Receptivity of the rotating disk boundary layer to traveling disturbances

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I. INTRODUCTION

Receptivity describes the initial stage of the laminar-turbulent transition process, in which external fluctuations (including sound and vortical waves and distributed and localized surface roughness) excite perturbations within the boundary layer. The initial conditions of the boundary layer perturbation are characterized by the receptivity mechanism. If the initial amplitude of the disturbance is small, the initial stages of transition can be described by the linearized Navier–Stokes (LNS) equations. On the other hand, for larger initial amplitudes, nonlinear mechanisms, secondary instabilities, and non-modal stability strategies are significant.

Assuming the early stages of transition are described by the linear stability theory and disturbance development is computed via the LNS equations, two- and three-dimensional perturbations emerge that depend on the flow geometry. Tollmien–Schlichting (TS) waves are found in the boundary layer that forms on a flat plate, while crossflow vortices develop on swept bodies, rotating disks, rotating cones, and rotating spheres. Linear perturbations may be decomposed as the product of a constant amplitude, $A$, and spatiotemporal function, $q = q(x, t)$, that describes the growth and evolution of the disturbance. Here, $x$ and $t$ represent the spatial coordinates and time, respectively. The spatiotemporal function, $q$, depends on the flow conditions, including the Reynolds number of the flow, spatial wavelength, and temporal periodicity. The disturbance amplitude, $A$, depends on the receptivity process and the location, shape, and form of the external fluctuations that excite the laminar-turbulent transition process.

The disturbance amplitude, $A$, can be determined by solving the LNS equations subject to boundary conditions that describe the environmental fluctuations. However, this particular approach requires the LNS equations be solved for each new set of boundary conditions. Thus, an LNS-based receptivity strategy is both computationally and time expensive.

Another receptivity strategy was developed by Goldstein and Ruban, who derived an asymptotic approach (i.e., large Reynolds number) for determining the amplitude of TS waves generated by the interaction of a free-stream acoustic wave with surface roughness. Similar asymptotic studies have modeled the interaction of acoustic waves and roughness in transonic and hypersonic flows, and...
several investigations have utilized an approach based on the Orr–Sommerfeld equation \(^{(19)}\) (i.e., finite Reynolds number).

A third approach that allows for a faster receptivity investigation and is the focus of the current study is the implementation of an adjoint-based formulation. Utilizing the earlier work of Salwen and Grosch,\(^{21}\) Hill\(^{22}\) derived the adjoint linearized Navier–Stokes (ALNS) equations in Cartesian coordinates and undertook a receptivity study of TS waves that develop in the Blasius flow over an infinitely long flat plate. In this way, Hill computed the sensitivity of TS waves to a broad range of external disturbances, including sources of momenta, mass, vorticity, and unsteady wall motion.

Following Hill’s analysis, Airiau\(^{27}\) developed the adjoint parabolized stability equations to study the receptivity of the non-parallel Blasius flow to both localized and non-localized acoustic waves. Related investigations by Dobrinsky and Collins\(^{28}\) and Dobrinsky\(^{29}\) applied the adjoint approach to determine the receptivity of the family of Falkner–Skan flows. Further adjoint-based studies have considered the receptivity of Görtler vortices,\(^{30}\) TS waves in the leading-edge region,\(^{31}\) and crossflow disturbances in swept wing boundary layers,\(^{32,33}\) as well as a strategy for optimizing the control of disturbance development.\(^{34–36}\) More recently, Thomas and Davies\(^{38}\) developed a vorticity form of the ALNS equations in cylindrical coordinates.

The boundary layer on a rotating disk develops when a disk of infinite radius rotates at a constant angular velocity below an otherwise stationary body of incompressible viscous fluid. As the disk rotates, fluid moves along the wall-normal direction toward the disk surface before being thrown radially outward. This particular flow was first described by von Kármán\(^{19}\) and displays transition mechanisms similar to that found in the boundary layer over swept-wing aircraft. The rotating disk boundary layer is susceptible to the crossflow instability, which takes the form of stationary spiral vortices.\(^{39}\) Crossflow is a convective form of instability where the disturbance propagates away from the initial source. Following the seminal investigation by Gregory et al.,\(^{39}\) the crossflow, or type-I instability, has been observed experimentally by many within the fluids community.\(^{40–43}\) In addition, Malik and coworkers\(^{44–46}\) applied the linear stability theory to determine the conditions for the onset of the stationary form of this instability.

In addition to the crossflow instability, the rotating disk boundary layer is susceptible to at least two other forms of instability: a type-II mode due to the Coriolis forces and curvature effects\(^{42}\) and a type-III spatially damped mode that propagates radially inwards.\(^{46}\) Hall\(^{19}\) determined the asymptotic structure of the stationary type-I and type-II modes, while Turkyilmazoglu and Gajjar\(^{47,48}\) extended the analysis to the traveling wave instabilities. Balakumar and Malik\(^{49}\) undertook a linear stability analysis of the traveling type-I and type-II instabilities before Lingwood\(^{50}\) showed the coalescence of the type-I and type-III modes results in the formation of absolute instability.

Balakumar et al.\(^{51}\) undertook a receptivity study of traveling disturbances in the non-parallel flow on a rotating disk. The effect of coupling high-frequency oscillations with surface roughness was modeled using parabolized stability equations. Using an ALNS vorticity formulation, Thomas and Davies\(^{52}\) computed the receptivity of stationary crossflow vortices excited by surface roughness in the rotating disk boundary layer.

The current paper aims to develop an adjoint-based strategy for measuring the receptivity of stationary and traveling disturbances in the rotating disk boundary layer to external forces, including sources of momenta, sources of mass, and unsteady wall motion. Thus, the following study extends the earlier analysis of Thomas and Davies,\(^{53}\) which was limited to the receptivity of stationary crossflow disturbances excited by surface roughness. Similar to those earlier linear stability studies undertaken by Malik\(^{54}\) and Lingwood,\(^{55}\) the parallel flow approximation is employed, which amounts to replacing the radius with the Reynolds number of the flow. A definition for the Reynolds number is provided below in (6)). Following the formulation of Hill\(^{22}\) in Cartesian coordinates, ALNS equations and a receptivity formula are derived for a cylindrical geometry.

The remainder of this paper is outlined as follows: Undisturbed flow equations are presented in Sec. II, with the LNS and ALNS formulation described in Sec. III. A receptivity formula is derived in Sec. IV for linear crossflow and Coriolis instabilities excited by external forces. The results of our investigation are discussed in Sec. V and conclusions are given in Sec. VI.

II. UNDISTURBED FLOW

Consider a disk of infinite radius that rotates in an infinite body of incompressible fluid at a constant angular velocity \(\Omega = (0, 0, \Omega_z)\). The modeling is carried out in cylindrical polar coordinates \(r = (r, \theta, z^\prime)\), where \(r\) is the radial distance from the vertical axis of rotation, \(\theta\) is the azimuthal angle, and \(z^\prime\) is the wall-normal direction. A sketch of the flow on a rotating disk is given in Fig. 1(a). (Here, an asterisk denotes dimensional quantities.) In what follows, all quantities are defined in the frame of reference that rotates with the disk.

The Navier–Stokes equations, in cylindrical coordinates, are defined as

\[
\frac{\partial U^*}{\partial t} + (U^* \cdot \nabla) U^* + 2\Omega \times U^* + \Omega^* \times (\Omega^* \times r^*) = -\frac{1}{\rho^*} \nabla^* P^* + \nu^* \nabla^2 U^* \tag{1a}
\]

and

\[\nabla^* \cdot U^* = 0. \tag{1b}\]

Here, \(t^*\) denotes dimensional time, while \(\nu^*\) and \(\rho^*\) represent the respective kinematic viscosity and density of the fluid. The dimensional velocity and pressure fields are denoted \(U^* = (U^*, V^*, W^*)\) and \(P^*, \) respectively. The third and fourth terms on the left-hand side of Eq. (1a) embody the Coriolis and centrifugal effects. Boundary conditions on the disk surface and in the far-field are defined as

\[U^* = 0 \quad \text{on} \quad z^\prime = 0, \tag{2a}\]

\[V^* = 0 \quad \text{on} \quad z^\prime = 0, \tag{2b}\]

\[W^* = 0 \quad \text{on} \quad z^\prime = 0, \tag{2c}\]

and

\[U^* \to 0 \quad \text{as} \quad z^\prime \to \infty, \tag{2d}\]

\[V^* \to -r^* \Omega^* \quad \text{as} \quad z^\prime \to \infty. \tag{2e}\]

The von Kármán\(^{19}\) similarity solution is found by setting

\[U^* = (r^* \Omega^* F(z), r^* \Omega^* G(z), \delta^* \Omega^* H(z)) \tag{3a}\]

and

\[P^* = \rho^* \nu^* \Lambda^* P(z). \tag{3b}\]
which is solved subject to the boundary conditions

\[ F = 0 \quad \text{on} \quad z = 0, \quad (4e) \]

\[ G = 0 \quad \text{on} \quad z = 0, \quad (4f) \]

and

\[ F \rightarrow 0 \quad \text{as} \quad z \rightarrow \infty, \quad (4h) \]

\[ G \rightarrow -1 \quad \text{as} \quad z \rightarrow \infty. \quad (4i) \]

Here, a prime denotes differentiation with respect to \( z \). Figure 1(b) depicts the von Kármán velocity profiles \( F, G, \) and \( H \) as functions of the wall-normal \( z \)-direction.

On non-dimensionalizing the velocity and pressure fields on the respective scales \( r^* \Omega^2 \) and \( \rho r^* \Omega^2 \), the undisturbed flow can be written as

\[
U_B(r, z) = \left( \frac{r}{Re} F(z), \frac{r}{Re} G(z), \frac{1}{Re} H(z) \right)
\]

and

\[
P_B(z) = \frac{1}{Re^2} P(z),
\]

where the Reynolds number

\[ Re = \frac{r^*}{\delta^*} \equiv r_\infty,
\]

for a non-dimensional reference radius \( r_\infty \).

### III. PERTURBATION EQUATIONS

#### A. Linearized Navier-Stokes equations

The undisturbed flow \((5)\) is perturbed by infinitesimally small disturbances, with the total velocity \( U \) and pressure \( P \) given as

\[
U = U_B + \varepsilon u, \quad (7a)
\]

\[
P = P_B + \varepsilon p, \quad (7b)
\]

where \( \varepsilon \ll 1 \) and \( u = (u, v, w) \). Substituting Eq. \((7)\) into the non-dimensional form of Eq. \((1)\) and linearizing with respect to \( \varepsilon \) gives the linearized Navier-Stokes (LNS) equations

\[
L_v(u, p) = \frac{\partial u}{\partial t} + \frac{\rho F}{Re} \frac{\partial u}{\partial r} + \frac{G}{Re} \frac{\partial u}{\partial \theta} + \frac{1}{Re} \left( \frac{\partial P}{\partial r} - \frac{u}{r^2} \right) = 0.
\]

\[
L_\theta(u, p) = \frac{\partial v}{\partial t} + \frac{\rho F}{Re} \frac{\partial v}{\partial r} + \frac{G}{Re} \frac{\partial v}{\partial \theta} + \frac{H}{Re} \frac{\partial v}{\partial z} + \frac{\rho^*}{Re} \left( \frac{\partial P}{\partial \theta} - \frac{v}{r^2} \right) = 0,
\]

\[
L_z(u, p) = \frac{\partial w}{\partial t} + \frac{\rho F}{Re} \frac{\partial w}{\partial r} + \frac{G}{Re} \frac{\partial w}{\partial \theta} + \frac{H}{Re} \frac{\partial w}{\partial z} + \frac{1}{Re} \left( \frac{\partial P}{\partial z} + \frac{w}{r^2} \right) = 0.
\]

and

\[
\nabla \cdot u = \frac{1}{r} \frac{\partial (ru)}{\partial r} + \frac{1}{r} \frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial z} = 0,
\]

where

\[ H = 0 \quad \text{on} \quad z = 0, \quad (4g) \]

\[ F \rightarrow 0 \quad \text{as} \quad z \rightarrow \infty, \quad (4h) \]

\[ G \rightarrow -1 \quad \text{as} \quad z \rightarrow \infty. \quad (4i) \]
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\[ \nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2} \]  

(9)

and the LNS operator \( L = (L_r, L_\theta, L_z) \). Boundary conditions on the disk surface and in the far-field are given as

\[ u = 0 \text{ on } z = 0 \]  

\[ u \rightarrow 0 \text{ as } z \rightarrow \infty, \]  

(10a)

and

\[ p \rightarrow 0 \text{ as } z \rightarrow \infty. \]  

(10c)

1. Simplification via the parallel flow approximation

The parallel flow approximation is implemented, whereby the radius, \( r \), is replaced with the Reynolds number, \( Re \). Additionally, as \( Re \gg 1 \), terms of order \( Re^{-2} \) and smaller are neglected, as per Malik and Balakumar and Malik.\(^{53} \) Replacing \( r \) by \( Re \) in Eq. (8) establishes a set of equations that are separable with respect to \( r, \theta, \) and \( t \), which allow perturbations to be decomposed in the normal mode form as

\[ \{u, \psi\}(r, \theta, z, t) = \{\tilde{u}(z), \tilde{\psi}(z)\} \exp(i(\kappa r + n \theta - \omega t)), \]  

(11)

where \( z \in \mathbb{C} \) denotes the radial wavenumber, \( \omega \in \mathbb{R} \) represents the frequency, and \( n = \beta Re \) denotes the integer-valued azimuthal mode number for the azimuthal wavenumber \( \beta \in \mathbb{R} \). The azimuthal mode number, \( n \), indicates the periodicity of the disturbance in the azimuthal direction. For instance, in relation to the type-I crossflow instability, \( n \) represents the number of crossflow vortices observed in experiments.\(^{19} \) (For a spatial linear stability analysis, \( \kappa_i < 0 \) corresponds to unstable behavior.)

Given the above assumptions, Eq. (8) becomes

\[ i(ax + \beta G - \omega)\tilde{u} + F'\tilde{w} + i\beta \tilde{p} \]

\[ = \frac{1}{Re} \left( \tilde{w}'' - \lambda^2 \tilde{u} - F\tilde{u} - H\tilde{u} + 2(G + 1)\tilde{v} \right), \]  

(12a)

and

\[ i(ax + \beta G - \omega)\tilde{w} + G'\tilde{w} + i\beta \tilde{p} \]

\[ = \frac{1}{Re} \left( \tilde{v}'' - \lambda^2 \tilde{w} - F\tilde{v} - H\tilde{v} + 2(G + 1)\tilde{u} \right), \]  

(12b)

\[ i(ax + \beta G - \omega)\tilde{u} + \tilde{p}' = \frac{1}{Re} \left( \tilde{w}' - \lambda^2 \tilde{u} \right), \]  

(12c)

\[ i\tilde{v} + i\beta \tilde{v} + \tilde{w}' = 0, \]  

(12d)

where \( \lambda^2 = ax^2 + \beta^2 \) and \( \bar{u} = z - i/Re \). Eliminating \( \tilde{p} \) and neglecting terms of order \( Re^{-2} \) and smaller gives

\[ \left[ i(D^2 - \lambda^2)(D^2 - \lambda^2) + Re(ax + \beta G - \omega)(D^2 - \lambda^2) \right. \]

\[ - Re(ax' + \beta G') - iHD(D^2 - \lambda^2) - iH' (D^2 - \lambda^2) \]

\[ - iF D^2 \tilde{w} + [2(G + 1)D + 2G']\tilde{w} = 0 \]  

(13a)

and

\[ [2(G + 1)D - iRe(ax' + \beta G')]\tilde{w} \]

\[ + \left[ i(D^2 - \lambda^2) + Re(ax + \beta G - \omega) - iHD - iH' \right] \tilde{w} = 0, \]  

(13b)

where \( D = d/dz, \lambda^2 = ax^2 + \beta^2, \) and \( \tilde{w} = \tilde{v} - \tilde{u} \). Equations (13) are valid to order \( Re^{-1} \) and are solved subject to the boundary conditions

\[ \tilde{w} = 0 \text{ on } z = 0, \]  

(14a)

\[ \tilde{w}' = 0 \text{ on } z = 0, \]  

(14b)

\[ \tilde{\eta} = 0 \text{ on } z = 0, \]  

(14c)

\[ \tilde{w} \rightarrow 0 \text{ as } z \rightarrow \infty, \]  

(14d)

\[ \tilde{w}' \rightarrow 0 \text{ as } z \rightarrow \infty, \]  

(14e)

\[ \tilde{\eta} \rightarrow 0 \text{ as } z \rightarrow \infty. \]  

(14f)

B. Lagrange identity

The adjoint linearized Navier–Stokes (ALNS) equations for the adjoint velocity \( \tilde{u} = (\tilde{u}, \tilde{v}, \tilde{w}) \) and pressure \( \tilde{p} \) perturbation fields, are derived by first formulating the Lagrange identity. This is achieved by taking the inner product of the adjoint fields with the LNS Eqs. (8) and integrating by parts to give

\[ L_u(\tilde{u}, \tilde{p}) \cdot \tilde{u} + \nabla \cdot \tilde{u} \tilde{p} + u \cdot L(\tilde{u}, \tilde{p}) + p \nabla \cdot \tilde{u} \]

\[ = \frac{\partial}{\partial t} \left( u \cdot \tilde{u} \right) + \nabla \cdot \left( f(\tilde{u}, \tilde{p}, \tilde{u}) \right), \]

(15)

where

\[ u \cdot \tilde{u} = u\tilde{u} + v\tilde{v} + w\tilde{w}. \]

C. Adjoint linearized Navier–Stokes equations

The ALNS system of equations are given by those expressions above the second underline in Eq. (15) that form an inner product with the linear perturbation field \( \{u, \psi\} \)

\[ \tilde{L}_u(\tilde{u}, \tilde{p}) = \frac{\partial}{\partial t} \tilde{u} - \frac{F'}{Re} \tilde{u} - \frac{G}{Re} \tilde{u} + \tilde{H} \tilde{u} + \frac{H}{Re} \tilde{u} + \frac{2}{Re} \frac{\partial \tilde{u}}{\partial \tilde{u}} = 0, \]

(16a)

\[ \tilde{L}_\psi(\tilde{u}, \tilde{p}) = \frac{\partial}{\partial t} \tilde{\psi} - \frac{\partial}{\partial t} \tilde{\psi} + \frac{F}{Re} \tilde{\psi} + \frac{G}{Re} \tilde{\psi} + \frac{H}{Re} \tilde{\psi} + \frac{2}{Re} \frac{\partial \tilde{\psi}}{\partial \tilde{\psi}} = 0, \]

(16b)

\[ \tilde{L}_\tilde{u}(\tilde{u}, \tilde{p}) = \frac{\partial}{\partial t} \tilde{u} - \frac{F'}{Re} \tilde{u} - \frac{G}{Re} \tilde{u} + \tilde{H} \tilde{u} + \frac{H}{Re} \tilde{u} + \frac{1}{Re} \frac{\partial \tilde{u}}{\partial t} + \frac{1}{Re} \left( \nabla^2 \tilde{v} - \frac{\tilde{v}^2}{\tilde{v}} + 2 \frac{\partial \tilde{u}}{\partial \tilde{u}} \right) = 0, \]

(16c)

\[ \tilde{L}_\tilde{\psi}(\tilde{u}, \tilde{p}) = \frac{\partial}{\partial t} \tilde{\psi} - \frac{\partial}{\partial t} \tilde{\psi} + \frac{F}{Re} \tilde{\psi} + \frac{G}{Re} \tilde{\psi} + \frac{H}{Re} \tilde{\psi} + \frac{\partial \tilde{\psi}}{\partial t} + \frac{1}{Re} \frac{\partial \tilde{\psi}}{\partial \tilde{\psi}} = 0, \]

(16d)

where the ALNS operator \( L = (L_r, L_\theta, L_z) \). The above ALNS system of equations are solved subject to the same boundary conditions imposed on the LNS Eqs. (8), namely,

\[ \tilde{u} = 0 \text{ on } z = 0, \]  

(17a)

\[ \tilde{u} \rightarrow 0 \text{ as } z \rightarrow \infty. \]  

(17b)
and
\[ \tilde{p} \to 0 \quad \text{as} \quad z \to \infty. \] (17c)

1. Simplification via the parallel flow approximation

Similar to the LNS scheme the parallel flow approximation is implemented, with adjoint perturbations decomposed as
\[ \{ \tilde{u}, \tilde{p} \} = \{ \tilde{u}(z), \tilde{p}(z) \} \exp(-i(\bar{x}r + \bar{n} \theta - \omega t)), \] (18)
where \( \bar{x} \in \mathbb{C} \) is the adjoint radial wavenumber, \( \bar{x} \in \mathbb{R} \) is the adjoint frequency, and \( \bar{n} = \tilde{\beta}/\Re \) is the adjoint azimuthal mode number for the adjoint azimuthal wavenumber \( \tilde{\beta} \in \mathbb{R} \). For the subsequent receptivity study of spatially evolving disturbances, \( \omega, \bar{n}, \tilde{\beta} \) are set equal to those values modeled in the LNS scheme (i.e., \( \omega = \omega, \bar{n} = \bar{n}, \tilde{\beta} = \tilde{\beta} \)), while \( \bar{x} \neq \bar{x} \). Additionally, a sign change arises for these three parameters as adjoint disturbances evolve backwards in time and in the spatially opposite sense to the LNS perturbations given by Eq. (11).

Given the parallel flow approximation and on ignoring all terms of order \( \Re^{-2} \) and smaller, Eq. (16) becomes
\[ i(\tilde{\bar{x}}F + \beta G - \omega)\tilde{u} + i\bar{x}\tilde{p} = \frac{1}{\Re} \left( \tilde{u}'' - \tilde{\lambda}^2 \tilde{u} - \tilde{F}\tilde{u} + H\tilde{t}'' - 2(\tilde{G} + 1)\tilde{v} \right), \] (19a)
\[ i(\tilde{\bar{x}}F + \beta G - \omega)\tilde{v} + i\bar{x}\tilde{p} = \frac{1}{\Re} \left( \tilde{v}'' - \tilde{\lambda}^2 \tilde{v} - \tilde{F}\tilde{v} + H\tilde{t}'' + 2(\tilde{G} + 1)\tilde{u} \right), \] (19b)
\[ \tilde{\bar{z}}\tilde{u} + \bar{n}\tilde{v} + \tilde{\lambda}\tilde{w} = \frac{1}{\Re} \left( \tilde{\lambda}^2 \tilde{u} + \bar{n}\tilde{v} + \tilde{\lambda}\tilde{w} \right), \] (19c)
where \( \tilde{\lambda} = \tilde{\lambda} + \beta F \) and \( \tilde{x} = \tilde{x} + i/\Re \). Eliminating \( \tilde{p} \) and neglecting terms of order \( \Re^{-2} \) and smaller gives
\[ \left[ i(D^2 - \tilde{\lambda}^2)(D^2 - \tilde{\lambda}^2) + \Re(\tilde{F}F + \beta G - \omega)(D^2 - \tilde{\lambda}^2) \right] \tilde{u} \]
\[ + 2\Re(\tilde{F}F + \beta G)D + 2\Re H\tilde{t}'' + i\Re F(D^2 - \tilde{\lambda}^2) \]
\[ - 2\Re H^2(D^2 - \tilde{\lambda}^2) - i\Re F^2 - i\Re^2 D \tilde{w} \]
\[ + \left[ 2(\tilde{G} + 1)D + 2\Re H + i\Re(\tilde{G}^2 - \beta F^2) \right] \tilde{w} \]
\[ \tilde{\bar{z}}\tilde{u} + i\bar{n}\tilde{v} + \tilde{\lambda}\tilde{w} = 0, \] (20a)
and
\[ \left[ 2(\tilde{G} + 1)D \tilde{w} + i(D^2 - \tilde{\lambda}^2) + \Re(\tilde{F}F + \beta G - \omega) + i\Re H - i\Re \right] \tilde{\bar{z}} = 0, \] (20b)
where \( \tilde{\lambda} = \tilde{\lambda} + \beta F \) and \( \tilde{\bar{z}} = \tilde{\bar{z}} - \tilde{\beta} \tilde{u} \). Similar to the LNS Eqs. (13), the final adjoint Eqs. (20) are valid to order \( \Re^{-1} \) and are solved subject to the boundary conditions
\[ \tilde{\bar{z}}' = 0 \quad \text{on} \quad z = 0, \] (21a)
\[ \tilde{\bar{z}}' = 0 \quad \text{on} \quad z = 0, \] (21b)
and
\[ \tilde{\bar{z}}' = 0 \quad \text{on} \quad z = \infty. \] (21c)

D. Bi-linear concomitant

The operator \( J = (J_1, J_0, J_1) \) on the right-hand side of the Lagrange identity (15) represents the bi-linear concomitant and is defined as
\[ J_1(u, p, \tilde{u}, \tilde{p}) = \frac{\Re F}{\Re} u \cdot \tilde{u} + \tilde{u} \cdot \tilde{p} + \frac{1}{\Re} \left( \tilde{u} \frac{\partial u}{\partial \tilde{u}} + \frac{\partial u}{\partial \tilde{u}} + \frac{\partial v}{\partial \tilde{v}} + \frac{\partial \tilde{v}}{\partial \tilde{v}} + \frac{\partial \tilde{w}}{\partial \tilde{w}} - \frac{\partial \tilde{w}}{\partial \tilde{w}} \right), \] (22a)
\[ J_0(u, p, \tilde{u}, \tilde{p}) = \frac{\Re G}{\Re} u \cdot \tilde{u} + \tilde{v} \cdot \tilde{v} + \frac{1}{\Re} \left( \tilde{u} \frac{\partial u}{\partial \tilde{u}} + \frac{\partial u}{\partial \tilde{u}} + \frac{\partial v}{\partial \tilde{v}} + \frac{\partial \tilde{v}}{\partial \tilde{v}} + \frac{\partial \tilde{w}}{\partial \tilde{w}} - \frac{\partial \tilde{w}}{\partial \tilde{w}} \right) \]
\[ J_2(u, p, \tilde{u}, \tilde{p}) = \frac{\Re H}{\Re} u \cdot \tilde{u} + \tilde{w} \cdot \tilde{w} + \frac{1}{\Re} \left( \tilde{u} \frac{\partial u}{\partial \tilde{u}} + \frac{\partial u}{\partial \tilde{u}} + \frac{\partial v}{\partial \tilde{v}} + \frac{\partial \tilde{v}}{\partial \tilde{v}} + \frac{\partial \tilde{w}}{\partial \tilde{w}} - \frac{\partial \tilde{w}}{\partial \tilde{w}} \right). \] (22b)

In Sec. IV, the \( J \)-operator is utilized to establish a receptivity formula for predicting the amplitude of linear disturbances to external forces, i.e., sources of momenta and sources of mass and unsteady wall motion.

E. Normalization

Assuming linear perturbation fields (11) and adjoint perturbation fields (18) are solutions of the respective LNS Eqs. (8) and ALNS Eqs. (16), the left-hand side of the Lagrange identity (15) is zero. Consequently, it can be shown that
\[ i(\bar{x} - \bar{x})(K(u, \tilde{u}, \tilde{p}, \tilde{p})) = 0, \] (23a)
where
\[ K(u, \tilde{u}, \tilde{p}, \tilde{p}) = \int_{0}^{\infty} \left( \tilde{F}\tilde{u} \cdot \tilde{u} + \tilde{p}\tilde{u} + \tilde{u}\tilde{p} - \frac{1}{\Re} (\bar{x} - \bar{\lambda}) \tilde{u} \cdot \tilde{u} \right) dz, \] (23b)
with boundary conditions (10) and (17) applied to the disk wall and in the far-field. Similar to the derivation of Eqs. (12) and (19), the parallel flow approximation has been implemented (that is, \( r = \Re \)) and terms of order \( \Re^{-2} \) and smaller are neglected. Moreover, \( \bar{x} \) and \( \bar{x} \) coincide.

Linear perturbation fields \( \{ u, p \} \) are then normalized by setting
\[ \max \frac{\bar{v}}{\bar{v}} = 1, \] (24a)
while the equivalent adjoint perturbation fields \( \{ \tilde{u}, \tilde{p} \} \) are normalized by fixing
\[
K(\tilde{u}, \tilde{p}, \tilde{u}, \tilde{p}) = 1.
\]
A similar strategy was implemented by Hill\(^{35}\) for modeling the receptivity of the parallel Blasius flow over an infinitely long flat plate.

**IV. RESPONSE TO AN EXTERNAL FORCE**

In this section, a receptivity formula is derived to predict the amplitude, \( A \), of the linear perturbation (11) generated by an external force. Sources of momenta and sources of mass, in addition to unsteady wall motion, are modeled.

The LNS Eqs. (8) are redefined as
\[
L(u, p) = \phi(r, \theta, z, t)
\]
and
\[
\nabla \cdot u = \psi(r, \theta, z, t),
\]
where \( \phi = (\phi_1, \phi_2, \phi_3) \) and \( \psi \) represent sources of momenta and mass, respectively. In addition, the boundary condition (10) at the wall is reformulated as
\[
u = u_w(r, \theta, t) \quad \text{on} \quad z = 0,
\]
for \( u_w = (u_w, v_w, w_w) \), which can be used to model wall roughness, mass transfer through the disk surface, and unsteady wall motion. All external sources are assumed to be localized and zero outside the radial interval \( r_1 \leq r \leq r_2 \).

Similar to the normal mode structure of the linear perturbations (11), sources of momenta and mass are decomposed as
\[
\{ \phi, \psi \}(r, \theta, z, t) = \{ \hat{\phi}(r, z), \hat{\psi}(r, z) \} \exp(i(n\theta - \omega t)),
\]
while the wall condition
\[
u_n(r, \theta, t) = \hat{u}_w(r) \exp(i(n\theta - \omega t)),
\]
where \( n \) and \( \omega \) denote the azimuthal mode number and frequency of the linear disturbance.

Given the above modifications, and on assuming that Eqs. (11) and (18) are solutions of the respective Eqs. (8) and (16), the Lagrange identity (15) reduces to the form
\[
\hat{\phi} \cdot \tilde{u} e^{-i\omega t} + \hat{\psi} \tilde{p} e^{-i\omega t} = \nabla \cdot \mathbf{J}.
\]

Integrating with respect to the wall-normal \( z \)-direction for \( z \in [0, \infty) \) gives
\[
\frac{\partial}{\partial r} \left( \int_0^\infty J_z \, dz \right) = \int_0^\infty \hat{\phi} \cdot \tilde{u} c e^{-i\omega t} \, dz + \int_0^\infty \hat{\psi} \tilde{p} e^{-i\omega t} \, dz + J_z |_{z=0} e^{-i\omega t}.
\]

Given the definition for \( J_z \) in Eq. (22c), and that \( \tilde{u} = 0 \) and \( u = u_w \) on \( z = 0 \), then
\[
J_z |_{z=0} = u_w \cdot \sigma,
\]
where
\[
\sigma = \left( \frac{1}{Re} \frac{d}{dz} + \frac{1}{Re} \nabla \cdot \tilde{p} \right) \bigg|_{z=0}.
\]
As per Hill\(^{35}\), the vector \( \sigma \) is defined as the adjoint stress.

Integrating the left-hand side of Eq. (29) from \( r = r_1 \) to \( r = r_2 \) that are located far upstream and downstream of any external sources yields the following:
\[
\left( \int_0^\infty J_z \, dz \right) |_{r_1}^{r_2} = (A_2 - A_1) K(\tilde{u}, \tilde{p}, \tilde{u}, \tilde{p}),
\]
where \( A_1 \) and \( A_2 \) denote the amplitude of the linear perturbation \( \{ u, p \} \) about the radial locations \( r_1 \) and \( r_2 \), respectively. On normalizing linear and adjoint perturbation fields via Eq. (24), and setting \( A = A_2 - A_1 \), it follows that
\[
A = \int_{r_1}^{r_2} \int_0^\infty \phi\tilde{u} e^{-i\omega t} \, dz \, dr + \int_{r_1}^{r_2} \int_0^\infty \psi\tilde{p} e^{-i\omega t} \, dz \, dr + \int_{r_1}^{r_2} \int_0^\infty J_z |_{z=0} e^{-i\omega t} \, dr.
\]

The right-hand side of Eq. (32) represents the change in amplitude of the linear disturbance between \( r = r_1 \) and \( r = r_2 \). If there are no external forces (i.e., sources of momenta, sources of mass, or wall motion), then the change in the disturbance amplitude, \( A \), is zero.

**V. RESULTS**

**A. Numerical scheme and validation**

The linear and adjoint perturbation fields required for the amplitude formula (32) may be computed by either solving Eqs. (12) and (19) or via the reduced Eqs. (13) and (20). (Differences between the two approaches, due to the step terms of \( O(Re^{-2}) \) and less are negligible.) In the subsequent study, the former set of Eqs. (12) and (19) were solved using the Chebyshev collocation method developed by Trefethen\(^{35}\). Derivatives along the wall-normal \( z \)-direction were replaced by Chebyshev matrix approximations, with \( N = 100 \) Chebyshev mesh points mapped from the semi-infinite physical domain \( z \in [0, \infty) \) onto a finite computational interval \( \xi \in [-1, 1] \) via the coordinate transformation
\[
\xi = \frac{L - z}{L + z},
\]
where \( L = 2 \) is a stretching parameter.

![FIG. 2. Neutral stability curves in the (Re, n)-plane, for five frequencies \( \omega = Re \).](image-url)
The numerical procedure was validated by undertaking a spatial linear stability analysis. Neutral conditions for linear instability (that is, \(Re, \alpha, n = \beta Re, \omega = \omega Re\)) were computed (i.e., \(z_1 = 0\)) at azimuthal wavenumber step intervals \(\Delta \beta = 10^{-4}\). Figure 2 displays neutral stability curves in the \((Re, n)\)-plane for five frequencies \(\omega = \omega Re\), matched to those modeled in the earlier investigation by Balakumar and Malik (see Figs. 2–5 of their study), with the upper and lower branches of the neutral stability curves matched to the type-I crossflow and type-II Coriolis instabilities, respectively. Table I presents the corresponding critical conditions for the onset of linear instability, with neutral conditions for the crossflow instability given in bold and the equivalent conditions for the Coriolis instability given in italics.

B. Traveling wave instability: \(\alpha = 7.9\)

1. Example of crossflow instability: \(n = 30\)

Figure 3 depicts the real and imaginary parts of the radial wavenumbers, \(\bar{\alpha}\) and \(\bar{\beta}\), as functions of the Reynolds number, \(Re\), for parameter settings \(\alpha = 7.9\) and \(n = 30\): (a) real part and (b) imaginary part.

The absolute value of the normalized linear and adjoint perturbation fields, \(\{u, p\}\) and \(\{\bar{u}, \bar{p}\}\), established for the parameter settings \((Re, \alpha, n) = (400, 7.9, 30)\), are plotted in Fig. 4. The hat notation has been dropped for simplicity in the following discussion. The corresponding linear and adjoint radial wavenumbers for this particular parameter set are \(\bar{\alpha} = \bar{\beta} = 0.565 - 0.007\), i.e., the traveling type-I crossflow instability is linearly unstable. Blue solid lines depict the radial components of the linear and adjoint velocity perturbation fields, and red dashed lines and yellow chain lines illustrate the equivalent azimuthal and wall-normal velocity fields, respectively. Purple dotted lines represent the corresponding pressure perturbation fields. The radial \(u\)-velocity field attains a maximum amplitude at \(z \approx 0.5\), while the azimuthal \(v\) and wall-normal \(w\) velocity fields attain maximum values near \(z \approx 1.6\) and \(z \approx 2.3\), respectively. The maximum absolute value of the pressure, \(p\), is located about the wall-normal height \(z \approx 1.7\).

The four adjoint perturbation fields achieve a maximum amplitude about \(z \approx 1.9\), which is near the wall-normal \(z\)-location of the critical layer, i.e., \(z_{crit} = 1.8\). This particular receptivity characteristic is consistent with the earlier observations of Hill (1970). TS waves in the
Blasius boundary layer attain a maximum amplitude for external sources centered near the critical layer. About this wall-normal z-location, the radial component of velocity, \( \tilde{u} \), achieves a significantly larger maximum than the other two adjoint velocity fields (that is, \( \tilde{v} \) and \( \tilde{w} \)) and the adjoint pressure, \( \tilde{p} \). For \( z > 4 \), both linear and adjoint disturbances decay toward zero.

Given the receptivity formula (32) for the disturbance amplitude, \( A \), the relative sizes of the adjoint velocity, \( \tilde{u} \), and pressure, \( \tilde{p} \), characterize the strength of those linear disturbances generated by unsteady localized sources of momenta, \( \phi \), and sources of mass, \( \psi \), respectively. A unit source of momenta, \( \phi \), acting along the radial \( r \)-direction, applied at a height \( z \approx 1.9 \) above the rotating disk, will excite a traveling crossflow instability with an amplitude \( A \approx 90 \). If the unit source of momenta is applied nearer to the disk surface or further along the wall-normal \( z \)-direction, the size of the disturbance is considerably reduced. Significantly smaller crossflow disturbances are established by momenta sources, \( \phi \), applied along the azimuthal \( \theta \)- and wall-normal \( z \)-directions. The crossflow instability achieves maximum amplitudes \( A \approx 15 \) and \( A \approx 30 \) when azimuthal and wall-normal point sources of momenta are centered near \( z \approx 1.9 \). Moreover, the magnitude of the adjoint pressure, \( \tilde{p} \), is noticeably smaller than all three velocity fields.

**TABLE I.** Critical conditions for linear instability. Type-I crossflow instability in bold, and type-II Coriolis instability in italics.

<table>
<thead>
<tr>
<th>( \hat{\omega} = \omega Re )</th>
<th>( Re_c )</th>
<th>( \hat{\beta} )</th>
<th>( \hat{n} = \hat{\beta} \hat{\omega} Re_c )</th>
<th>( \hat{z} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(-5)</td>
<td>284.98 ((-))</td>
<td>0.1074 ((-))</td>
<td>30.61 ((-))</td>
<td>0.3588 ((-))</td>
</tr>
<tr>
<td>0</td>
<td>286.06 (463.32)</td>
<td>0.0775 (0.0462)</td>
<td>22.17 (21.41)</td>
<td>0.3852 (0.1317)</td>
</tr>
<tr>
<td>4</td>
<td>297.62 (175.69)</td>
<td>0.0564 (0.0128)</td>
<td>16.79 (2.25)</td>
<td>0.4078 (0.1602)</td>
</tr>
<tr>
<td>7.9</td>
<td>316.59 (64.46)</td>
<td>0.0393 (0.1066)</td>
<td>12.44 (6.87)</td>
<td>0.4276 (0.4012)</td>
</tr>
<tr>
<td>10</td>
<td>329.36 (68.72)</td>
<td>0.0316 (0.1288)</td>
<td>10.41 (8.85)</td>
<td>0.4374 (0.4800)</td>
</tr>
</tbody>
</table>

**FIG. 5.** Radial wavenumbers, \( \hat{\alpha} \) and \( \hat{\beta} \), as functions of the Reynolds number, \( Re \), for parameter settings \( \hat{\omega} = 7.9 \) and \( \hat{n} = -5 \): (a) real part and (b) imaginary part.

**FIG. 6.** Absolute value of the velocity and pressure perturbation fields for parameter settings \( Re = 100 \), \( \hat{\omega} = 7.9 \), and \( \hat{n} = -5 \), with \( \hat{u} = \hat{\alpha} = 0.267 - 0.031 \): (a) Linear perturbations \( \{ u, p \} \) and (b) adjoint perturbations \( \{ \tilde{u}, \tilde{p} \} \). The vertical dotted line in (b) corresponds to the wall-normal z-location that the effective shear stress vanishes, i.e., \( \hat{z}_{sh} \approx 1.3 \), where \( \hat{\alpha r} F + \hat{\beta G} = 0 \).
Consequently, a localized unit source of mass, $\psi$, fixed about the same height, $z \approx 1.9$, will only establish a traveling crossflow instability of amplitude $A \approx 5$. A secondary, smaller peak in the adjoint pressure, $\tilde{p}$, is located on the disk wall (with $A \approx 2$) and characterizes the size of the linear disturbance established by unsteady wall motion, $u_w$ [recall Eqs. (26) and (30)].

**2. Example of Coriolis instability: $n = -5$**

Figures 5 and 6 depict a second set of radial wavenumbers, $\tilde{\alpha}$ and $\tilde{\beta}$, and linear and adjoint perturbation fields, $(u,p)$ and $(\tilde{u}, \tilde{p})$, for the frequency $\omega = 7.9$ and the azimuthal mode number $n = -5$. These particular flow conditions correspond to a type-II Coriolis instability. As before, $\tilde{\alpha} = \tilde{\beta}$, for the range of Reynolds numbers, $Re$, modeled in Fig. 5, as required for Eq. (23).

Perturbation fields plotted in Fig. 6 correspond to the unstable Reynolds number $Re = 100$ and azimuthal mode number $n = -5$, with line types the same as those presented in Fig. 4. Similar to those results displayed above for $n = 30$, the radial component of the adjoint velocity, $\tilde{u}$, achieves larger amplitudes than the other three adjoint perturbation fields. Consequently, a radial source of momenta will again establish a larger linear disturbance than that generated by other sources of momenta and mass. However, the maximum amplitude is significantly smaller than that obtained at the larger azimuthal mode number. In addition, the local maximum is now located about the smaller wall-normal height $z \approx 1.1$, which is near the location the effective shear stress vanishes, i.e., $\tilde{\alpha}F' + \beta G' = 0$. A secondary, smaller peak in $|\tilde{u}|$ is found about $z \approx 3.2$. Thus, for this parameter set, a unit source of radial momenta fixed about $z \approx 1.1$ establishes a Coriolis instability of amplitude $A \approx 7$. The relative sizes of those linear disturbances excited by other sources of momenta (i.e., azimuthal and wall-normal) and sources of mass are similarly diminished, with amplitudes $A < 3$ achieved about $z \approx 1.1$, $z \approx 2.8$, and on the disk surface $z = 0$, for a source of azimuthal momenta, wall-normal momenta, and mass, respectively.

**FIG. 7.** Maximum absolute value of the adjoint perturbation fields obtained for $\omega = 7.9$ and $n \in [-5, 60]$. (a) $\max_j |\tilde{u}_j|$, (b) $\max_j |\tilde{v}_j|$, (c) $\max_j |\tilde{w}_j|$, and (d) $\max_j |\tilde{p}_j|$. Black circular markers indicate the critical Reynolds number, $Re_c$, for each azimuthal mode number, $n$. 

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3. Azimuthal mode numbers $n \in [-5,60]$

The receptivity analysis for the frequency $\omega = 7.9$ is extended to Reynolds numbers $Re \in [50,600]$ and azimuthal mode numbers $n \in [-5,60]$ at the respective step intervals $\Delta Re = 1$ and $\Delta n = 5$. The range of $Re$ and $n$ includes both type-I crossflow and type-II Coriolis instabilities for conditions matched to linearly unstable and marginally stable behavior. Figure 7 depicts the maximum absolute value of the three adjoint velocity perturbation fields, $\max_j |\tilde{u}|$, and the adjoint pressure, $\max_j |\tilde{p}|$, as functions of the Reynolds number, $Re$. Black circular markers indicate the critical Reynolds number, $Re_c$, for linearly unstable behavior associated with each azimuthal mode number, $n$. (Results matched to the azimuthal mode number $n = 10$ are omitted from Fig. 7 as both type-I and type-II modes are of similar size, leading to irregular behavior. Further discussion on this particular receptivity feature is given below.) Like those results presented above in Figs. 4 and 6, the radial component of the adjoint velocity, $\tilde{u}$, achieves considerably larger amplitudes than the other three adjoint perturbation fields. The implication is that a unit source of momenta, $\rho \omega$, will be amplified by radial and azimuthal sources of momenta, $\phi$, and sources of mass, $\psi$, prescribed for Reynolds numbers, $Re$, near the onset of linear instability. (In the case of the wall-normal $w$-velocity, $\max_j |\tilde{w}|$ continuously decreases with the Reynolds number, $Re$, for the range of $Re$ and $n$ modeled. Indeed, $\max_j |\tilde{w}|$ always attains a peak value for linearly stable Reynolds numbers, suggesting that receptivity is enhanced by wall-normal sources of momenta at $Re < Re_c$.)

Solid lines in Fig. 8 display the wall-normal location, $z_{max}$, that the adjoint radial $\tilde{u}$-velocity field achieves those maximum absolute values, $\max_j |\tilde{u}|$, plotted in Fig. 7(a). The corresponding $z$-locations for the critical layer, $z_{cl}$ (i.e., $\tilde{x}_F + \beta G = \omega$), and for a vanishing effective shear stress, $z_{sh}$, are represented by the chain and dotted lines, respectively. For azimuthal mode numbers $n \in [20,60]$ matched to the type-I crossflow instability, the maximum occurs within the interval $1.8 \leq z_{max} \leq 2$, which is in the vicinity of the critical layer, $z_{cl}$. On the other hand, for the Coriolis instabilities $n \in [-5, 5]$, $0.8 \leq z_{max} \leq 1$, which is near the wall-normal $z$-location for a vanishing effective shear stress, $z_{sh}$. Thus, crossflow disturbances are enhanced by radial sources of momenta fixed near the wall-normal height $z \approx 2$, and Coriolis instabilities are amplified by momenta sources located near $z \approx 1$.

![Figure 8](image1.png)

**FIG. 8.** The wall-normal $z$-location, $z_{max}$ (solid lines), that the adjoint $\tilde{u}$-velocity field achieves those maximum amplitudes, $\max_j |\tilde{u}|$, plotted in Fig. 7. The frequency $\omega = 7.9$ and azimuthal mode number (a) $n = -5$, (b) $n = 10$, and (c) $n = 60$, $n = 40$, and $n = 20$. Chain and dotted lines represent the respective $z$-locations, $z_{cl}$ and $z_{sh}$, for the critical layer (i.e., $\tilde{x}_F + \beta G = \omega$) and for a vanishing effective shear stress (i.e., $\tilde{x}_F + \beta G = 0$).

![Figure 9](image2.png)

**FIG. 9.** Absolute value of the adjoint velocity, $\tilde{u}$, and pressure, $\tilde{p}$, perturbation fields for $\omega = 7.9$ and $n = 10$, and (a) $Re = 450$ and $\bar{a} = 0.190 - 0.015$, (b) $Re = 475$ and $\bar{a} = 0.175 - 0.020$, and (c) $Re = 500$ and $\bar{a} = 0.162 - 0.022$. Arrows indicate the location of $z_{max}$ for $|\tilde{u}|$. 

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**Physics of Fluids**

**ARTICLE**

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For the azimuthal mode number $n = 10$, $z_{\text{max}}$ shifts between those $z$-locations that are significant to the respective crossflow and Coriolis instabilities, with $z_{\text{max}} \in [1.8, 2]$ for Reynolds numbers $350 \leq Re \leq 475$, and $z_{\text{max}} \in [0.8, 0.9]$ for larger $Re$. This behavior is likely due to both type-I and type-II modes being of commensurate significance for this azimuthal mode number. Indeed, for these parameter settings, the amplitude of the adjoint radial velocity field, $|\tilde{u}|$, is characterized by two peaks of comparable magnitude, as demonstrated in Fig. 9 for Reynolds numbers $Re = 450$, $Re = 475$, and $Re = 500$. Arrows indicate the wall-normal location, $z_{\text{max}}$, that $|\tilde{u}|$ attains a maximum. As the Reynolds number, $Re$, increases, $z_{\text{max}}$ moves from the peak at $z/\gamma_{1} = 1.7$ (near the critical layer, $z_{\text{cri}}$) to the peak at $z/\gamma_{1} = 0.9$ (near the location, $z_{\text{sh}}$, that the effective shear stress vanishes). In the instance $Re = 475$, the amplitude of the two peaks is almost identical.

Comparable solutions for $z_{\text{max}}$ to that presented in Fig. 8, are obtained for the azimuthal component of the adjoint velocity field, $\tilde{v}$. The exception is that for the azimuthal mode number $n = 10$, $z_{\text{max}} < 1$ for the full range of Reynolds numbers investigated. This particular result is a consequence of the azimuthal $\tilde{v}$-velocity field having only one maximum as opposed to the two peaks found in the corresponding $\tilde{u}$-velocity field (see Fig. 9). Conversely, $z_{\text{max}} \approx 1.9$ for all results associated with the adjoint wall-normal $\tilde{w}$-velocity, while the adjoint pressure, $\tilde{p}$, features two maxima located near the height $z \approx 1.9$ and on the disk surface $z = 0$.

The absolute value of the adjoint pressure, $|\tilde{p}|$, on the disk wall $z = 0$, is plotted in Fig. 10 alongside the corresponding magnitudes $Re^{-1}|\tilde{u}|dz|$ and $Re^{-1}|\tilde{v}|dz|$, that make up the adjoint stress, $\tilde{\sigma}$, given in Eq. (30b). These three quantities characterize the strength of the crossflow and Coriolis instabilities excited by conditions imposed on the surface of the rotating disk, i.e., unsteady wall motion. The adjoint stress components $Re^{-1}|\tilde{u}|dz|$ and $Re^{-1}|\tilde{v}|dz|$ indicate how strongly disturbances are excited by radial and azimuthal velocity fields at a fixed point on the disk surface, respectively. Moreover, the adjoint pressure on the disk wall, $|\tilde{p}|$, specifies the same but for an unsteady velocity acting along the wall-normal $z$-direction. On comparing the three plots in Fig. 10, wall-normal motion on the disk surface establishes larger disturbances than that generated by equivalent motion acting along the radial $r$- and azimuthal $\theta$-directions; $|\tilde{p}|$ is typically of the

![Figure 10](https://example.com/fig10.png)

**FIG. 10.** Absolute value of the adjoint perturbation fields at the disk wall $z = 0$, for $n = 7.9$ and $n \in [-5, 60]$. (a) $Re^{-1}|\tilde{u}|dz|$, (b) $Re^{-1}|\tilde{v}|dz|$, and (c) $|\tilde{p}|$. Black circular markers indicate the critical Reynolds number, $Re_c$, for each azimuthal mode number, $n$. 

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order 10 and $10^2$ greater than $Re^{-1}\left|\frac{du}{dz}\right|$ and $Re^{-1}\left|\frac{dv}{dz}\right|$, respectively. In addition, a radial velocity acting on the disk surface generates a disturbance ten times greater than that established by an equivalent motion directed along the azimuthal $\theta$-direction. It is worth noting that the above observations are comparable with the adjoint computations undertaken by Hill\textsuperscript{26} for the Blasius flow over a flat plate: a wall-normal velocity on the plate surface generates TS waves 20 times larger than motion acting along the streamwise direction.

Solutions displayed in Fig. 10 again illustrate the contrasting receptivity characteristics of the type-I crossflow instability and the type-II Coriolis instability. For azimuthal mode numbers $n \leq 5$, matched to the Coriolis instability, the two adjoint stress quantities, $Re^{-1}\left|\frac{du}{dz}\right|$ and $Re^{-1}\left|\frac{dv}{dz}\right|$, achieve maximum amplitudes at stable Reynolds numbers, $Re$, with their respective amplitudes decreasing rapidly as $Re$ increases. On the other hand, the adjoint stress quantity, $|p|$, attains a maximum at Reynolds

![Graphs showing adjoint perturbation fields](image)

**FIG. 11.** Adjoint perturbation fields $\max|\tilde{u}|$, $\max|\tilde{v}|$, $Re^{-1}\left|\frac{du}{dz}\right|_{\omega=0}$, and $|p|_{\omega=0}$, for frequency $[(a), (d), (g), and (j)] \omega = 5$, $[(b), (e), (h), and (k)] \omega = 0$, and $[(c), (f), (i), and (l)] \omega = -5$. Black circular markers indicate the critical Reynolds number, $Re_c$, for each azimuthal mode number, $n$. 

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numbers, $Re$, marginally greater than critical conditions for linear instability.

For azimuthal mode numbers $n \geq 15$, matched to the crossflow instability, solutions display comparable receptivity characteristics for the range of Reynolds numbers, $Re$, shown. The adjoint stress quantity $|\sigma|$ is quantitatively the same for all $n$ modeled, with the amplitude increasing with $Re$. Whereas quantities $Re^{-1}(\partial u/\partial z)$ and $Re^{-1}(\partial v/\partial z)$ change in size as the azimuthal mode number, $n$, increases, with maximum values attained slightly inboard of the critical conditions for linear instability (that are again indicated by black circular markers).

Thus, radial and azimuthal wall-motion prescribed for Reynolds numbers, $Re$, near the onset of linearly unstable behavior will amplify the receptivity of linear crossflow disturbances. This particular receptivity characteristic is analogous to the experimental observations of Radetzky et al., on a swept wing and the adjoint-based numerical computations of Thomas et al., on a swept wing and Thomas and Davies, on a rotating disk for the receptivity of stationary crossflow disturbances: surface roughness located near the onset of linear instability generates the strongest receptivity response, with roughness further outboard having a negligible influence. Clearly, similar receptivity characteristics are found for traveling crossflow instabilities excited by unsteady wall motion, with receptivity enhanced by motion located near the critical conditions for linear instability.

C. Traveling and stationary wave instabilities: $\omega = \pm 5$ and $\omega = 0$

Figure 11 displays adjoint perturbation fields $\max|\tilde{u}|$ and $\max|\tilde{v}|$, and $Re^{-1}(\partial u/\partial z)$ and $|\sigma|$ on the disk surface, for traveling waves $\omega = 5$ (plots on the left-hand side) and $\omega = -5$ (plots on the right-hand side), and the stationary wave $\omega = 0$ (center plots). In each instance, results are plotted for four azimuthal mode numbers $n \in [30, 60]$, matched to the type-I crossflow instability, with black circular markers again indicating the critical Reynolds number, $Re$, for linear instability. Each adjoint quantity exhibits similar behavior and amplitudes to those results presented above for the frequency $\omega = 7.9$. As before, the adjoint radial $\tilde{u}$-velocity field attains larger amplitudes than the other perturbation fields, with the maximum again found for Reynolds numbers, $Re$, inboard of the critical conditions for linear instability and in the vicinity of the critical layer, $z_{crit}$. In addition, the maximum value increases as the azimuthal mode number, $n$, increases. The adjoint azimuthal $\tilde{v}$-velocity field displays similar receptivity characteristics but at lower amplitudes. In the case of $n = 30$ and $\omega = -5$ (purple dotted lines), solutions increase as $Re$ increases. Although this behavior seems out of character with the other trends shown in Fig. 11, this particular feature is likely due to the fact that disturbances are only linearly unstable on a finite interval for these parameter settings, i.e., $285 < Re < 384$.

Consequently, a radial momenta source, $\Phi$, located near the critical layer again enhances receptivity compared with other sources of momenta, $\Phi$, and mass, $\psi$. Similarly, the adjoint pressure, $\tilde{p}$, at the disk wall is larger than the other components of the adjoint stress, $\tilde{\sigma}$. Thus, wall-normal motion on the disk surface establishes larger disturbance amplitudes than radial and azimuthal wall motions.

Although the current study employs the parallel flow approximation, the results closely align with those of Thomas and Davies, who modeled the non-parallel flow. This is despite Thomas and Davies assuming a distinct modal structure for the linear and adjoint disturbances that encompassed a slowly varying shape function [see Eqs. (5.1) and (5.2) of Thomas and Davies] which required a different normalization procedure to that implemented herein [compare Eq. (3.13) of Thomas and Davies with the normalization procedure presented above in Sec. III E]. Concerning crossflow instabilities excited by surface roughness, Thomas and Davies showed that receptivity is amplified by roughness located near the onset of linearly unstable behavior. The results presented herein corroborate this observation. Specifically, external forces (including sources of momenta and mass and unsteady wall motion) fixed near the critical conditions for linear instability enhance receptivity.

D. Amplitude response to wall motion

The above adjoint solutions illustrate the amplitude response of a crossflow or Coriolis instability induced by a point source of momenta, mass, or unsteady wall motion. However, external sources that seed the receptivity process are typically distributed throughout the spatial domain. For instance, suppose the rotating disk is flexible about a specific radial interval, causing the disk wall to move only in the vertical direction. Here, the boundary condition (26) becomes

$$u = \left( -F(0)\tilde{h} - G'(0)\tilde{h} \frac{\partial \tilde{h}}{\partial r} \right) \bigg|_{z=0},$$

where the non-dimensional vertical wall displacement $\tilde{h} = \tilde{h}(r)e^{i(\omega t - \omega z)}$. Consequently, the amplitude, $A$, of the linear disturbance generated by the unsteady wall motion is given as

$$A = B \int_0^r \tilde{h}(r)e^{-i\omega r} dr,$$

FIG. 12. Amplitude, $A$, of the linear disturbance induced by unsteady wall motion with frequency (a) $\omega = 7.9$, (b) $\omega = 5$, (c) $\omega = 0$, and (d) $\omega = -5$. The amplitude is computed using Eq. (35) and black circular markers indicate the critical Reynolds number, $Re_c$, for each azimuthal mode number, $n$. 

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where

\[ B = -\frac{1}{Re} \left( F(\theta) \frac{d\theta}{dz} + G(\theta) \frac{d\theta}{dz} + i\omega \tilde{p}\right)_{z=0} \tag{35b} \]

and \( r_1 \) and \( r_2 \) again denote radial locations upstream and downstream of the wall motion, \( \tilde{h} \).

Setting \( \tilde{h}(r) = e^{-(r-r_2)^2/2} \) establishes a Gaussian-shaped wall motion centered about the radius \( r_2 \). Since the parallel flow approximation has been implemented herein, \( r_2 \) is set equal to zero for simplicity without any loss of generality. Upon substituting the above expression for \( \tilde{h} \) into Eq. (35) and computing the amplitude, \( A \), for Reynolds numbers \( Re \in [250, 600] \), azimuthal mode numbers \( n \in [30, 60] \) at step intervals of \( \Delta n = 10 \), and frequencies \( \omega = 7.9 \), \( \omega = 2.5 \), and \( \omega = 0 \), gives those results depicted in Fig. 12. (Note that the absolute value of \( A \) is displayed in the plot.) There are few variations in the amplitude, \( A \), for frequencies \( \omega = 7.9 \) and \( \omega = 5 \), with \( A \approx 0.06 \) for the range of Reynolds numbers and azimuthal mode numbers considered. Comparable results are realized at the other two frequencies. However, as noted earlier, the behavior associated with the smaller-valued azimuthal mode numbers for these two frequencies is likely a consequence of these disturbances being linearly unstable on a small interval of Reynolds numbers.

**VI. CONCLUSIONS**

A receptivity study of the rotating disk boundary layer was undertaken using an adjoint-based approach. The adjoint linearized Navier–Stokes equations were derived in cylindrical coordinates, for which the parallel flow approximation was employed. The amplitude of both linear crossflow and Coriolis instabilities was determined, subject to several types of external forcing, including sources of momenta and mass and unsteady wall motion. Receptivity characteristics were computed for a broad range of Reynolds numbers, \( Re \), azimuthal mode numbers \( n \), and radial wavenumbers, \( x \). Although the main focus of the study was on those traveling modes with frequency \( \omega = \omega/Re = 7.9 \), comparable receptivity characteristics emerged at other frequencies, including both traveling and stationary waves.

The type-I crossflow instability attained larger amplitudes compared to the type-II Coriolis instability. Moreover, radial sources of momenta induce considerably larger disturbances than equivalent sources of azimuthal momenta, wall-normal momenta, and mass. In addition, motion acting along the direction normal to the disk wall engineers a stronger receptivity response than wall motions directed along the radial and azimuthal directions.

Both traveling and stationary crossflow instabilities were amplified by external forces located near the critical layer, i.e., \( \alpha F + \beta G = 0 \). Moreover, the receptivity of crossflow disturbances peaked for Reynolds numbers, \( Re \), slightly below the threshold conditions for linear instability. This observation aligns with earlier experimental and numerical studies on crossflow disturbances due to surface roughness.\(^{31,32,58,59} \) Conversely, Coriolis instabilities were magnified by external sources located in the vicinity of a vanishing effective shear stress, i.e., \( \alpha F + \beta G = 0 \). This aligns with the asymptotic structure developed by Hall.\(^{53} \)

The results demonstrate that the amplitude of the type-I crossflow instability, induced by external sources of momenta and mass and unsteady wall motion, is amplified with increasing azimuthal mode number, \( n \). At first glance, we might infer that the larger-valued mode numbers have a more significant role in the laminar-turbulent transition process than the smaller-valued mode numbers. However, for a comprehensive understanding of the transition process on a rotating disk, it is necessary to couple the receptivity amplitude established by the external forces with the spatial growth of the linear disturbance, given by the normal mode form in Eq. (11). While the larger-valued azimuthal mode numbers, \( n \), yield larger receptivity amplitudes, it is the smaller-valued \( n \) that first exhibit linear instability. These smaller-valued modes can grow rapidly and achieve amplitudes significant enough to trigger the latter stages of transition, including nonlinear and secondary instability mechanisms.\(^{52,57} \)

The receptivity calculations presented herein may be used to predict the amplitude of linear disturbances generated by external forces in experiments and, thus, describe the early stages of laminar-turbulent transition. For instance, the adjoint stress quantity, \( \sigma \) [recall Eq. (30b)], may be used to predict the amplitude of stationary crossflow instabilities [see Fig. 11(h)] established by surface roughness. Roughness on an experimental rotating disk may be measured\(^{99,61} \) using a profilometer and combined with the adjoint receptivity formula (32) to ascertain the initial amplitude of the disturbance. These calculations may then be used as initial conditions in numerical computations that include nonlinear effects.\(^{62,63} \)

The receptivity calculations outlined in this study can be extended to determine the receptivity of traveling crossflow and Coriolis instabilities induced by the interaction of a free-stream acoustic wave and localized surface roughness. Similar to previous asymptotic and numerical investigations\(^{5,11} \) that modeled the generation of Tollmien–Schlichting waves through the interaction of acoustic waves with rough surfaces, the adjoint amplitude formula, Eq. (32), can be used to quantify the amplitude of the linear disturbance arising from this interaction. Following the methodology outlined in Hill\(^{26} \) (see Sec. 6.1 of that paper), an unsteady motion (that acts as the free-stream acoustic wave) is superimposed on the undisturbed flow, given by Eq. (5). The presence of surface roughness perturbs both the undisturbed flow and the unsteady flow, resulting in correction terms in Eqs. (4) and (8). The interaction between the acoustic wave and surface roughness establishes sources of momenta and wall motion that induce traveling disturbances, with the amplitude determined by Eq. (32).

The current study applies the parallel flow approximation, where the radial dependence of the undisturbed flow was neglected by setting the radius equal to the Reynolds number. For sufficiently large Reynolds numbers, such as those associated with the crossflow instability, non-parallel effects will be negligible.\(^{11} \) However, for traveling Coriolis instabilities that develop at smaller Reynolds numbers, non-parallel effects will be significant. Thus, future receptivity studies on traveling wave instabilities should be extended to the non-parallel flow and follow the strategies of Balakumar et al.\(^{54} \) and Thomas and Davies,\(^{98} \) who employed parabolized stability equations and numerical simulations in their respective receptivity investigations. Nonetheless, considering the observations from the current study based on the parallel flow approximation and the conclusions drawn by Thomas and Davies\(^{38} \) on the non-parallel flow and stationary disturbances, we can anticipate similar receptivity characteristics. Specifically, external forces located near the
critical conditions for linear instability establish greater amplitudes than those sources located further inboard or outboard.

AUTHOR DECLARATIONS
Conflict of Interest
The author has no conflicts to disclose.

Author Contributions
Christian Thomas: Investigation (lead).

DATA AVAILABILITY
The data that support the findings of this study are available from the corresponding author upon reasonable request.

REFERENCES


