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OPTIMAL MIXED IMPULSE-EQUITY INSURANCE CONTROL PROBLEM WITH REINSURANCE*

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Abstract. We investigate an optimal financing and dividend control problem of an insurance company facing fixed and proportional transaction costs. The goal of the company is to maximize the expected present value of future dividends after deduction of the equity issuance until the time of bankruptcy. We formulate the problem as a mixed classical-impulse control and discuss the problem using the HJB dynamic programming approach. A viscosity solution is considered and its uniqueness is established. We also give results for the regularity and structure of the value function and the optimal policy of the control problem.

Key words. excess of loss, impulse-equity control, optimal dividend control, optimal financing control, fixed transaction cost, HJB quasi-variational inequality, viscosity solution, regularity

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1. Introduction and mathematical model.

Reinsurance is an effective tool for insurance companies to manage and control their exposure to risk. An appropriate use of reinsurance protects an insurer against undesirable, or unexpected, potentially large losses, and hence reduces the insurer's earnings volatility. Different types of reinsurance policies have been adopted in practice. Among them, the proportional reinsurance and the excess-of-loss reinsurance are two popular types of reinsurance policies. These two types of reinsurance have been investigated in the actuarial science literature. For example, Schmidli [25], [26] considered proportional reinsurance and determined an optimal proportional reinsurance strategy by minimizing the probability of ruin. Taksar and Markussen [30] extended this analysis using a diffusion model with investment and proportional reinsurance. Schmidt [27] dealt with an optimal proportional reinsurance problem for dependent lines of business. Some other works on optimal proportion reinsurance include [29], [14], [2], and references therein.

The excess-of-loss reinsurance has also attracted interest among academia and practitioners. Asmussen, Højgaard, and Taksar [3] explored the excess-of-loss reinsurance and the dividend distribution policy in the context of maximizing the expected present value of dividends in a diffusion model. Choulli, Taksar, and Zhou [7] investigated the excess-of-loss reinsurance. They solved the problems of risk control and dividend optimization for a financial corporation facing a constant liability payment. Centeno [6] investigated optimal excess-of-loss retention limits for two dependent risks.

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Some other works on the excess-of-loss reinsurance include [23], [16], [21], [32], and [15].

Recently, research on optimal dividend payout policies related to ruin problems has received attention in the literature. Some examples include [14], [2], [3], and the survey paper [29]. In addition to studies of dividend payout policies, problems associated with equity issuance have also been considered in the literature. Sethi and Taksar [24] studied a model for a company that controls its risk exposure by issuing new equities and paying dividends. He and Liang [12], [13] investigated an optimal financing and dividend control problem of an insurance company with no transaction costs and fixed transaction costs, respectively. Motivated by these works, the objective of this paper is to provide an additional and rigorous analysis for an effective use of the excess-of-loss reinsurance policy taking into account the effects of dividend payout policies and equity issuance.

We now present our model. Let $(\Omega, \mathcal{F}, \{\mathcal{F}(t)\}, P)$ be a complete, filtered probability space, where P is a real-world probability. In the classical Cramér–Lundberg model, the reserve (or surplus) of an insurer at time t, denoted by $P(t)$, evolves over time as

$$P(t) = x + pt - \sum_{i=1}^{N(t)} Z_i,$$

where $x \geq 0$ is the initial level of reserve; $\{N(t)\}$ is a Poisson process with intensity parameter $\beta > 0$, where $N(t)$ counts the number of claims up to and including time t; $Z_i, i = 1, 2, \ldots$, are independent and identically distributed (i.i.d.) random variables with common continuous distribution F having a finite first moment μ_∞ and a finite second moment σ_∞^2, where Z_i is the size of the ith claim; and we assume that Z_i, $i = 1, 2, \ldots$, and $\{N(t)\}$ are independent under P. Let p be the constant premium rate, where $p > 0$. As usual, the premium rate p is determined by the expected value principle. That is,

$$p = (1 + \eta)\beta\mu_\infty,$$

where η is the relative safety loading and $\eta > 0$.

We now consider a modification of the above classical Cramér–Lundberg model that takes into account the presence of reinsurance. Recall that Z_i is the loss (or claim) random variable insured by the insurer without acquiring reinsurance. When reinsurance is present, we use the notation $\mathcal{H}(Z_i)$ to denote the portion of the claim Z_i which is retained by the insurer. In other words, $Z_i - \mathcal{H}(Z_i)$ is the residual part of Z_i that is covered by a reinsurer. It is not unreasonable to assume that $\mathcal{H}(x)$ is an increasing function in x and that $\mathcal{H}(0) = 0$ and $0 \leq \mathcal{H}(x) \leq x$. Write $\mathfrak{D}$ for the set of $\mathcal{H}$ satisfying the above conditions.

For a given reinsurance policy $\mathcal{H}$, let $P^{\mathcal{H}}(t)$ be the reserve at time t in the generalized Cramér–Lundberg model with reinsurance. Then we have

$$P^{\mathcal{H}}(t) = x + p^{\mathcal{H}} \cdot t - \sum_{i=1}^{N_t} \mathcal{H}(Z_i),$$

where $p^{\mathcal{H}}$ is the net premium rate reflecting the reinsurance premium that is paid by the insurer to the reinsurer.

Under the additional assumption that the reinsurer also uses the expected value principle with the same safety loading η as the insurer (i.e., cheap reinsurance), the reserve of the cedent is given by

$$P^{(\mathcal{H},\eta)}(t) = x + p^{(\mathcal{H},\eta)} \cdot t - \sum_{i=1}^{N_t} \mathcal{H}(Z_i),$$

where the premium rate is determined as

$$\begin{aligned} p^{(\mathcal{H},\eta)} &= (1+\eta)\beta\mu_\infty - (1+\eta)\beta[\mu_\infty - E\mathcal{H}(Z_i)] \\ &= (1+\eta)\beta E[\mathcal{H}(Z_i)]. \end{aligned}$$

Without loss of generality, we assume that $\beta = 1$. Then according to Grandell [10] (see also [3]),

$$\{\eta P^{\mathcal{H},\eta}(t/\eta^2) | t \geq 0\} \to r(t) := \int_0^t E[\mathcal{H}(Z)]ds + \int_0^t \sqrt{E[\mathcal{H}^2(Z)]}dB(s) \tag{1.1}$$

in $D[0,\infty)$ (i.e., the space of right continuous functions with left limits endowed with the Skorohod topology) as $\eta \downarrow 0$, where $\{B(t)|t \geq 0\}$ is a standard Brownian motion on $(\Omega, \mathcal{F}, \{\mathcal{F}(t)\}, P)$.

We now consider two special cases of the above generalized Cramér–Lundberg model by making additional assumptions on the structure of the reinsurance strategy $\mathcal{H}$. In particular, we consider the proportional reinsurance and the excess-of-loss reinsurance. For the proportional reinsurance with a proportional constant $0 < a < 1$, $\mathcal{H}(Z_i) = aZ_i$ so that $E(\mathcal{H}(Z)) = a\mu_\infty$ and $E(\mathcal{H}^2(Z)) = a^2\sigma_\infty^2$. With this specification, the diffusion process (1.1) is then simplified as

$$dr(t) = a\mu_\infty dt + a\sigma_\infty dB(t). \tag{1.2}$$

For the excess-of-loss reinsurance with retention level a (i.e., $\mathcal{H}(Z_i) := \min(Z_i, a) = Z_i \wedge a$),

$$\mu(a) := E[Z \wedge a] = \int_0^a \bar{F}(x)dx, \tag{1.3}$$

$$\sigma^2(a) := E[(Z \wedge a)^2] = \int_0^a 2x\bar{F}(x)dx, \tag{1.4}$$

where $\bar{F}(x) := P(Z > x)$. Furthermore, the diffusion process (1.1) becomes

$$dr(t) = \mu(a)dt + \sigma(a)dB(t). \tag{1.5}$$

Motivated by the works [19], [12], [13], we can regard the equity issuance and dividends payout as the absorbing and reflecting boundaries of the reserve process, respectively. We assume that the process $Q^\pi(t)$ is described by a sequence of increasing stopping times $\{\tau_i | i = 1, 2, \ldots\}$ and a sequence of random variables $\{\xi_i | i = 1, 2, \ldots\}$, which are associated with the times and the amounts of dividend payouts. We denote by $L^\pi(t)$ the total amount raised by issuing equity from time 0 up to time t. The dynamics of the controlled state process $\{R^\pi(t)\}$ are then given by

$$R^\pi(t) = x + \int_0^t E[\mathcal{H}_s^\pi(Z)]ds + \int_0^t \sqrt{E[\mathcal{H}_s^\pi(Z)]^2}dB(s) - \sum_{n=1}^{\infty} I_{\{\tau_n^\pi \leq t\}}\xi_n^\pi + L^\pi(t),$$

where $E(\mathcal{H}_t^\pi(Z)) = E(\mathcal{H}(Z))|_{\mathcal{H}=\mathcal{H}_t^\pi}$ and $E(\mathcal{H}_t^\pi(Z))^2 = E(\mathcal{H}(Z))^2|_{\mathcal{H}=\mathcal{H}_t^\pi}$.

We have the following definition of an admissible policy that can be selected by the insurer.

DEFINITION. *A triple*

$$\pi := \{\mathcal{H}^\pi; L^\pi; Q^\pi\} := \{\mathcal{H}^\pi; L^\pi; \tau_1^\pi, \tau_2^\pi, \dots, \tau_n^\pi, \dots; \xi_1^\pi, \xi_2^\pi, \dots, \xi_n^\pi, \dots\}$$

is an admissible policy if

1. *$\mathcal{H}$, Q, and L are $\{\mathcal{F}(t)\}$-adapted, increasing, right continuous with left limits with $0 < \xi_i^\pi \le R^\pi(\tau_i-)$, $\mathcal{H}_t^\pi(0) = 0$, $0 \le \mathcal{H}_t^\pi(x) \le x$;*
2. *τ_i, $i = 1, 2, \dots$, are $\{\mathcal{F}(t)\}$-stopping times;*
3. *$\xi_i^\pi \in \mathcal{F}(\tau_i)$, for each $i = 1, 2, \dots$.*

The class of admissible policies, or controls, is denoted by $\Pi(x)$.

With each admissible policy π, the associated ruin time of the insurer is given by

$$\tau^\pi := \inf\{t \ge 0 : R^\pi(t) < 0\},$$

and the performance function $J(x; \pi)$ associated with the policy π and the initial reserve x is defined as

$$J(x, \pi) := E\left[\sum_{n=1}^{\infty} e^{-c\tau_n^\pi}(-K + \beta_1 \xi_n^\pi) I_{\{\tau_n^\pi \le \tau^\pi\}} - \int_0^{\tau^\pi} e^{-cs}\beta_2 dL^\pi(s)\right].$$

This is the expected present value of the dividends payout minus the equity issuance before bankruptcy. In the dividends payout process, $\beta_1 < 1$ is attributed to proportional transaction costs generated by the tax; K is due to fixed transaction costs generated by advisory and consulting fees. In the equity issuance process, $\beta_2 > 1$ is due to proportional transaction costs generated by the tax.

The objective of the insurer is to find the optimal return function, or the value function, defined as

$$V(x) := \sup\{J(x, \pi); \pi \in \Pi(x)\}, \tag{1.6}$$

and to find an optimal strategy π^* such that $V(x) = J(x, \pi^*)$.

Remark 1. Due to the strict inequality in (1.6), τ^π is, indeed, an $\{\mathcal{F}(t)\}$-stopping time (see, for example, Theorem 1.1.27 in [18]).

The rest of the paper is structured as follows. Section 2 justifies the optimality of the excess-of-loss reinsurance policy. Section 3 gives some properties of the value function. In section 4, we study the uniqueness of the viscosity solution of the impulse control problem. Section 5 discusses the regularity of the value function. In section 6, we present some results on the structure of the value function and the regularity of the optimal policy. Section 7 summarizes the paper.

2. The gain of excess-of-loss reinsurance. In this section, we show that the optimal excess-of-loss reinsurance is better than any other reinsurance. The main result is presented in Theorem 1. We first give the following lemma, which is required to prove Theorem 1. The proof of this lemma can be found in Meng and Zhang [20].

LEMMA 1. *For a fixed function $\mathcal{H} \in \mathfrak{D}$, let a_1 be a constant satisfying the condition*

$$E(Z \wedge a_1)^2 - E[\mathcal{H}^2(Z)] = 0.$$

Then

$$E(Z \wedge a_1) - E[\mathcal{H}(Z)] \ge 0.$$

THEOREM 1. *Suppose $V(x)$ is the value function under the excess-of-loss reinsurance and $V_{\mathcal{H}}(x)$ is the maximum of the expected present value of the dividends payout minus the equity issuance under any reinsurance function $\mathcal{H}$. Then, for all $x \in \Re^+$,*

$$V(x) \geq V_{\mathcal{H}}(x). \tag{2.1}$$

Proof. Fix $\mathcal{H}_t^\pi(Z) \in \mathfrak{D}$, a reinsurance function. Let

$$\pi_{\mathcal{H}} := \{\mathcal{H}; L_{\mathcal{H}}; \tau_1, \tau_2, \ldots, \tau_n, \ldots; \xi_1, \xi_2, \ldots, \xi_n, \ldots\}$$

be the feedback risk policy under any reinsurance function $\mathcal{H}$. Then $0 \leq \sigma^2(\mathcal{H}_t^\pi(Z)) \leq \sigma_\infty^2$. We can choose a feedback control $a_{\pi_1}(t)$ in the excess-of-loss model in such a way that

$$\sigma^2(a_{\pi_1}(t)) = E[Z \wedge a_{\pi_1}(t)]^2 = E[\mathcal{H}_t^\pi(Z)]^2. \tag{2.2}$$

From the above lemma, we have

$$\mu(a_{\pi_1}(t)) = E[Z \wedge a_{\pi_1}(t)] \geq E[\mathcal{H}_t^\pi(Z)].$$

Take a control $L_1(t)$ such that $L_1(0) = L_{\mathcal{H}}(0)$ and

$$dL_{\mathcal{H}}(t) = dL_1(t) + [\mu(a_{\pi_1}(t)) - E(\mathcal{H}_t^\pi(Z))]dt.$$

Then

$$\begin{aligned} R^{\pi_{\mathcal{H}}}(t) &= x + \int_0^t E(\mathcal{H}_s^\pi(Z))ds + \int_0^t \sqrt{E(\mathcal{H}_s^\pi(Z))^2}ds - \sum_{n=1}^\infty I_{\{\tau_n \leq t\}}\xi_n + L_{\mathcal{H}}(t) \\ &= x + \int_0^t E(\mathcal{H}_s^\pi(Z))ds + \int_0^t \sqrt{E(\mathcal{H}_s^\pi(Z))^2}ds - \sum_{n=1}^\infty I_{\{\tau_n \leq t\}}\xi_n \\ &\quad + L_1(t) + \int_0^t [\mu(a_{\pi_1}(s)) - E(\mathcal{H}_s^\pi(Z))]ds \\ &= x + \int_0^t \mu(a_{\pi_1}(s))ds + \int_0^t \sigma(a_{\pi_1}(s))ds - \sum_{n=1}^\infty I_{\{\tau_n \leq t\}}\xi_n + L_1(t). \end{aligned}$$

Hence $R^{\pi_1}(t) = R^{\pi_{\mathcal{H}}}(t)$, while $L_1(t) \leq L_{\mathcal{H}}(t)$. This implies that $V(x) \geq J(x, \pi_1) \geq V_{\mathcal{H}}(x)$. ∎

From now on, we consider only the Cramér–Lundberg model with the excess-of-loss reinsurance.

3. Some properties of the value function. Motivated by Cadenillas et al. [5], we obtain a boundedness condition for the value function, which is presented in the following proposition.

PROPOSITION 1. *For each $x \in [0, \infty)$, the value function V defined by* (1.6) *satisfies the following boundedness condition:*

$$V(x) \leq \beta_2(x + \mu_\infty/c). \tag{3.1}$$

Proof. Define a stochastic process $\{U(t)\}$ by putting

$$U(t) := x + \int_0^t \mu(a(s))ds + \int_0^t \sigma(a(s))dB(s).$$

Then

$$E\left[\int_0^{\tau^\pi} e^{-cs}dU(s)\right] = E\left[\int_0^{\tau^\pi} e^{-cs}\mu(a(s))ds\right] \le \frac{\mu_\infty}{c}.$$

By the Itô's product rule,

$$e^{-c\tau^\pi}U(\tau^\pi) = x - c\int_0^{\tau^\pi} e^{-cs}U(s)ds + \int_0^{\tau^\pi} e^{-cs}dU(s).$$

Since $R(\tau^\pi) = 0$ and $R(t) \ge 0$, for $t \le \tau^\pi$,

$$-E\left[\int_0^{\tau^\pi} e^{-cs}dR(s)\right] = x - cE\left[\int_0^{\tau^\pi} e^{-cs}R(s)ds\right] \le x.$$

Obviously,

$$\begin{aligned} J(x,\pi) &\le \beta_2 E\left[\sum_{n=1}^{\infty} e^{-c\tau_n^\pi}\xi_n^\pi I_{\{\tau_n^\pi \le \tau^\pi\}} - \int_0^{\tau^\pi} e^{-cs}dL^\pi(s)\right] \\ &= \beta_2 E\left[\int_0^{\tau^\pi} e^{-cs}dU(s)\right] - \beta_2 E\left[\int_0^{\tau^\pi} e^{-cs}dR(s)\right] \\ &\le \beta_2(x + \mu_\infty/c). \end{aligned}$$

Hence the result follows. ∎

4. Viscosity solution to the Hamilton–Jacobi–Bellman equation and its properties. In this section we prove that the value function for the impulse control problem is the unique viscosity solution of the HJB equation under certain conditions.

Define the continuation region $\mathcal{C}$ and the action region $\mathcal{A}$, respectively, by

$$\begin{aligned} \mathcal{C} &:= \{x \in \Re^+ : \mathcal{M}V(x) < V(x)\}, \\ \mathcal{A} &:= \{x \in \Re^+ : \mathcal{M}V(x) = V(x)\}. \end{aligned}$$

Consider, for each $a \in [0, N]$, the following operators:

$$\begin{aligned} \mathcal{M}\nu(x) &= \sup_{0<\xi\le x}\{\nu(x-\xi) + \beta_1\xi - K\}, \\ \mathcal{L}^a\nu(x) &= \frac{1}{2}\sigma^2(a)\nu''(x) + \mu(a)\nu'(x) - c\nu(x). \end{aligned}$$

Using some standard results in stochastic optimal control (see, for example, Fleming and Soner [9]), the HJB equation of the impulse control problem is given by

$$\max\left\{\max_{a\in[0,N)}\mathcal{L}^a\nu(x), \nu'(x) - \beta_2, \mathcal{M}\nu(x) - \nu(x)\right\} = 0, \quad x > 0, \tag{4.1}$$

with the boundary condition

$$\nu(0) \ge 0. \tag{4.2}$$

The boundary condition can be explained as follows. Consider a strategy $\pi^\dagger$ such that $\tau_1^{\pi^\dagger} = \infty$ (i.e., no capital inflow, but only dividend payments). The strategy $\pi^\dagger$ is an admissible policy. Write $V^{\dagger}(x)$ for the value function of the impulse control with respect to the space of such strategy $\pi^{\dagger}$. Then

$$V(x) \geq V^{\dagger}(x).$$

The boundary condition then follows by putting $x = 0$ and noting that $V^{\dagger}(0) = 0$.

We shall show that the value function of the impulse control problem is, indeed, the unique viscosity solution to the above HJB equation.

LEMMA 2. *$V(x) \geq \mathcal{M}V(x)$ for all $x \in \Re^{+}$.*

Proof. Let

$$\pi \equiv \{\mathcal{H}^{\pi}; L^{\pi}; Q^{\pi}\} \equiv \{\mathcal{H}^{\pi}; L^{\pi}; \tau_1^{\pi}, \tau_2^{\pi}, \ldots, \tau_n^{\pi}, \ldots; \xi_1^{\pi}, \xi_2^{\pi}, \ldots, \xi_n^{\pi}, \ldots\}$$

be an admissible control policy. Then so is

$$\pi' \equiv \{\mathcal{H}^{\pi}; L^{\pi}; Q^{\pi}\} \equiv \{\mathcal{H}^{\pi}; L^{\pi}; 0, \tau_1^{\pi}, \tau_2^{\pi}, \ldots, \tau_n^{\pi}, \ldots; \xi, \xi_1^{\pi}, \xi_2^{\pi}, \ldots, \xi_n^{\pi}, \ldots\}.$$

Then

$$V(x) \geq J(x, \pi') = \beta_1 \xi - K + J(x - \xi, \pi).$$

The result then follows by taking the supremum over π and the maximum over $\xi \in (0, x]$. □

The following results are similar to those in Propositions 1 and 2 of Guo and Wu [11]. We state the results here without giving the proof.

LEMMA 3. *The continuation region $\mathcal{C}$ is open.*

LEMMA 4. *Suppose $x \in \mathcal{A}$ (i.e., x is in the action region). Then*

(1) *the set*

$$\Xi(x) := \{\xi \in \Re^{+} : \mathcal{M}V(x) = V(x - \xi) + \beta_1 \xi - K\} \tag{4.3}$$

is nonempty;

(2) *for any $\xi_x \in \Xi(x)$, $x - \xi_x \geq 0$, and*

$$V(x - \xi_x) > \mathcal{M}V(x - \xi_x),$$

we have $x - \xi_x \in \mathcal{C}$.

We now recall the following definitions of viscosity subsolution and supersolution.

DEFINITION 1. *A function ν is said to be a viscosity subsolution (supersolution, resp.) of the HJB equation if $\varphi \in C^2(\Re^{+})$, $\nu - \varphi$ has a global maximum (minimum, resp.) at x, and $\nu(x) = \varphi(x)$, then*

$$\max\left\{\max_{a \in [0,N)} \mathcal{L}^a \varphi(x), \varphi'(x) - \beta_2, \mathcal{M}\varphi(x) - \varphi(x)\right\} \geq 0 \quad (\leq 0 \textit{ resp.}). \tag{4.4}$$

A function ν is a viscosity solution if it is both a viscosity subsolution and a viscosity supersolution.

The global optimality conditions can be weakened to give the following definition.

DEFINITION 2. *A function ν is said to be a viscosity subsolution (supersolution, resp.) of the HJB equation if $\varphi \in C^2(\Re^{+})$, $\nu - \varphi$ has a local maximum (minimum, resp.) at x, and $\nu(x) = \varphi(x)$, then*

$$\max\left\{\max_{a \in [0,N)} \mathcal{L}^a \varphi(x), \varphi'(x) - \beta_2, \mathcal{M}\nu(x) - \nu(x)\right\} \geq 0 \quad (\leq 0 \textit{ resp.}). \tag{4.5}$$

Then the following theorem is similar to Theorem 3.1 of Guo and Wu [11]. We state only the result.

THEOREM 2. *The two definitions of viscosity subsolutions (supersolutions, resp.) are equivalent.*

The following theorem is the main result.

THEOREM 3. *The value function* V *defined by* (1.6) *is a viscosity solution of the HJB equation*

$$\max\left\{\max_{a\in[0,N)}\mathcal{L}^a\phi(x),\phi'(x)-\beta_2,\mathcal{M}\phi(x)-\phi(x)\right\}=0. \tag{4.6}$$

Proof. We first show that V is a viscosity supersolution. By Lemma 2, it suffices to prove that

$$\max\left\{\max_{a\in[0,N)}\mathcal{L}^a\varphi(x),\varphi'(x)-\beta_2\right\}\le 0. \tag{4.7}$$

Using similar arguments as in [9], we can show that V satisfies the dynamic programming principle

$$\begin{aligned}V(x)=\sup_{\pi\in\prod}E\bigg[&\sum_{n=1}^{\infty}e^{-c\tau_n^\pi}(-K+\beta_1\xi_n^\pi)I_{\{\tau_n^\pi\le\tau^\pi\wedge\varsigma\}}-\int_0^{\tau^\pi\wedge\varsigma}e^{-cs}\beta_2dL^\pi(s)\\&+I_{\{\varsigma<\tau^\pi\}}e^{-c\varsigma}V(R^\pi(\varsigma))\bigg]\end{aligned} \tag{4.8}$$

for any stopping time ς.

For any admissible strategy $\pi\in\Pi$ and $h\in(0,\infty)$, let $\vartheta_\pi^h=h\wedge\inf\{t:R^\pi(t)\notin(x-h,x+h)\}$. Then $\vartheta_\pi^h<\infty$ and $\vartheta_\pi^h\to 0$ as h does, almost surely. Define the strategy π by $a_\pi(t)=a$, $L^\pi(t)=Q^\pi(t)=0$ for $t<\vartheta_\pi^h$; $a_\pi(t)$, $L^\pi(t)$, $Q^\pi(t)$ take any admissible policy for $t\ge\vartheta_\pi^h$, where $a\in[0,N]$ is a constant. Choose $h<x$; then $\vartheta_\pi^h<\tau^\pi$ by the definition of τ^π. From (4.8), with $\varsigma=\vartheta_\pi^h$, we have

$$V(x)\ge E[e^{-c\vartheta_\pi^h}V(R^\pi(\vartheta_\pi^h))]. \tag{4.9}$$

For the function $\varphi\in\mathcal{C}^2(\Re^+)$, applying Itô's differentiation rule to φ and taking expectation then give

$$E[e^{-c\vartheta_\pi^h}\varphi(R^\pi(\vartheta_\pi^h))]-\varphi(x)=E\bigg[\int_0^{\vartheta_\pi^h}e^{-ct}\mathcal{L}^a\varphi(R^\pi(t))dt\bigg]. \tag{4.10}$$

Given that $V\ge\varphi$ in a neighborhood of x and that $V(x)=\varphi(x)$. Then combining (4.9) and (4.10) and sending $h\to 0$ give

$$\mathcal{L}^a\varphi(x)\le 0\quad\forall a\in[0,N).$$

For any $h>0$, we can find a policy π^x such that $J(x,\pi^x)\ge V(x)-h^2$. Let $0<h<x$, and define π by $L^\pi(t)=h+L^{\pi^{x+h}}(t)$, $a_\pi(t)=a_{\pi^{x+h}}(t)$, and $Q^\pi(t)=Q^{\pi^{x+h}}(t)$ for all $t\ge 0$. Then

$$V(x)\ge J(x,\pi)=-\beta_2h+J(x,\pi^{x+h})\ge-\beta_2h+V(x+h)-h^2.$$

Consequently,

$$-\beta_2 h - h^2 \le V(x) - V(x+h). \tag{4.11}$$

As $V - \varphi$ has a local minimum at x, we have

$$-\beta_2 h - h^2 \le V(x) - V(x+h) \le \varphi(x) - \varphi(x+h). \tag{4.12}$$

Dividing by h and sending $h \to 0$ yield

$$\varphi'(x) \le \beta_2. \tag{4.13}$$

It now remains to show that V is a viscosity subsolution at any $x > 0$.

Suppose $\varphi \in C^2(\Re^+)$, $V - \varphi$ has a local maximum at x, and $V(x) = \varphi(x)$. If either $\varphi'(x) \ge \beta_2$ or $\mathcal{M}V(x) = V(x)$, then (4.5) is trivially true. Thus, we assume that $V(x) > \mathcal{M}V(x)$ and $\varphi'(x) < \beta_2$. By the continuity of $\mathcal{M}$ and φ, there exist constants $\delta > 0$ and $\rho > 0$ such that if $|y - x| < \rho$,

$$V(y) \le \varphi(y), \quad V(y) - \mathcal{M}V(y) > \delta, \quad \varphi'(y) < \beta_2. \tag{4.14}$$

Define

$$\tau_\rho := \inf\{t > 0 : |R^\pi(t) - x| \ge \rho\}. \tag{4.15}$$

For any $\varepsilon > 0$, choose an ε-optimal control

$$\pi \equiv \{\mathcal{H}^\pi; L^\pi; Q^\pi\} \equiv \{\mathcal{H}^\pi; L^\pi; \tau_1^\pi, \tau_2^\pi, \ldots, \tau_n^\pi, \ldots; \xi_1^\pi, \xi_2^\pi, \ldots, \xi_n^\pi, \ldots\}$$

so that

$$J(x, \pi) \ge V(x) - \varepsilon.$$

We shall show that $\tau_1^\pi \ne 0$. If $\tau_1^\pi = 0$, then $R^\pi(t)$ makes an immediate jump from x to $x - \xi_1^\pi$, and hence,

$$V(x) - \varepsilon \le J(x - \xi_1^\pi) + \beta_1 \xi_1^\pi - K \le \mathcal{M}V(x), \tag{4.16}$$

which is a contradiction if $\varepsilon < V(x) - \mathcal{M}V(x)$.

Fix $R > 0$ and define

$$\theta := \tau_1^\pi \wedge R \wedge \tau_\rho. \tag{4.17}$$

By the dynamic programming principle,

$$V(x) \le E\left[-\beta_2 \int_0^\theta e^{-ct} dL^\pi(t) + e^{-c\theta} V(R^\pi(\theta))\right] + \epsilon. \tag{4.18}$$

Note that $V(R^\pi(\theta)) \le \varphi(R^\pi(\theta))$, and that $V(x) = \varphi(x)$. Applying (4.18) and Itô's differentiation rule to φ gives

$$E\left[\int_0^\theta \mathcal{L}^a \varphi(R^\pi(t)) dt\right] \ge -\epsilon.$$

The result then follows by dividing both sides of the inequality with $E[\theta]$ and sending $\rho \to 0$ and $\epsilon \to 0$. $\square$

Let $u : \Re^+ \to \Re$. For each $x \in \Re^+$, let

$$J^{2,+}u(x) := \left\{(p,A) \in (\Re^+)^2 | u(y) \le u(x) + p(y-x) + \frac{1}{2}A(y-x)^2 + o(|y-x|^2) \text{ as } \Re^+ \ni y \to x\right\},$$

$$\bar{J}^{2,+}u(x) := \{(p,A) \in (\Re^+)^2 | \exists (x_n, p_n, A_n) \in \Re^+ \times (\Re^+)^2 \text{ such that } (p_n, A_n) \in J^{2,+}u(x_n) \text{ and } (x_n, u(x_n), p_n, A_n) \to (x, u(x), p, A) \text{ as } n \to \infty\},$$

and

$$J^{2,-}u(x) := -J^{2,+}(-u(x)), \quad \bar{J}^{2,-}u(x) := -\bar{J}^{2,+}(-u(x)). \tag{4.19}$$

The following proposition is the comparison principle.

PROPOSITION 2. *Suppose $\underline{\nu}(x)$ is a subsolution of* (4.5) *and $\bar{\nu}(x)$ is a supersolution of* (4.5) *for $x > 0$. Then $\underline{\nu}(x) \le \bar{\nu}(x)$ for all $x \ge 0$.*

We first construct a strict supersolution of (4.5) by a perturbation of $\bar{\nu}(x)$. The result is given in the following lemma.

LEMMA 5. *Let*

$$g(x) := \ell x + \gamma,$$

where $\beta_1 < \ell < \beta_2$ and $\gamma > 0$.

Then $\bar{\nu}_m(x)$ defined by

$$\bar{\nu}_m(x) := \left(1 - \frac{1}{m}\right)\bar{\nu}(x) + \frac{g(x)}{m}, \quad m \in N_+,$$

is a strict supersolution; that is, there exists a constant $\rho > 0$ such that the test function φ satisfies

$$\max\left\{\max_{a\in[0,N)} \mathcal{L}^a\varphi(x), \varphi'(x) - \beta_2, \mathcal{M}\bar{\nu}_m(x) - \bar{\nu}_m(x)\right\} + \frac{\rho}{m} \le 0.$$

Proof. For any given $\varphi \in C^2(\Re^+)$, we suppose that $\bar{\nu}_m - \varphi$ attains a local minimum at x and $\bar{\nu}_m(x) = \varphi(x)$. Then

$$\begin{aligned}\bar{\nu}_m(y) - \varphi(y) &= \left(1 - \frac{1}{m}\right)\bar{\nu}(y) + \frac{g(y)}{m} - \varphi(y) \\ &= \left(1 - \frac{1}{m}\right)\left[\bar{\nu}(y) - \left(\frac{m}{m-1}\varphi(y) - \frac{\ell y + \gamma}{m-1}\right)\right],\end{aligned}$$

and we obtain that $\bar{\nu}(y) - [m/(m-1)\varphi(y) - (\ell y + \gamma)/(m-1)]$ attains a local minimum at x and $\bar{\nu}(x) = [m/(m-1)\varphi(x) - (\ell x + \gamma)/(m-1)]$. Consequently, since $\bar{\nu}$ is a viscosity supersolution,

$$\frac{m}{m-1}\sup_{a\in[0,N]} \mathcal{L}^a\varphi + \sup_{a\in[0,N]} \frac{-\mu(a)\ell}{m-1} + \frac{c(\ell x + \gamma)}{m-1} \le 0.$$

This implies that

$$\sup_{a\in[0,N]} \mathcal{L}^a\varphi + \frac{c\gamma}{m} < 0.$$

Furthermore,

$$\frac{m}{m-1}\varphi' - \frac{\ell}{m-1} - \beta_2 \leq 0,$$

so

$$\varphi' - \beta_2 + \frac{\beta_2 - \ell}{m} \leq 0.$$

Similar calculation then gives

$$\begin{aligned}\mathcal{M}\bar{\nu}_m(x) - \bar{\nu}_m(x) &= \sup_{0<\xi\leq x}\left\{\left(1-\frac{1}{m}\right)\bar{\nu}(x-\xi) + \frac{\ell(x-\xi)+\gamma}{m} + \beta_1\xi - K\right\}\\ &\quad -\left\{\left(1-\frac{1}{m}\right)\bar{\nu}(x) + \frac{\ell x+\gamma}{m}\right\}\\ &\leq \left(1-\frac{1}{m}\right)(\mathcal{M}\bar{\nu}(x) - \bar{\nu}(x)) + \sup_{0<\xi\leq x}\frac{(\beta_1-\ell)\xi}{m} - \frac{K}{m}\\ &\leq -\frac{K}{m}.\end{aligned}$$

The result then follows by taking $\rho = \min\{c\gamma, \beta_2 - \ell, K\}$. ∎

We now give the proof of Proposition 2.

Proof of Proposition 2. This proof is based on a technique of Crandall, Ishii, and Lions [8] and the proof of Theorem 3.8 in [22]. The proposition is proved by contradiction. Assume that

$$\sup_{x\in\Re^+}(\underline{\nu}(x) - \bar{\nu}(x)) > 0.$$

Choose m such that

$$\sup_{x\in\Re^+}(\underline{\nu}(x) - \bar{\nu}_m(x)) > 0.$$

Define

$$h(x) := \underline{\nu}(x) - \bar{\nu}_m(x).$$

For $j = 1, 2, \ldots$ and $(x, y) \in (\Re^+)^2$, let

$$H_j(x,y) := \underline{\nu}(x) - \bar{\nu}_m(y) - \frac{j}{2}|x-y|^2,$$

$$M_j := \sup_{(x,y)\in(\Re^+)^2} H_j(x,y),$$

and

$$M := \sup_{x\in\Re^+} h(x).$$

It can be shown that there exists $(x_j, y_j) \in (\Re^+)^2$ such that

$$M_j = H_j(x_j, y_j) < \infty.$$

Furthermore,

$$j|x_j - y_j|^2 \to 0 \text{ as } j \to \infty$$

and

$$M_j \to M \text{ as } j \to \infty.$$

Define

$$\text{argmax } h(x) := \left\{ \bar{\zeta} | h(\bar{\zeta}) = \sup_{\zeta \in \Re^+} h(\zeta) \right\}.$$

Choose $\hat{\zeta} \in (0, \infty)$ such that

$$M = h(\hat{\zeta}). \tag{4.20}$$

Then

$$(x_j, y_j) \in (0, \infty) \times (0, \infty) \text{ for a large enough } j.$$

By Theorem 3.2 of [8], there exist some constants A_j and B_j such that

$$(p_j, A_j) \in \bar{J}^{2,+}(\underline{\nu}(x_j)), (q_j, B_j) \in \bar{J}^{2,-}\bar{\nu}_m(y_j)$$

and

$$\begin{pmatrix} A_j & 0 \\ 0 & -B_j \end{pmatrix} \leq 3j \begin{pmatrix} I & -I \\ -I & I \end{pmatrix},$$

where $p_j = j(x_j - y_j)$ and $q_j = p_j$.

Since $\underline{\nu}$ is a subsolution and $\bar{\nu}_m$ is a strict supersolution, we obtain

$$\begin{aligned} &\max\left\{ \max_{a \in [0,N)} \left\{ \frac{1}{2}\sigma^2(a) A_j + \mu(a) p_j - c\underline{\nu}(x_j) \right\}, p_j - \beta_2, \mathcal{M}\underline{\nu}(x_j) - \underline{\nu}(x_j) \right\} \geq 0, \\ &\max\left\{ \max_{a \in [0,N)} \left\{ \frac{1}{2}\sigma^2(a) B_j + \mu(a) q_j - c\bar{\nu}_m(y_j) \right\}, q_j - \beta_2, \mathcal{M}\bar{\nu}_m(y_j) - \bar{\nu}_m(y_j) \right\} \leq -\frac{\rho}{m}. \end{aligned} \tag{4.21}$$

Obviously, $p_j - \beta_2 = q_j - \beta_2 \leq -\frac{\rho}{m}$. Suppose $\mathcal{M}\underline{\nu}(x_j) - \underline{\nu}(x_j) \geq 0$. From (4.21),

$$\mathcal{M}\bar{\nu}_m(y_j) - \bar{\nu}_m(y_j) \leq -\frac{\rho}{m}.$$

Combining the above gives

$$M_j < \underline{\nu}(x_j) - \bar{\nu}_m(y_j) < \mathcal{M}\underline{\nu}(x_j) - \mathcal{M}\bar{\nu}_m(y_j) - \frac{\rho}{m}.$$

Then we can prove the desired result by contradiction as follows:

$$\begin{aligned} M = \lim_{j\to\infty} M_j &\leq \lim_{j\to\infty} \left(\mathcal{M}\underline{\nu}(x_j) - \mathcal{M}\bar{\nu}_m(y_j) - \frac{\rho}{m} \right) \\ &\leq \mathcal{M}\underline{\nu}(\hat{\zeta}) - \mathcal{M}\bar{\nu}_m(\hat{\zeta}) - \frac{\rho}{m} \\ &\leq \sup_{0\leq\xi\leq\hat{\zeta}} \{\underline{\nu}(\hat{\zeta} - \xi) + \beta_1\xi - K\} - \sup_{0\leq\xi\leq\hat{\zeta}} \{\bar{\nu}_m(\hat{\zeta} - \xi) + \beta_1\xi - K\} - \frac{\rho}{m} \\ &\leq \sup_{0\leq\xi\leq\hat{\zeta}} \{\underline{\nu}(\hat{\zeta} - \xi) - \bar{\nu}_m(\hat{\zeta} - \xi)\} - \frac{\rho}{m} \\ &\leq M - \frac{\rho}{m}. \end{aligned}$$

Suppose that $\max_{a\in[0,N)}\{\frac{1}{2}\sigma^2(a)A_j + \mu(a)p_j - c\underline{\nu}(x_j)\} \geq 0$. From (4.21),

$$\max_{a\in[0,N)}\left\{\frac{1}{2}\sigma^2(a)B_j + \mu(a)q_j - c\bar{\nu}_m(y_j)\right\} \leq -\frac{\rho}{m}.$$

Consequently,

$$\begin{aligned}
0 &< cM \\
&= \lim_{j\to\infty} c(\underline{\nu}(x_j) - \bar{\nu}_m(y_j)) \\
&\leq \lim_{j\to\infty}\left\{\max_{a\in[0,N)}\left\{\frac{1}{2}\sigma^2(a)A_j + \mu(a)p_j\right\} - \max_{a\in[0,N)}\left\{\frac{1}{2}\sigma^2(a)B_j + \mu(a)q_j\right\}\right\} - \frac{\rho}{m} \\
&\leq \lim_{j\to\infty}\max_{a\in[0,N)}\left\{\frac{1}{2}\sigma^2(a)(A_j - B_j)\right\} - \frac{\rho}{m} \\
&\leq -\frac{\rho}{m},
\end{aligned}$$

which results in a contradiction. This contradiction shows that the assumption (4.20) does not hold. Therefore we must have

$$\hat{\zeta} = 0. \tag{4.22}$$

To treat the boundary point, we proceed as in the proof of Theorem 6.7 of [1]. There exists a sequence $\{x_j\} \in (0,\infty)$ converging to $\hat{\zeta}$ such that $\bar{\nu}_m(x_j) \to \bar{\nu}_m(\hat{\zeta})$ when $j \to \infty$. Let $\epsilon_j := |x_j - \hat{\zeta}|$. We consider the test function

$$w_j(x,y) := \underline{\nu}(x) - \bar{\nu}_m(y) - \theta_j(x,y), \quad (x,y) \in (\Re^+)^2,$$

where

$$\theta_j(x,y) := \frac{|x-y|^2}{2\epsilon_j} + \frac{1}{4}\left(\frac{d(y)-d(x)}{d(x_j)} - 1\right)^4 + \frac{1}{4}|x-\hat{\zeta}|^4.$$

Here $d(y)$ is the distance between y and 0, and similarly for $d(x)$ and $d(x_j)$. The proof of this proposition can then be completed by following the same steps as in [1], which then lead to a contradiction. ∎

PROPOSITION 3. *Suppose $\bar{\nu}(x)$ is a supersolution of* (4.5). *Then $\bar{\nu}(x) \geq V(x)$ for all $x \in \Re^+$.*

Proof. As in Wheeden and Zygmund [31] or Azcue and Muler [4], we can show that there exists a sequence $\{\nu_n(x)\}_{n=1}^{\infty}$ of mapping $\nu_n(x) \in C^2(\Re^+)$ such that

1. $\nu_n(x) \to \bar{\nu}(x)$, $\mathcal{L}^a\nu_n(x) \to \mathcal{L}^a\bar{\nu}(x)$ uniformly on compact subsets of $\Re^+$ as $n \to \infty$;
2. $\nu_n(x)$ satisfies the growth condition $\nu_n(x) \leq \beta_2(x + \mu_\infty/c)$;
3. for sufficiently large n,

$$\max\left\{\max_{a\in[0,N)}\mathcal{L}^a\nu_n(x), \nu_n'(x) - \beta_2, \mathcal{M}\nu_n(x) - \nu_n(x)\right\} \leq 0. \tag{4.23}$$

Fix an admissible policy $\pi \in \Pi$, and define $A^\pi(t) := \sum_{n=1}^{\infty} I_{\{\tau_n^\pi \leq t\}}\xi_n^\pi$. Let $\wedge := \{s : L^\pi(s-) \neq L^\pi(s)\}$ and $\wedge' := \{s : A^\pi(s-) \neq A^\pi(s)\}$. Write, for each t, $\hat{L}^\pi(t) := \sum_{s\in\wedge, s\leq t}(L^\pi(s) - L^\pi(s-))$ so that $\{\hat{L}^\pi(t)\}$ is the discontinuous part of $\{L^\pi(t)\}$. Then the continuous part $\{\tilde{L}^\pi(t)\}$ of $\{L^\pi(t)\}$ is given by

$$\tilde{L}^\pi(t) = L^\pi(t) - \hat{L}^\pi(t).$$

Similarly, $\{\hat{A}(t)\}$ and $\{\tilde{A}(t)\}$ are the discontinuous and continuous parts of $\{A^\pi(t)\}$, respectively. Define, for $\epsilon > 0$,

$$\tau^\epsilon := \inf\{t \geq 0 : R^\pi(t) \in (0, \epsilon]\}.$$

Applying the generalized Itô's differentiation rule to $\nu_n(x)$ gives

$$\begin{aligned} &e^{-c(t\wedge\tau^\epsilon)}\nu_n(R^\pi(t\wedge\tau^\epsilon\wedge\tau_n)) - \nu_n(x) \\ &\quad= \int_0^{t\wedge\tau^\epsilon} e^{-cs}\mathcal{L}^a\nu_n(R^\pi(s))ds + \int_0^{t\wedge\tau^\epsilon}\sigma(a_\pi)e^{-cs}\nu_n'(R^\pi(s))dB(s) \\ &\qquad+\int_0^{t\wedge\tau^\epsilon} e^{-cs}\nu_n'(R^\pi(s))d\tilde{L}^\pi(s) + \sum_{s\in\wedge\cup\wedge', s\leq t\wedge\tau^\epsilon} e^{-cs}[\nu_n(R^\pi(s)) - \nu_n(R^\pi(s-))]. \end{aligned} \tag{4.24}$$

Note that $R^\pi(s) - R^\pi(s-) = (L^\pi(s) - L^\pi(s-)) - (A^\pi(s) - A^\pi(s-))$. Then for sufficiently large n,

$$\begin{aligned} &\nu_n(R^\pi(s)) - \nu_n(R^\pi(s-)) \\ &\quad= [\nu_n(R^\pi(s)) - \nu_n(R^\pi(s-))]I_{\{(L^\pi(s)-L^\pi(s-))-(A^\pi(s)-A^\pi(s-))\geq 0\}} \\ &\qquad+[\nu_n(R^\pi(s)) - \nu_n(R^\pi(s-))]I_{\{(L^\pi(s)-L^\pi(s-))-(A^\pi(s)-A^\pi(s-))<0\}} \\ &\quad\leq \beta_2[(L^\pi(s)-L^\pi(s-)) - (A^\pi(s)-A^\pi(s-))]I_{\{(L^\pi(s)-L^\pi(s-))-(A^\pi(s)-A^\pi(s-))>0\}} \\ &\qquad+[K - \beta_1((A^\pi(s)-A^\pi(s-)) - (L^\pi(s)-L^\pi(s-)))] \\ &\qquad\times I_{\{(L^\pi(s)-L^\pi(s-))-(A^\pi(s)-A^\pi(s-))\leq 0\}} \\ &\quad\leq K - \beta_1(A^\pi(s)-A^\pi(s-)) + \beta_2(L^\pi(s)-L^\pi(s-)). \end{aligned}$$

Since $\nu_n(x)$ satisfies (4.23), the first term in the right-hand side of (4.24) is nonpositive. Since $0 \leq \nu_n'(R^\pi(s)) < \beta_2$ on $(0, \tau^\epsilon)$, the second term in the right-hand side of (4.24) is a square-integrable martingale.

Taking expectation on both sides of (4.24) then gives

$$\begin{aligned} &E\{e^{-c(t\wedge\tau^\epsilon)}\nu_n(R^\pi(t\wedge\tau^\epsilon))\} - \nu_n(x) \\ &\quad\leq E\left[\int_0^{t\wedge\tau^\epsilon} e^{-cs}\beta_2 dL^\pi(s)\right] - E\left[\sum_{j=1}^\infty e^{-c(t\wedge\tau^\epsilon)}(-K+\beta_1\xi_j)\right] \\ &\qquad\times I_{\{\tau_j^\pi < t\wedge\tau^\epsilon\}}. \end{aligned} \tag{4.25}$$

Letting $\epsilon \to 0, t \to \infty$ in (4.25) and using the definition of τ^π and the boundedness condition of $\nu_n'(x)$ on $(0,\infty)$, it is not difficult to show that

$$\begin{aligned} &\liminf_{t\to\infty} e^{-c(t\wedge\tau^\pi)}\nu_n(R^\pi(t\wedge\tau^\pi)) \\ &\quad= e^{-c\tau^\pi}\nu_n(0)I_{\{\tau^\pi<\infty\}} + \liminf_{t\to\infty} e^{-c(t\wedge\tau^\pi)}\nu_n(R^\pi(t\wedge\tau^\pi))I_{\{\tau^\pi=\infty\}} \\ &\quad\geq e^{-c\tau^\pi}\nu_n(0)I_{\{\tau^\pi<\infty\}} \geq 0. \end{aligned} \tag{4.26}$$

Letting $\epsilon \to 0$, $n \to \infty$, and $t \to \infty$, we deduce from (4.25) and (4.26) that

$$J(x,\pi) = E\left[\sum_{n=1}^\infty e^{-c\tau_n^\pi}(-K+\beta_1\xi_n^\pi)I_{\{\tau_n^\pi<\tau^\pi\}} - \int_0^{\tau^\pi} e^{-cs}\beta_2 dL^\pi(s)\right] \leq \bar{\nu}(x),$$

and the result follows. ∎

THEOREM 4. *The optimal return function, or the value function, V can be characterized as the unique viscosity solution* (4.5) *with the boundary condition*

$$V(0) := \inf\{\bar{\nu}(0) : \bar{\nu}(x) \text{ is a viscosity supersolution of } (4.5)\}.$$

Proof. The proof follows from Propositions 2 and 3.

5. Regularity of the value function. In this section we study the regularity of the value function of the optimization problem of the insurance company. We start by defining the following function spaces:

1. $W^{k,p}(U)$ is the space of all L^p functions with βth-order weak partial derivatives belonging to L^p, for all $|\beta| \leq k$, where $U \subset \Re^+$;
2. $C^k(U)$ is the space of all real-valued, kth-order continuously differentiable functions defined on U;
3. $C_c^\infty(U) = \{f \in C^\infty(U) : f \text{ has compact support in } U\}$;
4. $$C^{k,\alpha}(U) := \left\{ f \in C^k(U) : \sup_{x,y\in U} \left\{ \frac{D^\beta f(x) - D^\beta f(y)}{|x-y|^\alpha} \right\} < \infty \quad \forall |\beta| \leq k \right\}.$$

In what follows, we show that the value function V is in the Sobolev space $W^{2,p}(\mathcal{O})$ for any open bounded region $\mathcal{O}$ in $\Re^+$ and any $p < \infty$. In particular, $V \in C^1(\Re^+)$. The developments for the results and proofs in this section follow those in section 4 of Guo and Wu [11].

The following lemma gives a C^2-regularity result for the value function V in the continuation region $\mathcal{C}$.

LEMMA 6. (1) *The value function $V \in C^2(\mathcal{C})$ satisfies the following differential equation in the classical sense:*

$$\max_{a\in[0,N]} \mathcal{L}^a V(x) = 0, \quad x \in \mathcal{C}. \tag{5.1}$$

(2) *Suppose $D \subset \Re^+$ and that there exists a compact set F such that $D \subset F \subset \mathcal{C}$ (i.e., D is compactly contained in $\mathcal{C}$). Write $\bar{D}$ for the closure of D. Then*

$$V \in C^{2,\alpha}(\bar{D})$$

for some $\alpha > 0$.

Proof. First, from Theorem 3, V satisfies $\max_{a\in[0,N]} \mathcal{L}^a V(x) = 0$ in $\mathcal{C}$ in a viscosity sense. For any open ball $B \subset \mathcal{C}$ and each $a \in [0, N]$, we consider the following Dirichlet problem:

$$\mathcal{L}^a w^a(x) = 0, \quad x \in B,$$
$$w^a(x) = V(x), \quad x \in \partial B,$$

where ∂B is the boundary of B.

Then using classical Schauder estimates, for each $a \in [0, N]$, the associated Dirichlet problem has a solution in $C^{2,\alpha_a}(B)$ for some $\alpha_a > 0$. Take $\alpha := \max_{a\in[0,N]} \alpha_a$. Then the following Dirichlet problem has a solution in $C^{2,\alpha}(B)$:

$$\max_{a\in[0,N]} \mathcal{L}^a w(x) = 0, \quad x \in B,$$
$$w(x) = V(x), \quad x \in \partial B.$$

Consequently, w satisfies the differential equation in a viscosity sense and by classical uniqueness results of a viscosity solution of the linear PDE $w(x) = V(x)$ for all $x \in B$. Therefore, $V \in C^{2,\alpha}(\mathcal{O})$.

The second statement can be proved by noting that $\bar{D}$ can be covered by finitely many open balls contained in $\mathcal{C}$. ∎

The following theorem is the main regularity result and shows that the value function V is in the Sobolev space $W^{2,p}(\mathcal{O})$ for any open bounded region $\mathcal{O}$.

THEOREM 5. *For any bounded open set $\mathcal{O} \subset \Re^+$ and $p < \infty$,*

$$V \in W^{2,p}(\mathcal{O}).$$

In particular, by the Sobolev embedding, $V \in C^1(\Re^+)$.

Proof. Let $\mathcal{O}$ be any given bounded open set in $\Re^+$. Define $\mathcal{C}'$ and $\mathcal{A}'$, respectively, by

$$\mathcal{C}' := \mathcal{C} \cap \mathcal{O}, \quad \mathcal{A}' := \mathcal{A} \cap \mathcal{O}.$$

We wish to prove that there is a constant $K_{\mathcal{O}}$ depending on the region $\mathcal{O}$ such that

$$-K_{\mathcal{O}} \le \max_{[0,N]} \mathcal{L}^a V \le K_{\mathcal{O}},$$

in the sense of distribution, or in a weak sense.

That is, for any smooth, continuous test function $\phi \in C_c^\infty(\mathcal{O})$ with $\phi \le 0$,

$$-K_{\mathcal{O}} \int_{\mathcal{O}} \phi dx \le \int_{\mathcal{O}} \max_{[0,N]} \left(\frac{1}{2}\sigma^2(a) V' \phi' + \mu(a) V' \phi - cV\phi \right) dx \le K_{\mathcal{O}} \int_{\mathcal{O}} \phi dx.$$

Here V' is the first-order weak derivative of V. This weak derivative is well defined since V is Lipschitz in $\mathcal{O}$ (i.e., $V \in W^{1,\infty}(\mathcal{O})$).

Since V is a viscosity solution of the HJB equation, it is a viscosity supersolution of $\max_{a\in[0,N]} \mathcal{L}^a V = 0$ in $\mathcal{O}$, and hence a viscosity supersolution of $\mathcal{L}^a V = 0$ in $\mathcal{O}$, for each $a \in [0, N]$. By Theorem 1 of [17], V is also a weak, or distribution, supersolution of $\mathcal{L}^a V = 0$ in $\mathcal{O}$ for each $a \in [0, N]$. Consequently,

$$\mathcal{L}^a V \le 0,$$

in a weak sense.

Let $V^\epsilon := V * \eta^\epsilon \in C^\infty$, which is the mollification of V; η^ϵ is the standard mollifier and $\eta^\epsilon(x) := \eta(x/\epsilon)/\epsilon^n$. In what follows, we shall show that, for each $\bar{x}_0 \in \mathcal{O}$,

$$\max_{[0,N]} \mathcal{L}^a V^\epsilon(\bar{x}_0) \ge -\bar{M}$$

for some positive constant $\bar{M}$, which is independent of $\bar{x}_0$.

When $\bar{x}_0 \in \mathcal{C}'$, by applying the same arguments used in Lemma 6,

$$\max_{[0,N]} \mathcal{L}^a V(\bar{x}_0) = 0$$

in the classical sense. Consequently,

$$|\max_{[0,N]} \mathcal{L}^a V^\epsilon(\bar{x}_0)| = 0 \le \bar{M},$$

and so $\max_{[0,N]} \mathcal{L}^a V^\epsilon(\bar{x}_0) \geq -\bar{M}$.

When $\bar{x}_0 \in \partial\mathcal{A}'$, by the previous case, we can find a sequence $\{\bar{x}_n\} \subset \mathcal{C}'$ converging to $\bar{x}_0$ such that

$$\max_{[0,N]} \mathcal{L}^a V^\epsilon(\bar{x}_n) \geq -\bar{M}.$$

The result is obtained by taking the limit as $n \to \infty$.

In what follows, we consider the case that $\bar{x}_0 \in \mathcal{A}'_0$ (i.e., the interior of $\mathcal{A}'$). Note that by definition, $\mathcal{A}' \in \mathcal{O}$ and that $|\beta_1\xi - K| \to \infty$ as $|\xi| \to \infty$. Then we can find an open ball $\mathcal{O}'$ covering $\mathcal{O}$ so that $\xi_x \in \mathcal{O}'$ for all $x \in \mathcal{A}'$ and $\xi_x \in \Xi(x)$ since

$$\beta_1\xi_x - K = \mathcal{M}V(x) - V(x-\xi) \leq \mathcal{M}V(x) \leq \sup_{\bar{\mathcal{O}}'} \mathcal{M}V.$$

Define the set $\mathcal{D}$ by

$$\mathcal{D} := \left\{ x \in \mathcal{O}' | \mathcal{M}V(x) < V(x) - \frac{L}{2} \right\}.$$

It is then clear that $\mathcal{D}$ is compactly contained in $\mathcal{C}$. By Lemma 6,

$$V \in \mathcal{C}^{2,\alpha}(\bar{\mathcal{D}}).$$

For each $y \in \mathcal{A}'_0$, suppose $B_{\rho_1}(y) \subset \mathcal{A}'_0$. Take $\xi_y \in \Xi(y)$. Then $x := y + \Xi(y)$ satisfies

$$\mathcal{M}V(x) - V(x) \leq L$$

by Lemma 4. This implies that $x \in \mathcal{D}$.

Since $\mathcal{M}V - V$ is uniformly continuous on $\bar{\mathcal{O}}'$, there is $\rho_2 > 0$ such that if

$$|x - x'| \leq \rho_2,$$

then

$$|\mathcal{M}V(x') - V(x') - (\mathcal{M}V(x) - V(x))| \leq \frac{L}{4}.$$

Consequently, for any constant $\lambda \in [-1,1]$, $x' = x + \lambda\rho_2$ satisfies

$$\mathcal{M}V(x') - V(x') \leq \mathcal{M}V(x) - V(x) + \frac{L}{4} < -\frac{L}{2}.$$

So if we let $\rho \in (0, \rho_1 \wedge \rho_2]$, then for all $\lambda \in [-1,1]$, we have

$$\begin{aligned} &x = y + \xi_y \in \mathcal{D}, \quad x' = x + \lambda\rho \in \mathcal{D}, \\ &y \in \mathcal{A}'_0\ , \quad y + \lambda\rho \in \mathcal{A}'_0. \end{aligned}$$

By definition,

$$\begin{aligned} &V(y) = \mathcal{M}V(y) = V(y - \xi_y) + \beta\xi_y - K, \\ &V(y \pm \rho) = \mathcal{M}V(y \pm \rho) \leq V(y \pm \rho - \xi_y) + \beta\xi_y - K. \end{aligned}$$

Consequently,

$$\begin{aligned}
&\frac{1}{\rho^2}|V(y+\rho)+V(y-\rho)-2V(y)| \\
&\quad \leq \frac{1}{\rho^2}|V(x+\rho)+V(x-\rho)-2V(x)| \\
&\quad = \frac{1}{|\rho|}\int_0^1 |DV(x+\lambda\rho)-DV(x-\lambda\rho)|d\lambda \\
&\quad \leq C_{\mathcal{D}},
\end{aligned}$$

where $C_{\mathcal{D}} := \sup_{x\in\bar{\mathcal{D}}}|D^2V(x)| \leq ||V||_{\mathcal{C}^{2,\alpha}(\bar{\mathcal{D}})}$. Given $\bar{x}_0 \in \mathcal{A}_0'$, let $B_\theta(\bar{x}_0) \subset \mathcal{A}_0'$. Then for any $\epsilon \in (0, \frac{\theta}{2})$, $\rho_1 = \frac{\theta}{2}$, $z \in B_\epsilon(0)$, we have $B_{\rho_1}(\bar{x}_0 - z) \subset \mathcal{A}_0'$. Consequently,

$$\begin{aligned}
&\frac{1}{\rho^2}|V^\epsilon(\bar{x}_0+\rho)+V^\epsilon(\bar{x}_0-\rho)-2V^\epsilon(\bar{x}_0)| \\
&\quad = \frac{1}{\rho^2}\int_{B_\epsilon(0)}[V(\bar{x}_0-z+\rho)+V(\bar{x}_0-z-\rho)-2V(\bar{x}_0-z)]\eta^\epsilon(z)dz \\
&\quad \leq C_{\mathcal{D}}\int_{B_\epsilon(0)}\eta^\epsilon(z)dz = C_{\mathcal{D}}.
\end{aligned}$$

Letting $\rho \to 0$,

$$|D^2V^\epsilon(\bar{x}_0)| \leq C_{\mathcal{D}},$$

and so, for all $a \in [0, N]$,

$$|\sigma^2(a)D^2V^\epsilon| \leq C^\dagger$$

for some positive constant $C^\dagger$.

Note that $|V^\epsilon(\bar{x}_0)| + ||DV^\epsilon(\bar{x}_0)|| \leq ||V||_{W^{1,\infty}(\bar{\mathcal{O}})}$, so

$$\mathcal{L}^aV^\epsilon(\bar{x}_0) = \frac{1}{2}\sigma^2(a)D^2V^\epsilon(\bar{x}_0) + \mu(a)DV^\epsilon(\bar{x}_0) - cV^\epsilon(\bar{x}_0) \geq -\bar{M}$$

for some positive constant $\bar{M}$ independent of $\bar{x}_0$, but depending on $\mathcal{O}$.

Therefore,

$$\max_{[0,N]}\mathcal{L}^aV^\epsilon(\bar{x}_0) \geq -\bar{M} \quad \forall \bar{x}_0 \in \mathcal{O}$$

for some positive constant M independent of $\bar{x}_0$. This implies that for any smooth, continuous test function $\phi \in C_c^\infty(\mathcal{O})$ with $\phi \geq 0$,

$$\int_{\mathcal{O}}\max_{[0,N]}\left(\frac{1}{2}\sigma^2(a)(V^\epsilon)'\phi' + \mu(a)(V^\epsilon)'\phi - cV^\epsilon\phi\right)dx \geq -\bar{M}\int_{\mathcal{O}}\phi dx.$$

Since $V \in W^{1,2}(\mathcal{O})$, $V^\epsilon \to V$ and $(V^\epsilon)' \to V'$ in $L^2(\mathcal{O})$. Letting $\epsilon \to 0$, for some positive constant $\bar{M}$ depending on $\mathcal{O}$,

$$-\bar{M} \leq \max_{[0,N]}\mathcal{L}^aV \leq \bar{M}$$

in the sense of distribution. Therefore,

$$\max_{a\in[0,N]}\mathcal{L}^aV \in L^\infty(\mathcal{O}).$$

Using the Calderon–Zygmund estimate, $V \in W^{2,p}(\mathcal{O})$ for all $p < \infty$. ∎

6. Structure of the value function and the optimal policy. In this section we characterize the structure of the value function and the continuation and action regions of the impulse control problem. First, let $N := \sup\{y : F(y) < 1\} \leq \infty$ and

$$G(z) := \int_0^z \frac{\sigma^2(y)}{2cy^2 + 2y\mu(y) - \sigma^2(y)} dy. \tag{6.1}$$

It is not difficult to show that the integrand in the right-hand side of (6.1) is positive. Consequently, the inverse function $G^{-1}(z)$ of $G(z)$ exists. From Lemma 5.2 in [2], $G(\infty) < \infty$. Therefore, there exists an $x_0 < \infty$ such that $G^{-1}(x_0-) = N$.

6.1. Structure of the value function. The following theorem characterizes the structure of the solution of the impulse control problem.

THEOREM 6. (i) *There exists a constant x_1 such that*

$$\mathcal{C} := \{x \in \Re^+ : \mathcal{M}V(x) < V(x)\} = (0, x_1),$$
$$\mathcal{A} := \{x \in \Re^+ : \mathcal{M}V(x) = V(x)\} = [x_1, \infty).$$

(ii) *The value function V defined by* (1.6) *satisfies*

$$\begin{cases} \sup_{a\in[0,N]} \mathcal{L}^a V(x) = 0, & 0 \leq x < x_1, \\ V(x) = V(x_1) + \beta_1(x - x_1), & x \geq x_1. \end{cases} \tag{6.2}$$

(iii) *There is $\tilde{x} \in (0, x_1)$ such that*

$$V'(x_1) = V'(\tilde{x}) = \beta_1, \quad V(x_1) = V(\tilde{x}) + \beta_1(x_1 - \tilde{x}) - K, \quad V'(0) = \beta_2.$$

We need the following lemmas before proving the above theorem.

LEMMA 7. *For any $x \in \mathcal{A}$,*

$$V'(x) = \beta_1.$$

Proof. First, by Lemma 4, there exists ξ_x such that

$$V(x) = \mathcal{M}V(x) = V(x - \xi_x) + \beta_1\xi_x - K,$$

which means ξ_x is a global maximum of the function $V(x - y) + \beta_1 y - K$ of y. The first-order condition then gives

$$V'(x - \xi_x) = \beta_1. \tag{6.3}$$

Now, for any $\delta \neq 0$,

$$V(x + \delta) \geq \mathcal{M}V(x + \delta) \geq V(x + \delta - \xi_x) + \beta_1\xi_x - K.$$

Then

$$V(x + \delta) - V(x + \delta - \xi_x) \geq \beta_1\xi_x - K = V(x) - V(x - \xi_x).$$

Consequently,

$$\frac{V(x + \delta) - V(x)}{\delta} \geq \frac{V(x + \delta - \xi_x) - V(x - \xi_x)}{\delta}, \ \delta > 0,$$

$$\frac{V(x + \delta) - V(x)}{\delta} \leq \frac{V(x + \delta - \xi_x) - V(x - \xi_x)}{\delta}, \ \delta < 0.$$

The result then follows by sending $\delta \to 0^+$ ($\delta \to 0^-$) and using (6.3). ∎

LEMMA 8. *There exists a constant b such that $\mathcal{C}$ contains the interval $(0, b)$ with $b > 0$.*

Proof. We prove this lemma by contradiction. Suppose the statement is not true. Then we consider the following cases:

- For $b_1 > 0$, there exists $x_1 \in (0, b_1]$ such that $V(x_1) = \mathcal{M}V(x_1)$.
- For $b_2 \le \min\{\frac{1}{2}, x_1\}$, there exists $x_2 \in (0, b_2]$ such that $V(x_2) = \mathcal{M}V(x_2)$.
- For $b_n \le \min\{\frac{1}{n}, x_{n-1}\}$, there exists $x_n \in (0, b_n]$ such that $V(x_n) = \mathcal{M}V(x_n)$, $n = 2, 3, \dots$.

Consequently, $x_n \to 0$, and there exists a sufficiently large n_1 such that x_{n_1} satisfying

$$V(x_{n_1}) - V(x_{n_1} - \xi) - \beta_1 \xi \ge -\frac{K}{2} \quad \forall\, \xi \in (0, x_{n_1}]. \tag{6.4}$$

Thus we have

$$V(x_{n_1}) \ge \mathcal{M}V(x_{n_1}) + \frac{K}{2}, \tag{6.5}$$

which is a contradiction. ∎

The following lemma resembles Lemma 5.5 of Guo and Wu [11]. We state the result without giving the proof.

LEMMA 9. *$\mathcal{C}$ is connected.*

Proof of Theorem 6. (1) By Lemmas 8 and 9, $\mathcal{C} = (0, x_1)$ for some $x_1 \in (0, \infty)$.

(2) Suppose $x \ge x_1$. Since $V'(x) = \beta_1$, $V(x) = V(x_1) + \beta_1(x - x_1)$.

(3) Let $\xi_{x_1} \in \Xi(x_1)$ and $\tilde{x} := x_1 - \xi_{x_1}$. Then $\tilde{x} \in (0, x_1)$, $V'(\tilde{x}) = V'(x_1) = \beta_1$, and

$$V(x_1) = \mathcal{M}V(x_1) = V(\tilde{x}) + \beta_1(x_1 - \tilde{x}) - K. \tag{6.6}$$

From Theorem 4, the optimal return function $V(x)$ is the minimal viscosity supersolution, so $V'(0) = \beta_2$. ∎

6.2. The optimal return function and policy. First, we note that the HJBQVI (quasi-variational inequality) equation does not explicitly depend on x. Motivated by [28], we observe that $V(x)$ can be represented as $V(x) = \psi(x + h)$, where $h \ge 0$ and $\psi(x)$ is the viscosity solution of the equation

$$\max\left\{ \max_{a \in [0,N)} \mathcal{L}^a \psi(x), \mathcal{M}\psi(x) - \psi(x) \right\} = 0, \tag{6.7}$$

with $\psi(0) = 0$.

The value function of the optimal dividend-reinvestment problem can then be obtained by shifting the value function of the optimal dividend problem h units to the left, where h is determined by $\psi'(h) = \beta_2$.

Define

$$\mathcal{C}_\psi := \{x \in \Re^+ : \mathcal{M}\psi(x) < \psi(x)\},$$
$$\mathcal{A}_\psi := \{x \in \Re^+ : \mathcal{M}\psi(x) = \psi(x)\}.$$

By the analysis as above, $\mathcal{C}_\psi$ and $\mathcal{A}_\psi$ have, respectively, the following forms:

$$\mathcal{C}_\psi = [0, x_\psi), \quad \mathcal{A}_\psi = [x_\psi, \infty)$$

for some $x_\psi \in (0,\infty)$.

Let $a(x)$ be the maximum point of $\max_{a\in[0,N)} \mathcal{L}^a\psi(x) = 0$. The first-order condition then gives

$$a(x) = -\frac{\psi'(x)}{\psi''(x)}. \tag{6.8}$$

Substituting (6.8) into (6.7) leads to

$$\left[-\frac{1}{2a}\sigma^2(a) + \mu(a)\right]\psi'(x) - c\psi(x) = 0 \tag{6.9}$$

for $x < x_\psi$.

Differentiating with respect to x and using (6.8) once more give

$$\left[\frac{\sigma^2(a)a'(x)}{2a^2} - c\right]\psi'(x) + \left[-\frac{1}{2a}\sigma^2(a) + \mu(a)\right]\psi''(x) = 0,$$

and this then leads to

$$\left[\frac{\sigma^2(a)a'(x)}{2a^2} - c + \frac{1}{2a^2}\sigma^2(a) - \frac{\mu(a)}{a}\right]\psi'(x) = 0.$$

We divide by $\psi'(x)$ and obtain a differential equation

$$a'(x) = \frac{2ca^2(x) + 2a(x)\mu(a(x)) - \sigma^2(a(x))}{\sigma^2(a(x))}.$$

Then we see that

$$a(x) = G^{-1}(x).$$

If $0 \le x \le x_0$, one obtains $a(x) \in [0,N]$. From (6.8), we have

$$\psi(x) = C\int_0^x e^{\int_z^{x_0} \frac{1}{G^{-1}(y)}dy} dz, \qquad 0 \le x \le x_0,$$

where C is an unknown constant.

From (iii) of Theorem 6, we know that $[0,x_0] \subset [0,x_\psi)$. Then we have the following theorem.

THEOREM 7. *$a(x) \equiv N$ for $x \in [x_0, x_\psi)$.*

Proof. Suppose there exists $\eta_1 > x_0$ such that $a(\eta_1) < N$. There is a ball $B_\varepsilon(\eta_1)$ with center η_1 and radius ε such that for all $x \in B_\varepsilon(\eta_1)$, $a(x) < N$. Let $\eta_2 := \sup\{x < \eta_1 : a(x) = N\}$. Then $x_0 \le \eta_2 < \eta_1 < \eta_1 + \varepsilon$ and $a(\eta_2) = N$. Since $a(x) \le N$ for all $x \in [\eta_2, \eta_1+\varepsilon)$, $a(x)$ must satisfy

$$a(\eta_2) = N, \quad a'(x) = \frac{2ca^2(x) + 2a(x)\mu(a(x))}{\sigma^2(a(x))} - 1, \qquad x \in [\eta_2, \eta_1+\varepsilon).$$

Consequently, noting that the integral of the function $G(z)$ is positive, we see that

$$\begin{aligned} a(x) &= a(\eta_2) + \int_{\eta_2}^x \left(\frac{2ca^2(y) + 2a(y)\mu(a(y)) - \sigma^2(a(y))}{\sigma^2(a(y))}\right) dy \\ &= N + \int_{\eta_2}^x \left(\frac{2ca^2(y) + 2a(y)\mu(a(y)) - \sigma^2(a(y))}{\sigma^2(a(y))}\right) dy \\ &> N, \end{aligned}$$

which then leads to a contradiction. ∎

Thus, for $x \in (x_0, x_\psi)$, $a(x) = N$ (i.e., no reinsurance at all). Consequently, the HJBQVI equation takes the following form:

$$\frac{\sigma_\infty^2}{2}\psi''(x) + \mu_\infty \psi'(x) - c\psi(x) = 0 \tag{6.10}$$

for all $x \in (x_0, x_\psi)$.

Solving (6.10) yields

$$\psi(x) = C_1 e^{d_+(x-x_0)} + C_2 e^{d_-(x-x_0)},\ x_0 < x < x_1,$$

where C_1 and C_2 are unknown constants, and

$$d_+ = \frac{-\mu_\infty + \sqrt{\mu_\infty^2 + 2c\sigma_\infty^2}}{\sigma_\infty^2},\quad d_- = \frac{-\mu_\infty - \sqrt{\mu_\infty^2 + 2c\sigma_\infty^2}}{\sigma_\infty^2}.$$

The continuity of the first- and second-order derivatives of ψ at x_0 gives

$$\begin{aligned} C_1 d_+ + C_2 d_- &= C, \\ C_1 d_+^2 + C_2 d_-^2 &= -\frac{C}{N}, \end{aligned}$$

which results in

$$C_1 = Cb_1,\ C_2 = Cb_2,$$

where

$$\begin{aligned} b_1 &= -\frac{\frac{1}{N} + d_-}{d_+(d_+ - d_-)} > 0, \\ b_2 &= \frac{\frac{1}{N} + d_+}{d_-(d_+ - d_-)} < 0. \end{aligned}$$

Therefore, the solution has the following form:

$$\psi(x) = \begin{cases} C\int_0^x e^{\int_z^{x_0} \frac{1}{G^{-1}(y)}dy} dz, & 0 \le x < x_0, \\ C(b_1 e^{d_+(x-x_0)} + b_2 e^{d_-(x-x_0)}), & x_0 \le x \le x_\psi, \\ \beta_1(x - \tilde{x}_\psi) + \psi(\tilde{x}_\psi) - K, & x > x_\psi. \end{cases} \tag{6.11}$$

In what follows, we shall determine C, $\tilde{x}_\psi$, and x_ψ by the conditions

$$\psi'(\tilde{x}_\psi) = \psi'(x_\psi) = \beta_1,\ \psi(x_\psi) = \psi(\tilde{x}_\psi) + \beta_1(x_\psi - \tilde{x}_\psi) - K.$$

We start by constructing the following function $H(x)$, $x > 0$:

$$H(x) := \begin{cases} e^{\int_x^{x_0} \frac{1}{G^{-1}(y)}dy}, & x < x_0, \\ b_1 d_+ e^{d_+(x-x_0)} + b_2 d_- e^{d_-(x-x_0)}, & x > x_0. \end{cases} \tag{6.12}$$

From the above analysis, H is a continuously differentiable convex function. It is easy to see that

$$\lim_{x\to 0} H(x) = \lim_{x\to\infty} H(x) = \infty.$$

Consequently, there exists a point $x^* > x_0$ such that

$$H'(x^*) = 0.$$

Let $\alpha = H(x^*) > 0$. If $0 < C < \beta_1/\alpha$, then there exist two points $\tilde{x}_\psi^C < x^* < x_\psi^C$ such that $CH(\tilde{x}_\psi^C) = CH(x_\psi^C) = \beta_1$. Obviously, if $C = \beta_1/\alpha$, then $\tilde{x}_\psi^C = x_\psi^C = x^*$. It is easy to see that $\tilde{x}_\psi^C$ is an increasing function of C, while x_ψ^C is a decreasing function of C, for $C \in (0, \beta_1/\alpha]$.

Define

$$I(C) := \int_{\tilde{x}_\psi^C}^{x_\psi^C} (\beta_1 - CH(y))dy.$$

We can easily show that $I(C) \to \infty$ as $C \to 0$. Since $I(\beta_1/\alpha) = 0$, there exists $\tilde{C} < \beta_1/\alpha$ such that

$$I(\tilde{C}) := \int_{\tilde{x}_\psi^{\tilde{C}}}^{x_\psi^{\tilde{C}}} (\beta_1 - CH(y))dy = K$$

for a given positive constant K.

Define the function

$$\hat{T}(x) := \begin{cases} \tilde{C}H(x), & x < x_\psi^{\tilde{C}}, \\ \beta_1, & x \geq x_\psi^{\tilde{C}} \end{cases}$$

and

$$\psi(x) := \int_0^x \hat{T}(y)dy.$$

Then

$$\psi(x) = \begin{cases} \tilde{C}\int_0^x e^{\int_y^{x_0} \frac{1}{G^{-1}(z)}dz}dy, & 0 \leq x < x_0, \\ \tilde{C}(b_1 e^{d_+(x-x_0)} + b_2 e^{d_-(x-x_0)}), & x_0 \leq x < x_\psi^{\tilde{C}}, \\ \psi(x_\psi^{\tilde{C}}) + \beta_1(x - x_\psi^{\tilde{C}}), & x \geq x_\psi^{\tilde{C}}. \end{cases} \tag{6.13}$$

LEMMA 10. *If $\tilde{C} \leq \beta_2$, then there exists a unique $h_1 \in (0, \min\{x_0, \tilde{x}_\psi^{\tilde{C}}\})$ such that h_1 satisfies the following equation in x:*

$$\tilde{C}e^{\int_x^{x_0} \frac{1}{G^{-1}(y)}dy} = \beta_2. \tag{6.14}$$

Proof. Denote the left-hand side of (6.14) by $k(x)$. Obviously, $k(x)$ is a decreasing function, and $\min\{k(x_0), k(\tilde{x})\} = \min\{\tilde{C}, \beta_1\} \leq \beta_2$. From [3, p. 310],

$$\tilde{C}e^{\int_x^{x_0} \frac{1}{G^{-1}(y)}dy} \sim \tilde{C}x^{-1/(1+2c)} \to \infty \text{ as } x \to 0.$$

Therefore, (6.14) has a unique solution h_1. ∎

Similarly, we have the following lemma.

LEMMA 11. *If $\tilde{C} > \beta_2$, then there exists a unique $h_2 \in (x_0, \tilde{x}_\psi^{\tilde{C}})$ such that h_2 satisfies the following equation in x:*

$$\tilde{C}\left(-\frac{\frac{1}{N} + d_-}{(d_+ - d_-)}e^{d_+(x-x_0)} + \frac{\frac{1}{N} + d_+}{(d_+ - d_-)}e^{d_-(x-x_0)}\right) = \beta_2.$$

Then we have the following theorem, which gives the value function and the optimal policy of the impulse control problem.

THEOREM 8. *Let*

$$a_1(x) = \begin{cases} G^{-1}(x+h_1), & x < x_0 - h_1, \\ N, & x_0 - h_1 \le x \le x_\psi^{\tilde{C}} - h_1, \\ \textit{any strategy}, & x > x_\psi^{\tilde{C}} - h_1 \end{cases}$$

and

$$a_2(x) = \begin{cases} N, & 0 \le x \le x_\psi^{\tilde{C}} - h_2, \\ \textit{any strategy}, & x > x_\psi^{\tilde{C}} - h_2. \end{cases}$$

Suppose the condition that $\tilde{C} \le \beta_2$ holds true. Then the value function $V(x)$ is given by

$$V(x) = \psi(x+h_1) = \begin{cases} \tilde{C}\int_0^{x+h_1} e^{\int_y^{x_0} \frac{1}{G^{-1}(z)}dz} dy, & 0 \le x < x_0 - h_1, \\ \tilde{C}(b_1 e^{d_+(x+h_1-x_0)} + b_2 e^{d_-(x+h_1-x_0)}), & x_0 - h_1 \le x \le x_\psi^{\tilde{C}} - h_1, \\ \psi(x_\psi^{\tilde{C}}) + \beta_1(x + h_1 - x_\psi^{\tilde{C}}), & x \ge x_\psi^{\tilde{C}} - h_1. \end{cases}$$

The optimal policy $\pi^* := (a_{\pi^*}, L^{\pi^*}, Q^{\pi^*})$ satisfies

$$\begin{cases} R^{\pi^*}(t) = x + \int_0^t \mu(a(R^{\pi^*}(s)))ds + \int_0^t \sigma(a(R^{\pi^*}(s)))dB(s) - \sum_{n=1}^\infty I_{\{\tau_n^{\pi^*} \le t\}} \xi_n^{\pi^*} + L^{\pi^*}(t), \\ 0 \le R^{\pi^*}(t) \le x_\psi^{\tilde{C}} - h_1, \\ \int_0^\infty I_{\{t: R^{\pi^*}(t) < x_\psi^{\tilde{C}} - h_1\}} dQ^{\pi^*}(t) = 0, \\ \int_0^\infty I_{\{t: R^{\pi^*}(t) > 0\}} dL^{\pi^*}(t) = 0, \end{cases}$$

where $a_{\pi^*}(t) := a_1(R^{\pi^*}(t))$, $\tau_0^{\pi^*} := 0$, $\tau_i^{\pi^*} := \inf\{t > \tau_{i-1}^{\pi^*} : R^{\pi^*}(t-) = x_\psi^{\tilde{C}} - h_1\}$, and $\xi_i^{\pi^*} := x_\psi^{\tilde{C}} - \tilde{x}_\psi^{\tilde{C}}$, $i = 1, 2, \dots$.

Suppose the condition that $\tilde{C} > \beta_2$ holds true. Then the value function $V(x)$ is given by

$$V(x) = \psi(x+h_2) = \begin{cases} \tilde{C}(b_1 e^{d_+(x+h_2-x_0)} + b_2 e^{d_-(x+h_2-x_0)}), & 0 \le x \le x_\psi^{\tilde{C}} - h_2, \\ \psi(x_\psi^{\tilde{C}}) + \beta_1(x + h_2 - x_\psi^{\tilde{C}}), & x \ge x_\psi^{\tilde{C}} - h_2. \end{cases}$$

The optimal policy $\pi^{**} := (a_{\pi^{**}}, L^{\pi^{**}}, G^{\pi^{**}})$ satisfies

$$\begin{cases} R^{\pi^{**}}(t) = x + \int_0^t \mu(a(R^{\pi^{**}}(s)))ds + \int_0^t \sigma(a(R^{\pi^{**}}(s)))dB(s) \\ \qquad\qquad - \sum_{n=1}^\infty I_{\{\tau_n^{\pi^{**}} \le t\}} \xi_n^{\pi^{**}} + L^{\pi^{**}}(t), \\ 0 \le R^{\pi^{**}}(t) \le x_\psi^{\tilde{C}} - h_2, \\ \int_0^\infty I_{\{t: R^{\pi^{**}}(t) < x_\psi^{\tilde{C}} - h_2\}} dQ^{\pi^{**}}(t) = 0, \\ \int_0^\infty I_{\{t: R^{\pi^{**}}(t) > 0\}} dL^{\pi^{**}}(t) = 0, \end{cases}$$

where $a_{\pi^{**}}(t) := a_2(R^{\pi^{**}}(t))$, $\tau_0^{\pi^{**}} := 0$, $\tau_i^{\pi^{**}} := \inf\{t > \tau_{i-1}^{\pi^{**}} : R^{\pi^{**}}(t-) = x_\psi^{\tilde{C}} - h_2\}$, and $\xi_i^{\pi^{**}} := x_\psi^{\tilde{C}} - \tilde{x}_\psi^{\tilde{C}}$, $i = 1, 2, \dots$.

Remark 2. If $N = \infty$, then $x^* = x_0$, and $\alpha = H(x^*) = 1$. Therefore, $0 < \tilde{C} \le \beta_1 < \beta_2$, which rules out the possibility that $\tilde{C} > \beta_2$.

7. Conclusion. The quest for optimal management strategies of insurance companies has remained an active area of research in the past few decades. We provided here a detailed and rigorous mathematical analysis for the combined optimal dividend-financing-reinsurance problem, where the goal is to maximize the expected present value of dividend payouts minus the equity issuance until the bankruptcy time. The optimality of the excess-of-loss reinsurance was established and this reinsurance policy was then considered in the paper. We used the HJB dynamic programming principle to discuss the impulse control problem corresponding to the combined optimal dividend-financing-reinsurance problem. The uniqueness of the viscosity solution of the HJBQVI equation associated with the impulse control was established. We also gave some regularity results for the value function. The structure of the value function and the regularity of the optimal policies were discussed. For some cases, explicit results are derived. For example, the optimal retention in the excess-of-loss reinsurance strategy and the value function were given explicitly in these cases. When the distribution function of claim sizes has a bounded support, we solved the general impulse control problem with a nonzero boundary condition. We also showed that the optimal risk model is one which imposes no risk of bankruptcy with equity issuance. The corresponding results were also obtained when the distribution function of claim sizes has an unbounded support.

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