Fault Tolerance with Noisy and Slow Measurements and Preparation

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It is not so well known that measurement-free quantum error correction protocols can be designed to achieve fault-tolerant quantum computing. Despite their potential advantages in terms of the relaxation of accuracy, speed, and addressing requirements, they have usually been overlooked since they are expected to yield a very bad threshold. We show that this is not the case. We design fault-tolerant circuits for the 9-qubit Bacon-Shor code and find an error threshold for unitary gates and preparation of \( p_{\text{thresh}} = 3.76 \times 10^{-5} \) (30% of the best known result for the same code using measurement) while admitting up to 1/3 error rates for measurements and allocating no constraints on measurement speed. We further show that demanding gate error rates sufficiently below the threshold pushes the preparation threshold up to \( p_{\text{thresh}} = 1/3 \).

An ideal quantum computer is a theoretical object capable of highly efficient computation. A major difficulty with the realization of such a powerful theoretical object is that physical implementations of any quantum operation will be noisy. However, with the use of quantum error correction (QEC) codes, with the use of fault-tolerantly designed circuits, and provided that error rates are below some threshold value, one is still able to efficiently simulate a quantum computation with an arbitrarily high accuracy [1–3]. Experimental state of the art results show that the error rates and execution times required for operations in order to achieve the fault-tolerant regime are not currently available. The results in this Letter will alleviate part of this constraint by pushing required error rates a step closer to current technology.

In many physical systems, measurements pose a potential bottleneck for scalable fault-tolerant quantum computation because they are slower and/or noisier than gates or preparation [4,5]. However, they are central in the readout stage and are widely used in QEC routines as a way of extracting error syndrome information in order to correct the quantum data. Slow measurements have been shown to be a surmountable issue by using error correction where measured error syndromes can be classically postprocessed at the end of a round of gates to execute a compensating Pauli frame rotation [6], with the caveat that there can be a significant time lag during classical processing [7]. Regarding noise, measurement error rates cannot usually be improved by noise suppression techniques, i.e., dynamical decoupling, whereas gates can be [8,9]. Furthermore, measurement results must be distinguishable in every time step; i.e., one must be able to discriminate between results from different measurements repeatedly over the computation, which leads to further constraints on the physical processes executing the measurements, e.g., measurements relying on photon scattering as in ion traps [10].

In this Letter, we overcome these problems by eliminating most measurements during fault-tolerant computation.

It is well known [1,11] that this is possible for Calderbank-Shor-Steane (CSS) codes [12]; however, “The penalty paid in the stringency of the threshold has never been quantified, but it is expected that replacing measurement by coherent operations decreases the noise threshold by a large amount” [6]. We show that, contrary to these conjectures, coherent fault-tolerant QEC suffers only slightly in regards to the threshold and brings substantial rewards.

We begin by setting up our scenario and introducing measurement-free error correction (EC) routines for the Bacon-Shor code. We then show how to execute fault-tolerant Clifford operations and derive a threshold error rate which is stringent for preparation and gates but as high as 1/3 for measurement. Universality follows by executing fault-tolerant non-Clifford gates enabled by encoded magic states, \( |H_L\rangle = (|0_L\rangle + |1_L\rangle)/\sqrt{2} \), which can be prepared by a distillation protocol [7,13] using exclusively fault-tolerant Clifford operations. Moreover, we show how to relax the threshold value for preparation, by using a variant of algorithmic cooling and demanding a gate error rate \( p_{(g)} \) sufficiently below the threshold. Thus fault-tolerant universal quantum computing (FTUQC) can be achieved with measurement and preparation error rates \( p_{(p)} \) and \( p_{(m)} \), respectively, which are already within reach of current technology.

We demonstrate our scheme for the 9-qubit Bacon-Shor (BS) subsystem code [14], but our tools can be adapted to other CSS codes (see [15]). The BS code is defined by the stabilizer set on a two-dimensional array

\[
\begin{align*}
X X I & I I X X Z Z Z I I I \\
X X I & I I X X Z Z Z I I I \\
X I I & X I X I I I Z Z Z Z Z Z
\end{align*}
\]

For this code, logical Pauli operators are given by \( X_L = \prod_{i=1}^3 X_{1,i} Z_L = \prod_{i=1}^3 Z_{1,i} \mod \text{stabilizer operations} \); i.e., \( X_L \) (\( Z_L \)) acts on a column (row) of the array. This subsystem code is invariant under pairs of \( X \) (\( Z \)) operators along any given row (column) because they act only on
gauge degrees of freedom. Given the subsystem structure of the code, one is able to correct acting on only one row (for X errors) and one column (for Z errors). The library of physical (level-0) gates we use is $\{X,Z,H,CNOT,\text{TOFFOLI},Z-\text{TOFFOLI}=H^\otimes 3(\text{TOFFOLI})H^\otimes 3,|0\rangle\text{ preparation},|+\rangle=|(0) + (1))/2\text{ preparation},|H\rangle\text{ preparation},Z\text{ measurement},X\text{ measurement}\}$, allowing also for nonlocal interactions. We adopt an adversarial, local, stochastic error model [16].

The first obstacle is of course to design an EC routine or gadget which uses coherent feedback instead of measurements and feedback. One needs to use more gates within the EC gadgets to execute the coherent feedback, and, in particular, one would typically need fault-tolerant implementations of TOFFOLI gates at every level. This would yield a very bad threshold value [1,6]. However, during QEC we do not really need a full-fledged TOFFOLI gate since it will be controlled only by ancillas containing the syndrome, i.e., classical, information. For instance, when correcting $X$ errors, a $Z$ error in the ancillas is irrelevant; thus, we can map a BS encoded ancilla to a quantum repetition (QR), i.e., bit-flip, code which protects against $X$ errors but that is vulnerable to $Z$ errors. Using the QR encoded controls, and the structure of the logical operators in the BS code, we can use bitwise TOFFOLI ($b$TOFFOLI) gates to implement the needed operation [see Fig. 1].

The mapping between the BS code and the QR code, of the same level of concatenation, is achieved by using the gate $\mathcal{N}(k): |s_L^{(k)}\rangle \rightarrow |s_L^{(k)}\rangle$, where $|s_L^{(k)}\rangle$ is encoded in the $k$-concatenated BS code ($9^k$ physical qubits) and $|s_L^{(k)}\rangle$ denotes a bit encoded in $k$-concatenated QR code ($3^k$ physical qubits). From our joint use of BS and QR codes, we must also introduce an error correction measurement-free routine for the QR code, i.e., states of the form $a|0,0,0\rangle + b|1,1,1\rangle$. We build a majority voting gadget, which we dub the $\mathcal{M}$ gate [Fig. 1(a)]. In the QR code all gates involved in the $\mathcal{M}$ gate are transversal, and thus we can use this circuit as an EC gadget for this code at any level of concatenation. Moreover, through the $\mathcal{N}$ gate we can also use $\mathcal{M}$ as an encoded majority voting gadget, i.e., acting on a state of the form $a|0_L^{(k)},0_L^{(k)},0_L^{(k)}\rangle + b|1_L^{(k)},1_L^{(k)},1_L^{(k)}\rangle$. Given that the Bacon-Shor code is, in essence, a composition of $X$ and $Z$ basis QC codes, we can use $\mathcal{M}&\mathcal{N}$ as the building block for the BS EC gadget.

Schematically, the BS QEC routine works as follows. The dashed part of Figs. 1(a) and 1(c) is a syndrome extraction stage and turns the ancilla, initially in a $0$ state, into a string which contains the error information. We adapt this method to the BS code. To correct for $X$ errors, we execute an extraction stage in every column of the BS state and get three strings $(s_1, s_2, \text{ and } s_3)$. We use them to vote into a fourth one $s_4 = s_1 \oplus s_2 \oplus s_3$, which will control final correction via $\mathcal{N}$ and $b$TOFFOLI. A single error in, e.g., column one of the BS state leads to $s_4 = s_1$, which would correctly execute the correction by virtue of the gauge freedom; on the other hand, a gauge operation, e.g., two $X$ errors in the same row, leads to $s_4 = s \otimes s = 0$, which correctly implies an identity correction operation. An analogous analysis holds for $Z$-error correction.

Now the $X$ and $Z$ correction stages of the BS QEC routine are essentially equivalent but have some differences. Because the syndrome information after the syndrome extraction stage is different in both cases, we define $\mathcal{N}^{(X)}$ (and $\mathcal{N}^{(Z)}$) gates for the BS code at level $k$ of concatenation [see Fig. 1(b)]: $\mathcal{N}^{(X)}(k) \equiv \prod_{i \in \text{rows}}(V\mathcal{N}^{(X)})(k - 1) = \prod_{i}^{c\text{NOT}}(i,3,3)(k - 1)\prod_{i}^{c\text{NOT}}(i,2,3)(k - 1)\prod_{i}^{X}(k - 1)$, $\mathcal{N}^{(Z)}(k) \equiv \prod_{i \in \text{columns}}(V\mathcal{N}^{(Z)})(k - 1) = \prod_{i}^{c\text{NOT}}(3,0,1,3)(k - 1)\prod_{i}^{c\text{NOT}}(3,0,2,3)(k - 1)\prod_{i}^{Z}(3,0,3)(k - 1)$, where $A_{(r,c)}$ denotes gate $A$ acting on the qubit in row $r$. 

FIG. 1. Measurement-free QEC routines for the QR and BS code. The inputs are $|0^{(k)}\rangle = |000\rangle^{3^{k+1}}$ and $|+^{(k)}\rangle = |++\rangle^{3^{k+1}}$. (a) The $\mathcal{M}$ gate. An $X$-encoded majority voting gadget of level $(k + 1)$ of concatenation. Here all CNOTs are bitwise; i.e., each CNOT depicted corresponds to three CNOT($k$), and subscript $R$ corresponds to a cyclic $k$-encoded rotation of the targets of the corresponding gate. In the QR code the TOFFOLI gate depicted is bitwise. The $\mathcal{M}$ gate can also be designed for a $Z$-encoded quantum majority voting, with $|+^{(k)}\rangle$ ancillas and the obvious Hadamard conjunction of gates. When the need to distinguish them arises, we shall denote $X$- and $Z$-encoded majority voting $\mathcal{M}^{(X)}$ and $\mathcal{M}^{(Z)}$, respectively. (b) A subroutine acting on ancilla for processing error syndrome information extracted from the data. The circuit shows one row, $(V\mathcal{N})(k)$, of the fully contracted exRec $V\mathcal{N}(k)$ representing a collection of $k$-level protected gates acting on row $i$ of ancilla which take part in an EC($k + 1$) step. Note that in this circuit the output top lines are discarded so no EC gadget must protect them. With this, the exRec corresponding to $V\mathcal{N}$ at degree of concatenation $k$ is $\mathcal{N}(k) = EC(k) \times \prod_{i \in \text{rows}}(V\mathcal{N})(k - 1) \times EC(k)$.

In our circuits $[G(k)]$ denotes the implementation of gate $G$, in terms of level-$(k - 1)$ gates, without the prepended and appended EC($k$) routines, and $W$ denotes a waiting gate. (c) Full EC gadget for the BS code. Here, a TOFFOLI with $\diamond$ controls is a $Z$-TOFFOLI; $CX^{(k)} = \prod_{i=1}^{c\text{NOT}}(i,3,0,1,3)$ is a set of transversal CNOTs, $CX^{(k)}_{[3]} = \prod_{i=1}^{c\text{NOT}}(i,3,0,2,3)$, and $CX^{(k)}_{[3]} = \prod_{i=1}^{c\text{NOT}}(i,3,0,3)$. The control of the gates in boxes is always the top input of the gate. The last gate is a $b$TOFFOLI.
puts. The exRec with the largest number of malignant appending error correction routines on the inputs and outputs. An exRec of a gate is constructed by prepending and or column of the concatenation. Given their form, measuring encoded logical measurements.—They are required only at the highest level of execution whenever one executes the gate. (III)

Implementing the non-Clifford operation CNOT and X/C30L j þ 1 , for k large enough p(k) is vanishingly small and the terms O(p(k)) can be neglected. Then the threshold condition for X and Z measurements is p(m)thresh = 1/3.

Encoded non-Clifford operations.—The missing component to achieve universality is the fault-tolerant execution of a non-Clifford gate. Using the circuit in Fig. 2(b), we translate the problem into preparing the [H1] ancilla. To create an ancilla at the highest level we will use an encoder circuit. To encode an arbitrary state we start with the level-0 state |ψ⟩ we want to encode and 8 |0⟩ states, and then (ii) we use CNOT gates, including waiting times such that never in one step does one qubit interact with more than one qubit, to create the state [ψL]3×3 = a[ψL]3×3 + b[1]3×3. Finally, (iii) we execute a M(2) gate in every row, to create the state |ψL⟩ = a|0⟩L + b|1⟩L. We can recursively use the same algorithm to create the state at any level of concatenation k. Repeating this process recursively yields an error rate for the encoding at the highest level of concatenation k = L: p(anc) ≤ 10p(0) + 108∑j=0 p(j). Clearly, p(anc) cannot be made arbitrarily small; however, provided p(anc) ≤ pthresh, it can be made small enough to give p(anc) ≤ sin(π/8), and then one can use magic state distillation to achieve FTUQC [13].

Additionally, we promised that preparation errors can in fact be much higher than gate error rates. The argument proceeds by using a variant of the algorithmic cooling algorithm introduced in Ref. [18]. For a group of three qubits (a, b, and c) with identical probabilities p(ρ) = κ(0) < 1/2, to be in the erroneous state |1⟩, we apply TOFFOLI[(c,b,a)]CNOT(a,c)CNOT(a,b). The reduced state of qu-
bit $a$ is colder, i.e., has lower error ($e^{(1)} < e^{(0)}$). By concatenating the process, after $j$ rounds using a total of $3^j$ qubits, the final error of the one output qubit satisfies the recursion relation $e^{(j)} = (e^{(j-1)})^2(3 - 2e^{(j-1)})$. Including gate errors, the total error of this preparation process is $p_{(g)}^{(j)} \leq e^{(j)} + \frac{1}{2}(3^j - 1)p_{(g)}^{(0)}$.

We are now ready to combine our tools. If we are sensibly below threshold, say, with $p_{(g, p)} = 0.75p_{(g, p)\text{thresh}} = 2.82 \times 10^{-5}$, then with $p_{(m)} = 33\%$ we get $p_{(g)}^{(6)} \sim 10^{-13}$ and $p_{\text{anc}}^{(6)} = 8.32 \times 10^{-3}$, which is safely below the 14.6% needed for $|H_L\rangle$ distillation (and certainly below the 50% needed for the $|+\rangle$ distillation [3]). Thus FTUQC is achievable with noisy and currently achievable measurement error rates but with only a small impact to the threshold value as compared to the best known result ($1.26 \times 10^{-4}$) for the same code allowing measurements [3]. One can go further and use algorithmic cooling to also push preparation error rates within reach of current technology. We find that if one has physical preparation error rates of $p_{(p)} = 1\%$, then two rounds of AC and physical gate error rates $p_{(g)} = 2.32 \times 10^{-6}$ allow for FTUQC. Preparation rates as high as 1/3 can also be allowed, at the cost of demanding a lower gate error rate. For $p_{(p)} \geq 1/3$, one can instead use noisy measurement since measurement followed by a unitary is preparation.

To put this result in perspective, notice that $p_{(g)} = 1.39 \times 10^{-6}$ is not a threshold value but the required value such that effective preparation and gate error rates are sensibly below our threshold ($0.75 \times p_{\text{thresh}}$). In comparison, under the same assumptions the best known result [3] implies that quantum computing is possible, with reasonable overhead, when $p_{(p, g, m)} \sim 9.5 \times 10^{-5}$. So the price we pay to push measurement and error rates within reach of current technology (an improvement of 3 and 2 orders of magnitude, respectively), is demanding roughly 2 orders of magnitude more stringent gate error rates. Furthermore, note that the required measurement and preparation error rates have already been reported: In trapped ions [4] $p_{(m)} = 2.3 \times 10^{-3}$, while in quantum dots [5] $p_{(m)} = 3 \times 10^{-2}$.

We point out that the threshold value for gates computed here is by no means tight as we wanted to keep calculations simple. We have overcounted malignant pairs of locations, and certainly the design of our circuits may not be the optimal one in terms of error locations; thus, in principle, the threshold can be improved. On the other hand, restricting ourselves to two-qubit interactions only and decomposing TOFFOLI gates into one- and two-qubit gates degrade the gate and preparation threshold value to $2.69 \times 10^{-5}$. Also restricting to nearest-neighbor-only interactions will degrade the threshold value [19]. In our circuits ancillas can be prepared offline, and we have been careful to limit measurement only to when the data are encoded (at the highest level of concatenation); thus, physical systems with slow measurement or preparation are allowed.

In conclusion, we have shown that measurement-free QEC is viable, considerably relaxing the time and error rate constraints on preparation and measurement operations, and pushing them within reach of current technology, while yielding only a small penalty to the gate threshold. This small penalty seems even less relevant if one considers recent results showing that arbitrarily accurate unitary gates (and not preparation or measurement) can, in principle, be achieved by using open system control [9]. Those results complement the methods developed here and bring fault-tolerant quantum computing closer to reality.

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