathbf{J}$, $\mathbf{J}^\dagger$, and $\mathbf{J}_z$ act on the system Hilbert space $\mathcal{H}\equiv\otimes^N\mathbb{C}^2$. From the representation theory of su(2), $\mathcal{H}$ can be decomposed into irreducible components [25]—see Fig. 3 for a three qubit example. Each irreducible representation (irrep) V is generated by a unique lowest weight vector v satisfying $\mathbf{J}\cdot v\equiv\mathbf{J}_-\cdot v^-=0$. The vector v^- is an eigenvector of

$$\mathbf{J}_z = \sum_{i=1}^N \hat{\sigma}_{iz}. \quad (13)$$

The vector $\mathbf{J}^k\cdot v$, for k applications of $\mathbf{J}$ on v is an eigenvector of $\mathbf{J}_z$ with eigenvalue $-(N-2k)$. The decoherence operator $\mathbf{J}$ causes the system to decay to its lowest weight. The DF condition $L_D[V]=0$ for V a subspace implies that $\mathbf{K}\cdot V=0$ for all jump operators $\mathbf{K}$ [7]. Here, we only have one jump operator $\mathbf{J}$, so the subspace V satisfies $\mathbf{J}\cdot V=0$, and consists of combinations of lowest weight vectors. For example, the states $|\mathsf{b}\rangle$ and $|\mathsf{c}\rangle$ in Fig. 3 satisfy $\mathbf{J}|\mathsf{b}\rangle=0$ and $\mathbf{J}|\mathsf{c}\rangle=0$. They are states that do not decay to $|\mathsf{a}\rangle$ and so are DF. The operator $\mathbf{J}^\dagger$ acts like $\mathbf{J}^\dagger|\mathsf{b}\rangle=|\mathsf{e}\rangle$ and $\mathbf{J}^\dagger|\mathsf{c}\rangle=|\mathsf{f}\rangle$, but does not cause transitions from $|\mathsf{b}\rangle$ to $|\mathsf{f}\rangle$, or from $|\mathsf{c}\rangle$ to $|\mathsf{e}\rangle$.

The definition for DF dynamics [9]—a subspace W of states is DF if any $\rho(0)\in W$ evolves into a state $\rho(t)$ such that the evolution map $\rho(0)\rightarrow\rho(t)$ is unitary $\forall\ t$—is relevant here. We are interested in the evolution of quantum information under the action of both L_D and $\mathbf{H}_S$. Ensuring $L_D[\rho]$

=0 for some subspace is not sufficient to guarantee DF dynamics for all time. We call a subspace W completely decoherence-free (CDF) if it satisfies the following conditions: (1) $L_D[W]=0$, and (2) $\rho(t)\in W\ \forall\ t$.

These conditions ensure that the evolution is unitary. We note that the result at the replica symmetric point derived in Refs. [4,6] ensures CDF dynamics, and that the conditions stated here still permit the transfer of encoded information between states within W. For the purposes of this paper, when we refer to transfer of encoded information, we are referring to the effect of the off-diagonal terms of $\mathbf{H}_S$ in the Clebsch-Gordan basis on quantum information. A basic characterization of completely decoherence-free subspaces is given below.

Proposition 1. Let V be the subspace of lowest weight vectors. A necessary and sufficient condition that V contain a CDF subspace W is $\mathbf{H}_S\cdot W\subset W$. In particular, $\mathbf{H}_S$ can be diagonalized in W.

Proof. Define $\rho'(t)=e^{i\mathbf{H}_S t}\rho(t)e^{-i\mathbf{H}_S t}$ for some Hamiltonian $\mathbf{H}_S$. The equation satisfied by ρ' is

$$\frac{\partial\rho'}{\partial t} = L_D'[\rho'], \quad (14)$$

where $L_D'[\mathbf{J}]=L_D[e^{i\mathbf{H}_S t}\mathbf{J}e^{-i\mathbf{H}_S t}]$. So, in this picture $\rho'(0)$ is DF if and only if $L_D'[\rho'(0)]=0$. The generic DFSs are spanned by vectors $|x\rangle$ such that $\mathbf{J}'|x\rangle=e^{i\mathbf{H}_S t}\mathbf{J}e^{-i\mathbf{H}_S t}|x\rangle=0$. Let W be the subspace consisting of all such vectors. This must be satisfied for all t, so $\mathbf{H}_S\cdot W\subset W$. Conversely, if this is satisfied then

$$\mathbf{J}e^{-i\mathbf{H}_S t}|x\rangle = 0. \quad (15)$$

∎

This condition is weaker than that derived in Ref. [15], but stronger than that derived in Ref. [7]. Throughout this paper, it is implicitly assumed that if Eq. (15) is satisfied, then other techniques, such as quantum error correction, are applied to account for transfer of encoded information between states within a CDFS.

We have observed that DFSs are lowest weight vectors in the decomposition of $\mathbb{C}^{2^N}$ into irreps of su(2). The weight space $W(k)$ of weight k in $\mathcal{H}$ is the eigenspace of $\mathbf{J}_z$ with eigenvalue k. Since $[\mathbf{J}_z,\mathbf{H}_S]=0$ weight spaces $W(k)$ are left invariant by $\mathbf{H}_S$. Hence, it is sufficient to consider subspaces with a fixed weight; that is, all states in the subspace have the same number of excited (physical) qubits. Note that these nontrivial subspaces span across several irreps, so it is necessary to combine irreps if one is to satisfy Eq. (15). From the representation theory of su(2) [25], it follows that the weight spaces have weights $-N,-(N-2),\dots,N$. For $k\leq N/2$ the dimension of space $W(k)$ for k excitations with weight $-(N-2k)$ is $\binom{N}{k}$. The condition $\mathbf{J}\cdot V(k)=0$, $V(k)\subset W(k)$ allows us to write the dimensions of $V(k)$ as

$$\dim[V(k)] = \binom{N}{k} - \binom{N}{k-1} = \frac{N!(N-2k+1)}{k!(N-k+1)!}. \quad (16)$$

IV. SCALABILITY PROPERTIES AND LEAKAGE TIME SCALE

A sufficient condition for CDF dynamics derived in Ref. [4] for N qubits that are prepared in a DFS is as follows. If the unitary coupling satisfies $b_{\alpha\beta} \equiv b$, then the Hamiltonian can be written

$$\mathbf{H}_S = b \sum_{\alpha,\beta} \mathbf{S}_\alpha^\dagger \mathbf{S}_\beta = b \mathbf{J}^\dagger \mathbf{J}. \tag{17}$$

Thus, $\mathbf{H}_S$ is a product of the total operators $\mathbf{J}$ and $\mathbf{J}^\dagger$, and the irreps are left invariant by $\mathbf{H}_S$. An example of this is N co-located qubits that are described by the master equation derived in Ref. [16]. In this instance, the multiqubit level shift is the same for all qubits. Note that if $b_{\alpha\beta} \equiv b$ is satisfied, then DFSs are stable to a symmetry breaking perturbation [20].

For N qubits, one can estimate the effect of $\mathbf{H}_S$ on encoded quantum information by examining the proportion of the Hilbert space that consists of DF states relative to the proportion that consists of non-DF states. We consider vectors of the same weight, or equivalently, the same level of excitations. For $V(k)$ the DF subspace in $W(k)$ [with weight $-(N-2k)$, $2k \leq N$] the encoding efficiency is defined as the number of logical qubits per number of physical qubits, and r is independent of N. So, using Eq. (16) we define

$$d_{\mathrm{DF}} \equiv \frac{1}{N} \log_2 \dim[V(k)] = \frac{1}{N} \log_2 \left[\frac{N!(N-2k+1)}{k!(N-k+1)!} \right]. \tag{18}$$

The encoding efficiency d_{DF} measures how many DF qubits can be encoded into a Hilbert space $\mathcal{H} \equiv \otimes^N \mathbb{C}^2$, and is unity for scalable encoding. Writing $k = rN$, where $r \leq 1/2$, and taking the limit $N \to \infty$ gives

$$d_{\mathrm{DF}} \overset{N \to \infty}{\longrightarrow} -r \log_2 r - (1-r)\log_2(1-r), \tag{19}$$

where N is the number of physical qubits, and r is independent of N. For $N \to \infty$, d_{DF} is a maximum for $r = 1/2$. This is the canonical strong-collective DFS [26].

Encoding efficiency alone is not sufficient as a measure of scalability when the Hamiltonian $\mathbf{H}_S$ can cause leakage of quantum information from DF to non-DF states. So, as a measure of the likelihood that an arbitrary $\mathbf{H}_S$ causes quantum information to transfer between states within a particular excitation, we define

$$p_{\mathrm{DF}} \equiv \frac{\dim[V(k)]}{\dim[W(k)]} = 1 + \frac{k}{k-1-N}, \tag{20}$$

which is simply the fraction of a particular weight space that satisfies $\mathbf{J} \cdot V(k) = 0$. So, p_{DF} measures the proportion of DF states relative to non-DF states for a particular excitation. For $N \to \infty$, p_{DF} is

$$p_{\mathrm{DF}} \overset{N \to \infty}{\longrightarrow} \frac{1-2r}{1-r}. \tag{21}$$

The smaller p_{DF}, the more likely that $\mathbf{H}_S$ will cause quantum information to evolve into non-DF states in that particular
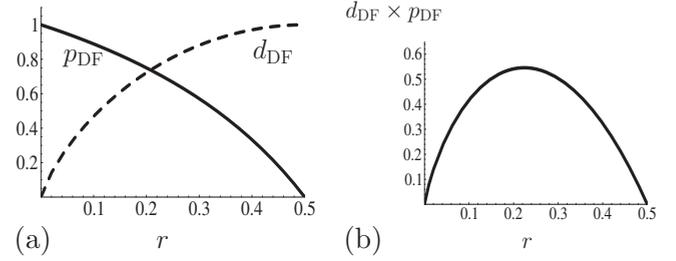

FIG. 1. (a) Plot of d_{DF} (dotted line) and p_{DF} (solid line) and (b) $d_{\mathrm{DF}} \times p_{\mathrm{DF}}$ for large N for d_{DF} and p_{DF} as defined in Eqs. (19) and (21), respectively. In (b), $d_{\mathrm{DF}} \times p_{\mathrm{DF}}$ is a maximum for $r \sim \frac{2}{9}$.

weight space. We show d_{DF} and p_{DF} for large N in Fig. 1(a), and the product $d_{\mathrm{DF}} \times p_{\mathrm{DF}}$ in Fig. 1(b). For an arbitrary $\mathbf{H}_S$ that causes quantum information to transfer between states within some weight space, the subspace which maximizes $d_{\mathrm{DF}} \times p_{\mathrm{DF}}$ is one for which $k \sim \frac{2}{9}N$. Of course, this assumes that one can account for the unitary evolution caused by $\mathbf{H}_S$ within the CDFSs. For particular forms of $\mathbf{H}_S$, it might be more appropriate to encode in other weight subspaces, but if scalability in this context is important, then care must be taken to ensure $d_{\mathrm{DF}} \times p_{\mathrm{DF}}$ is large.

As an example, we focus on the case of strong-collective decoherence [26]. The DF subspace in this instance has dimension [4,7,9]

$$\dim[\mathrm{DFS}(N)] = \frac{N!}{(N/2 + 1)!(N/2)!}, \tag{22}$$

for the collective basis $|J, m_J\rangle$, where $|0\rangle (|1\rangle)$ represents a $|j = \frac{1}{2}, m_j = -\frac{1}{2}\rangle (|j = \frac{1}{2}, m_j = \frac{1}{2}\rangle)$ state. Note that we are not referring here to the physical angular momentum of a particle, as in Ref. [26] we are simply using the notation for convenience. For large N, the encoding efficiency d_{DF} is asymptotically unity. However, for $J = 0$ and $r = 1/2$, $p_{\mathrm{DF}} \overset{N \to \infty}{\longrightarrow} 0$. So, the proportion of DF states in the $N/2$ subspace is asymptotically zero, implying that an arbitrary $\mathbf{H}_S$ that causes information to leak from DF states to non-DF states with the same weight negates the encoding efficiency of strong-collective DF subspaces. We consider three cases: (i) $\mathbf{H}_S$ causes DF information to leak into all other states in the $N/2$ subspace; (ii) $\mathbf{H}_S$ causes DF information to leak from $J = 0$ to $J = 1$ and $J = 2$ states; and (iii) $\mathbf{H}_S$ causes DF information to leak from $J = 0$ to $J = 1$ states. We define

$$p_{\mathrm{DF}, J_{\mathrm{tot}}} \equiv \frac{\dim[\mathrm{DFS}(N)]}{\dim[\text{non-DFS}(N)]}, \tag{23}$$

for

$$\dim[\text{non-DFS}(N)] = \sum_{J=1}^{J_{\mathrm{tot}}} \frac{(2J+1)N!}{(N/2 + J + 1)!(N/2 - J)!}. \tag{24}$$

Note that the quantity $p_{\mathrm{DF}, J_{\mathrm{tot}}}$ equals unity for equal amounts of DF and non-DF states, whereas for the same instance p_{DF} equals $1/2$. Figure 2 shows $p_{\mathrm{DF}, J_{\mathrm{tot}}}$ for $J_{\mathrm{tot}} = 1$, $J_{\mathrm{tot}} = 2$, and $J_{\mathrm{tot}} = N/2$ for 500 qubits. Allowing quantum information to transfer to just one other subspace reduces the encoding ef-
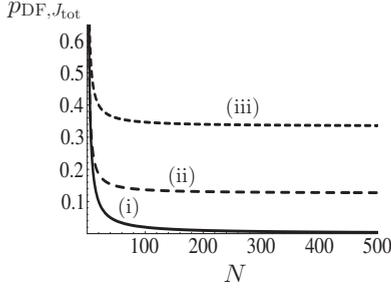
FIG. 2. Ratio $p_{\mathrm{DF},J_{\mathrm{tot}}}$ for (i) $J=1,\dots,N/2$, (ii) $J=1,2$, and (iii) $J=1$ for N qubits, where $p_{\mathrm{DF},J_{\mathrm{tot}}}$ is defined in Eq. (23).

ficiency. So, understanding the effect of $\mathbf{H}_S$ in particular physical realizations is important for scalability.

The effect of variations away from $b_{\alpha\beta}\equiv b$ can be quantified as follows. The fidelity $F(t)=\mathrm{Tr}[\rho_U(t)\rho(t)]$ for $\rho_U(t)$ the unwanted unitary evolution and $\rho(t)$ the desired evolution, is a measure of the effect of the unitary evolution on encoded quantum information. If $\rho_U(t)=\rho(t)$, then $F(t)=1$ (for pure states), and the system serves as a perfect quantum memory. The fidelity can be expanded as

$$F(t) = \sum_n \frac{1}{n!}\left(\frac{t}{\tau_n}\right)^n, \qquad (25)$$

for

$$\left(\frac{1}{\tau_n}\right)^n = \mathrm{Tr}[\{\rho_U(t)\rho(t)\}^{(n)}], \qquad (26)$$

where the superscript (n) denotes nth derivative [7,20]. The time scale $\tau_1^{-1}=0$ for any $\mathbf{H}_S$, so we focus on τ_2^{-1}. This is an estimate of the time scale over which quantum error correction (or some other technique, e.g., a corrective pulse sequence) will have to be applied [27]. So,

$$\frac{1}{2}\left(\frac{1}{\tau_2}\right)^2 = \langle\psi|\mathbf{H}_S|\psi\rangle^2 - \langle\psi|\mathbf{H}_S^2|\psi\rangle, \qquad (27)$$

for $|\psi\rangle$ within a DFS. We have assumed the system begins in a pure state in a DFS and that the desired evolution satisfies $\rho(t)=\rho(0)$. We are interested in the transfer time scale, so we assume that $L_D[\rho_U(t)]\neq 0$ only for $t>\tau_2$. If the states $|\psi\rangle$ are eigenstates of $\mathbf{H}_S$, then $\tau_2^{-1}=0$. The time scale in Eq. (27) also applies to transferring quantum information between states within a DFS.

V. SMALL QUBIT SYSTEMS

In practical applications and in many theoretical proposals, the number of qubits that are controlled in order to process quantum information is small. Here, we examine in detail systems that might be realizable in a laboratory, and give conditions on $b_{\alpha\beta}$ that lead to robust quantum information storage. We focus on three and four qubits because of the possibility that $b_{\alpha\beta}$ is spatially dependent (which is the case for dipole-coupled qubits). If so, experimental control of $b_{\alpha\beta}$ can be obtained through varying the spatial arrangement of the qubits.
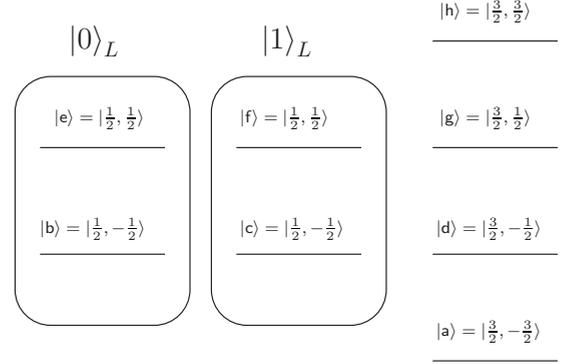

FIG. 3. DF encoding for three qubits. The states are labeled according to $|J,m_J\rangle$. The two isolated subspaces are circled according to the logical basis. See Eqs. (28) and (29) for details. The splitting of the degeneracy due to the non-nearest-neighbor unitary interaction is not included.

A. Three qubits

Three qubits is the smallest number of qubits that supports a DF qubit [8]. The eigenbasis is given in Fig. 3. The qubit is encoded as

$$|1\rangle_L = \begin{cases} |\mathsf{c}\rangle = \dfrac{1}{\sqrt{2}}(|010\rangle - |100\rangle), \\[2mm] |\mathsf{f}\rangle = \dfrac{1}{\sqrt{2}}(|011\rangle - |101\rangle), \end{cases} \qquad (28)$$

$$|0\rangle_L = \begin{cases} |\mathsf{b}\rangle = \dfrac{1}{\sqrt{6}}(-2|001\rangle + |010\rangle + |100\rangle), \\[2mm] |\mathsf{e}\rangle = \dfrac{1}{\sqrt{6}}(2|110\rangle - |101\rangle - |011\rangle), \end{cases} \qquad (29)$$

where the logical groupings are indicated in the first column, the basis used in this section is given in the second column, and the states are expanded in terms of the single-particle basis in the third column. Transitions are only allowed between states with the same symmetry, i.e., the jump operator $\mathbf{J}$ does not cause quantum information to decay from $|\mathsf{e}\rangle$ to $|\mathsf{c}\rangle$ or $|\mathsf{d}\rangle$, or from $|\mathsf{f}\rangle$ to $|\mathsf{b}\rangle$ or $|\mathsf{d}\rangle$. It only acts within the logical groupings. The degeneracy for each J is given by Eq. (24).

The off-diagonal elements of the Hamiltonian

$$\mathbf{H}_S = \sum_{\alpha,\beta=1}^{3} b_{\alpha\beta}\hat{\sigma}_{\alpha+}\hat{\sigma}_{\beta-}, \qquad (30)$$

where α and β label the qubit, in the one-excitation subspace in the collective basis are

$$\mathbf{H}_S^{\mathrm{I}} = \frac{1}{\sqrt{6}}(b_{23}-b_{13})|\mathsf{d}\rangle\langle\mathsf{c}| + \frac{1}{\sqrt{3}}(b_{13}-b_{23})|\mathsf{c}\rangle\langle\mathsf{b}| + \frac{1}{3\sqrt{2}}(2b_{12}$$
$$- b_{13} - b_{23})|\mathsf{d}\rangle\langle\mathsf{b}| + \mathrm{H.c.} \qquad (31)$$

For dipole-coupled qubits, the coefficient $b_{\alpha\beta}$ describes the dipole-dipole interactions. One possible physical system that might satisfy the required conditions is nitrogen-vacancy

(NV) centers in diamond. These can be manufactured in such a manner that the expansion detailed in Eqs. (9) and (10) adequately describes the physical system [28].

A consequence of $[\mathbf{J}_z, \mathbf{H}_S] = 0$ is that the Hamiltonian is block diagonal in the collective basis. We examine the three cases: (i) $b_{\alpha\beta} = b$, (ii) $b_{12} = b_{23} \neq b_{13}$, and (iii) $b_{12} \neq b_{23} \neq b_{13}$. It can be seen immediately for $b_{\alpha\beta} = b$ that $\mathbf{H}_S^I = 0$, and Eq. (15) is satisfied. It can also be seen for $b_{23} = b_{13}$ that $|c\rangle$ is not acted upon by $\mathbf{H}_S$ and is CDF.

We consider $b_{12} = b_{23} \neq b_{13}$. Introducing the states $|u\rangle = \frac{1}{2}|c\rangle - \frac{\sqrt{3}}{2}|b\rangle$ and $|v\rangle = \frac{\sqrt{3}}{2}|c\rangle + \frac{1}{2}|b\rangle$, the off-diagonal system Hamiltonian becomes

$$\mathbf{H}_S^I = \frac{\sqrt{2}}{3}(b_{23} - b_{13})|d\rangle\langle v| + \text{H.c.}, \qquad (32)$$

meaning that $|u\rangle$ is not acted upon by $\mathbf{H}_S$ and is CDF. So, if state $c_u|u\rangle + c_v|v\rangle$ for $|c_u|^2 + |c_v|^2 = 1$ is prepared, the second order transfer rate is

$$\left(\frac{1}{\tau_2}\right)^2 = 2\left[b_{13}c_u^2 - \frac{1}{3}(b_{13} - 4b_{12})c_v^2\right]^2 - 2b_{13}^2 c_u^2$$
$$- \frac{2}{3}(6b_{12}^2 - 4b_{12}b_{13} + b_{13}^2)c_v^2, \qquad (33)$$

which implies that smaller differences between the elements of $(b_{\alpha\beta})$ leads to more robust quantum information storage. The limiting case $b_{23} \to b$, $b_{13} \to b$ recovers (i).

Consider the most general case: $b_{12} \neq b_{23} \neq b_{13}$. It can be seen that there are no stationary states using

$$\dot{\rho} = \mathcal{M}\rho, \qquad (34)$$

where $\mathcal{M}$ is the restriction of $\mathbf{H}_S$ and L_D to the one-excitation subspace. For the steady state $\dot{\rho} = 0$, and there exists a nontrivial solution to $\mathcal{M}\rho = 0$ if and only if $\det(\mathcal{M}) = 0$. This is calculated to be

$$\det(\mathcal{M}) = -\frac{4\lambda^3}{27}(b_{12} - b_{23})^2(b_{12} - b_{13})^2(b_{23} - b_{13})^2, \qquad (35)$$

where λ is the nonzero eigenvalue of $(a_{\alpha\beta})$. This shows that for $\det(\mathcal{M}) = 0$, two elements of $(b_{\alpha\beta})$ must be equal, which is case (ii) above. Note that there are no CDF subspaces for the general case.

B. Four qubits

In order to exploit the collective properties of a system of qubits, τ_2^{-1} gives the time scale over which one would expect to be able to encode information without loss. However, this time scale might be much faster than the time scales given by the eigenvalues of $(a_{\alpha\beta})$. In fact, for a generic system of dipole-coupled qubits the time scale for unitary evolution is $\sim 10^8$ times faster than the decay time scale [16]. This difference in time scales remains for a fully renormalized theory, so for dipole-coupled qubits DF quantum information will evolve into non-DF states much faster than the decay rate of the system.
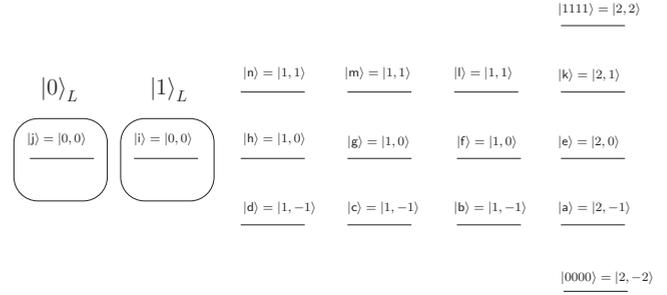

FIG. 4. Eigenbasis for four qubits, labeled $|J, m_J\rangle$, with the logical DFS explicitly labeled. The splitting of the degeneracy due to the non-nearest-neighbor unitary interaction is not included.

This time-scale difference implies that one should encode to protect against the effect of $\mathbf{H}_S$ before one encodes against nonunitary decoherence. Here, we give explicit examples for different forms of $b_{\alpha\beta}$ and show how one could encode against unitary evolution. A significant result presented here is an encoding that protects against an arbitrary $b_{\alpha\beta}$ in the two-excitation subspace. The Hamiltonian

$$\mathbf{H}_S = \sum_{\alpha,\beta=1}^{4} b_{\alpha\beta}\hat{\sigma}_{\alpha+}\hat{\sigma}_{\beta-} \qquad (36)$$

causes evolution between states that have the same value of m_J. So, encoding in the strong-collective DFS spanned by $\{|i\rangle, |j\rangle\}$, where

$$|i\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)(|01\rangle - |10\rangle), \qquad (37)$$

$$|j\rangle = \frac{1}{\sqrt{12}}(2|0011\rangle + 2|1100\rangle - |0101\rangle - |1010\rangle - |0110\rangle - |1001\rangle), \qquad (38)$$

will not guarantee stable quantum memory.

Focusing first on the one-excitation subspace, we notice that if $b_{14} = b_{23}$ and $b_{13} = b_{24}$, then

$$\mathbf{H}_S^I = \frac{1}{2}(b_{12} - b_{34})|a\rangle\langle d| + (b_{24} - b_{23})|b\rangle\langle c| + \text{H.c.} \qquad (39)$$

for the basis defined in Fig. 4. So, a logical state encoded across $\{|b\rangle, |c\rangle\}$ is not acted upon by $\mathbf{H}_S^I$, and satisfies Eq. (15). Under the same conditions, in the two-excitation subspace the off-diagonal terms of the Hamiltonian become

$$\mathbf{H}_S^{II} = \sqrt{\frac{2}{3}}(b_{23} - b_{24})(|e\rangle\langle i| + \sqrt{2}|i\rangle\langle j|)$$
$$+ \frac{\sqrt{2}}{3}(b_{23} - b_{12} + b_{24} - b_{34})|e\rangle\langle j| + \text{H.c.}, \qquad (40)$$

which shows that the DFS $\{|i\rangle, |j\rangle\}$ is coupled to the symmetric state. The states that are not acted upon by $\mathbf{H}_S^{II}$ are not lowest weight states, and so decay through the action of L_D. A further condition is required to decouple the DF state $|i\rangle$, namely, $b_{23} = b_{24}$. Then, there are three states that satisfy Eq. (15): $|b\rangle$, $|c\rangle$, and $|i\rangle$, and a logical qubit can be encoded

using the most convenient states for practical applications. Note that we are interested in storage times, and do not consider ease of preparation and manipulation. See Ref. [14] for one possible method of preparing and manipulating a logical qubit in a collection of dipole-coupled qubits using globally addressed bichromatic incident fields.

It should be mentioned that since the coefficient of the operator $|j\rangle\langle e|$ depends on all values of $b_{\alpha\beta}$, any perturbation away from $b_{\alpha\beta} \equiv b$ will cause information encoded in $|j\rangle$ to decohere. This would prevent the use of an experimentally controlled ($b_{\alpha\beta}$) being used as a single-qubit gate on $\{|i\rangle, |j\rangle\}$.

We now relax the constraints on the system Hamiltonian, and consider arbitrary values of $b_{\alpha\beta}$. We concentrate on the two-excitation subspace. The Hamiltonian can be split into two parts,

$$\mathbf{H}_S^{\text{II}} = \mathbf{H}_S^{\text{fgh}} + \mathbf{H}_S^{\text{eij}}, \tag{41}$$

where $\mathbf{H}_S^{\text{fgh}}$ ($\mathbf{H}_S^{\text{eij}}$) acts only on $\{|f\rangle, |g\rangle, |h\rangle\}$ ($\{|e\rangle, |i\rangle, |j\rangle\}$). The antisymmetric DF states $\{|i\rangle, |j\rangle\}$ are coupled to the symmetric state $|e\rangle$ that undergoes nonunitary evolution. The states of interest in the single-particle basis are

$$|f\rangle = \frac{1}{\sqrt{2}}(|1100\rangle - |0011\rangle), \tag{42}$$

$$|g\rangle = \frac{1}{2}(|0110\rangle + |0101\rangle - |1010\rangle - |1001\rangle), \tag{43}$$

$$|h\rangle = \frac{1}{2}(|1001\rangle - |1010\rangle + |0101\rangle - |0110\rangle). \tag{44}$$

In the collective basis, the zero-logical state in the two-excitation subspace that, before a jump occurs, is immune to an arbitrary $b_{\alpha\beta}$ is

$$|0\rangle_L = \frac{1}{\sqrt{\Omega_1^2 + \Omega_2^2}}(\Omega_1|g\rangle - \Omega_2|h\rangle), \tag{45}$$

for $\Omega_1 = \frac{1}{\sqrt{2}}(b_{14} - b_{13} - b_{23} + b_{24})$, $\Omega_2 = \frac{1}{\sqrt{2}}(b_{13} + b_{14} - b_{23} - b_{24})$, and the temporal evolution associated with the diagonal terms in $\mathbf{H}_S^{\text{fgh}}$ has been absorbed into $|g\rangle$ and $|h\rangle$. The one-logical state is then a combination of the two remaining eigenstates of $\mathbf{H}_S^{\text{fgh}}$,

$$|1\rangle_L = \begin{cases} \dfrac{1}{\sqrt{2(\Omega_1^2 + \Omega_2^2)}}(\sqrt{\Omega_1^2 + \Omega_2^2}|f\rangle + \Omega_2|g\rangle + \Omega_1|h\rangle), \\[2ex] \dfrac{1}{\sqrt{2(\Omega_1^2 + \Omega_2^2)}}(\Omega_2|g\rangle + \Omega_1|h\rangle - \sqrt{\Omega_1^2 + \Omega_2^2}|f\rangle). \end{cases} \tag{46}$$

The states $\{|f\rangle, |g\rangle, |h\rangle\}$ are coupled to each other, but are not coupled with states $\{|e\rangle, |l\rangle, |j\rangle\}$. Using the encoding given in Eqs. (45) and (46) means that, up until $\mathbf{J}$ acts on the two-excitation subspace, the qubit will be immune to an arbitrary environment-induced non-nearest-neighbor evolution. For the physical example of dipole-coupled qubits this is surprising, particularly since it was shown in Ref. [29] that including the dipole-dipole interaction destroyed any collective-emission behavior. Also, for dipole-coupled qubits in this regime, the time scale for unitary evolution is typically $\sim 10^8$ times quicker than that for decay, so using the encoding in Eqs. (45) and (46) might have immediate benefits to applications. We should emphasize that quantum information will decay within the $J=1$ irreps, so the proposed encoding does not support perfect quantum memory for infinite time.

VI. CONCLUSION

It is well known that the detrimental effect of bath-induced Hamiltonians is not accounted for by requiring $L_D[\tilde{\rho}]=0$, for $\tilde{\rho}$ a DFS [7]. In this paper, we stated a condition, similar to the conditions in Refs. [4,7,15], that ensures persistent DF quantum information in the presence of a non-nearest-neighbor bath-induced system Hamiltonian. We showed that, in light of an arbitrary system Hamiltonian, as the size of the Hilbert space increased, the strong-collective DFS is the least suitable subspace for quantum information storage. The most suitable place to store quantum information in N qubits—if scalability is important—is probably the subspace with $\frac{2}{9}N$ excitations. We then gave a time scale over which other methods would have to be applied to account for $\mathbf{H}_S$. We then concentrated on small qubit systems, giving specific examples for three and four qubits that we hope will have immediate benefit to applications. A particularly interesting result for four qubits was the encoding that eliminates the need to correct for $\mathbf{H}_S$ while the qubit remains in the two-excitation subspace.

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