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High school mathematics teachers' changes in beliefs and knowledge during lesson study

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Abstract

This research investigates how a lesson study (LS) on designing and implementing challenging tasks impacts Vietnamese high school mathematics teacher knowledge and beliefs. Its contribution highlights cultural considerations when adopting LS to the forefront to contextualize the impacts. The results show that the teachers developed their *specialized content knowledge* by attending to students' mathematics and creating cognitive conflicts building on student responses. The teachers changed their *curriculum knowledge* from *implementer* to *transformer*, improved *knowledge about content and students* attending to difficulties and misconceptions, and enhanced their *knowledge about content and teaching* in ways they designed, sequenced, and evaluated approaches that fit student learning. Finally, they changed their beliefs about mathematics to a comprehensive view of knowledge, *mathematical proficiency*, and *sophisticated beliefs* of teaching and learning. Discussion about the essence of LS when adopting it to different cultures is included.

Keywords Lesson study · Cultural considerations · Teacher beliefs · Teacher knowledge

Introduction

Lesson study (LS) is associated with Japanese student success in international comparisons (Stigler & Hieber, 1999). It provides an approach to enhancing learning when teachers collaboratively engage in detailed analyses of mathematics and student learning. During LS, teachers use their classroom contexts and draw on resources to study their practices.

The premise of improving instruction motivates educators/researchers to adopt LS in other countries, such as the USA (Fernandez & Yoshida, 2012), the UK (Dudley, 2013), Australia (Doig & Groves, 2011), and Asian countries (Saito & Atencio, 2013). When adopting LS, researchers ask: "Should lesson study be changed to fit existing cultural

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beliefs and practices, or should lesson study be used as a tool for changing beliefs, especially those more grounded in tradition than evidence?" (Stigler & Hiebert, 2016, pp. 583). In addition, Lewis et al. (2006) raise the conjecture that "LS strengthens pathways to instructional improvement [via] teachers' knowledge" (p. 5). The current study contributes to this line of research by investigating how an LS on designing and implementing challenging mathematics tasks impacts teacher knowledge and beliefs. It addresses the research questions:

a. How does teacher knowledge change during the LS?
b. How do teacher beliefs about mathematics and mathematics teaching and learning change during the LS?

Many studies explore the impact of LS on developing in-service (e.g., Doig & Groves, 2011) and preservice teacher knowledge (e.g., Corcoran & Pepperell, 2011). However, underlying cultural considerations when adopting LS are not explicitly addressed to contextualize the changes, and very few studies focus on teacher beliefs in LS (Bartolini Bussi et al., 2020). Moreover, LS research is scarce at the high school level when mathematics content is quite abstract, and high-stakes exams are demanding. This study addresses that gap through a case study of three teachers in one suburban school in Vietnam.

Cultural aspects when adopting LS

When teachers in a different country experience LS, it may be unfamiliar to them. Therefore, the participation allows them to reflect upon and rethink their own culture, referred to as *cultural transposition* (Mellone et al., 2019). This process is activated by researchers, teacher educators, and teachers through deconstructing adopted practices to reconsider the issues of their educational intentionality at different stratified levels. Different cultural backgrounds generate possibilities for meaning-making and perspectives. Bartolini Bussie et al. (2020) discussed the mismatch between the time planned and the actual time of the observed lesson in Japan and Italy to illustrate this process. In Japan, the focus was on improving the use of time to keep the planned harmony (cf. Fernandez & Yoshida, 2004), which reflects the underlying principle of *shikata*, "the way things are supposed to be done, both the form and the order, as a means of expressing and maintaining harmony in society and the universe" (De Mente, 1990, pp. 13–14). In contrast, in Italy, teachers found whether time devoted to each step in the real lesson was suitable for the intended process, reflecting the beliefs of good teaching that allow students to have enough time to delve deeply into their learning (Bartolini Bussi et al., 2017).

Andrews (2011) suggested three different levels of curricula when focusing on the cultural aspects of teaching and learning. First, the established beliefs about effective learning and teaching should be contemplated. Andrews (2011) conjectured that teachers' actions reflect their goals, and their goals reflect an idealized view of what their students need to achieve (*idealized curriculum*). These beliefs also function under the resources and constraints in the educational system, such as textbooks (*intended curriculum,* cf. Andrews). Finally, some researchers have demonstrated unnoticed but culturally-located practices of the systems (Stigler & Hiebert, 1999) (*received curriculum*, cf. Andrews). We focus next on research on cultural factors when conducting LS in Vietnam.

Research on cultural factors when adopting LS in Vietnam

The premise of LS is on connecting with students, and this view must be considered culturally dependent. Saito and Tsukui (2008) observed that Vietnamese teachers hold authoritarian views toward students. They regularly use a command style of teaching—akin to giving orders. Also, teaching is more important than learning, where teachers determine the pace and direction of lessons without paying attention to students' needs and interests (Saito & Atencio, 2013). Group learning activities are not expected (Saito & Atencio, 2013).

In LS, building a learning community to support teacher inquiry allows teachers to observe and jointly reflect on teaching. However, in Vietnam, the lesson is open to the public, mainly in teacher contests that rank teachers according to perceived effectiveness (Saito & Atencio, 2013). Therefore, there are reservations about LS, as lesson observers can be highly critical of the teacher being observed, and peer criticism can lead to mistrust and a lack of respect. In addition, teacher evaluation and lesson observation can be completely integrated (ibid.). Therefore, engaging in a community of learners can be challenging.

Research suggests that external consultants or *knowledgeable others* (Lewis, 2002) are essential to enhancing the quality of LS in emerging countries. However, the *power relationship* between the outsider and the teachers can be problematic. For example, Sato (2005) regularly observed that teachers gaze at external consultants, as they perceive consultants to have positions to surveil and objectify them, often in negative ways. In some instances, these consultants can criticize negatively, which is detrimental to the emergence of a learning community. Saito and Atencio (2013) report that in developing Asian countries, experts experienced suspicion and cynicism from the teachers and encountered resistance, not in heated debates but displayed through silence and lack of attention.

When implementing LS in a different culture, it is crucial to (a) emphasize cultural transposition, (b) make the idealized, received, and intended curricula explicit, and (c) contemplate power relationships between teacher-student, teacher-teacher, and teacher-researcher. These cultural aspects need to be addressed when studying the impacts of LS on teacher knowledge and beliefs.

Teacher knowledge and teacher beliefs are important (Potari & Chapman, 2020; Tatto et al., 2012) as they impact teaching and student learning. However, there have been diverse ways in which researchers conceptualize and operationalize them. A comprehensive review of this issue is out of the scope of this study (see Rowland & Ruthven, 2011; Depaepe et al., 2013, collective discussions about relevant constructs). In the current study, we focus on (a) teacher knowledge through the mathematical knowledge for teaching (MKT) framework (e.g., Ball et al., 2008), and (b) beliefs—convictions about mathematics and mathematics teaching and learning informed by the TEDS-M study (e.g., Tatto et al., 2012).

Teacher knowledge

Focusing on the structure of teacher knowledge, Ball et al. (2008) conceptualized MKT as including subject matter knowledge (SMK) and pedagogical content knowledge (PCK). They distinguished between common content knowledge (CCK) and specialized content knowledge (SCK) in the SMK domain. Ball et al. defined SCK as the knowledge that is only needed or used in mathematics teaching settings, which differs from CCK, which is used in a wide variety of settings, not unique to teaching. Their conceptualization is built on the work of primary teachers; therefore, researchers have argued that it is not straightforward to apply that conceptualization to secondary and college mathematics (Speer et al., 2015). These researchers argue that SCK is the ability to (a) unpack students' solutions, (b) follow students' mathematical thinking, (c) evaluate the validity of student-generated strategies, (d) make sense of student-generated solutions paths (Hill et al., 2008), (e) interpret students' representations, and (f) evaluate how well students' solutions link to lesson goals (Speer et al., 2015). In short, SCK is the knowledge needed to *think about* and *figure out* something about the mathematics students are discussing (Speer & Wagner, 2009).

Ball et al. (2008) highlighted *knowledge of content and student* (KCS) and *knowledge of content and teaching* (KCT) and provisionally included *knowledge about curriculum* (KC) without elaborating on this concept in PCK. The teacher must know the mathematics concerning their students, how they learn, their common misconceptions and conceptions (KCS), and link them with appropriate ways of teaching the content that addresses student misconceptions and difficulties (KCT). KCT includes selecting and sequencing examples, evaluating the advantages and disadvantages of representations used to teach specific ideas, and identifying how different methods and procedures can afford instruction.

Mathematics education researchers have conceptualized two approaches to operationalizing teacher knowledge, one as "in-the-head-of-teacher" and the other as situated and distributed (Depaepe et al., 2013; Kaiser et al., 2017). The first approach tends to measure teacher knowledge by using test items. In contrast, the second focuses on describing teacher knowledge through their use in practice by examining lesson plans, classroom observations, and interviews. Because teachers do not work in a silo, their work should be considered in association with the resources and the requirements and constraints of the system they function (Hodgen, 2011). This situated view requires researchers to describe teacher knowledge as teachers engage in teaching. This perspective helps explain why and how teachers did what they did in the circumstances and ways to support them. However, Deapaepe et al. (2013) argued that the pitfalls of the situated approach were that researchers do not often elaborate on how they defined and operationalized PCK. We addressed this concern by adopting the MKT framework. This adoption is appropriate as the MKT framework was developed based on the analysis of actual teaching practices and was used to identify the *knowledge needed for teaching*. It is also helpful to study the knowledge teachers use in practice (Speer et al., 2015).

Vietnamese mathematics teacher knowledge

Adopting this teacher knowledge lens helps us, researchers, reflect on the current situation of mathematics education in Vietnam (cultural transposition). We maintain that Vietnamese secondary mathematics teachers have strong mathematics knowledge (i.e., CCK as conceptualized by Speer and Wagner (2009). Their teacher education training includes advanced mathematics (equivalent to a master's degree in mathematics in the USA[1]) and school mathematics-from-an-advanced-standpoint courses. However, the same cannot be said for SCK and PCK. Current research literature in Vietnamese does not have equivalent terms, which might reflect the underlying cultural school customs that do not support teachers in developing these types of knowledge. As teaching practice tends to be teachers

[1] Personal examination of the programs.

presenting precise solutions for students to follow (received curriculum), SCK seems irrelevant when teachers judge the correctness of the solution as to how well it aligns with their (formal) presented approach. As prevailing educational practice tends to ignore student interest and needs (received curriculum), KCS may seem unnecessary. Effective teaching is judged on how teachers explain concepts and skills clearly and logically without acknowledging student difficulties, misconceptions, and ways of learning (idealized curriculum). Regarding KCT, in the teacher education program, PSTs are taught ways to teach mathematical concepts, theorems, and mathematics exercises (Nguyễn, 2015). Differing teaching approaches are not relevant, as teachers strictly follow prescribed textbooks (received curriculum). Therefore, their evaluation of different approaches considering student learning is not required. Finally, as Vietnamese teachers strictly follow textbooks (intended curriculum), different approaches to constructing curriculum knowledge are not pertinent.

Beliefs about mathematics and mathematics teaching and learning

The TEDS-M study (Tatto et al., 2012) describes beliefs as "understanding, premises or propositions about the world that are felt to be true" (Richardson, 1996, pp. 103). However, due to beliefs' experiential and context-bound nature (Schoenfeld, 1998), they are changeable when challenged and exposed. During cultural transposition, teachers can reflect on their beliefs about what important mathematics is and how to teach it effectively. One way to conceptualize beliefs about mathematics ranges from a-set-of-rules-and-procedures to a process-of-inquiry (Tatto et al., 2012). The beliefs about mathematics impact how a teacher views teaching and learning, either as following teacher direction and listening carefully to instruction or as an inquiry process, where students are free to make and learn from mistakes to construct their knowledge. Reformed curriculum (e.g., NCTM,) advocated for more active learning when students engage in the mathematical behaviors of doing mathematics—teachers display *sophisticated beliefs* instead of *naïve beliefs*–mastering skills and procedures (Francisco, 2013).

Researchers maintain that beliefs are best studied indirectly, inferred rather than directly accessed (e.g., Ambrose et al., 2003). Also, "responses, written or oral, tell only part of the story about one's beliefs" (Wilson & Cooney, 2002, pp. 135). Therefore, using a phenomenological approach to study beliefs is appropriate, where beliefs are inferred from an analysis of individuals' accounts of their lived (educational) experiences obtained from in-depth interviews. In addition, Hofer and Pintrich (1997) urge researchers to gather evidence with higher ecological validity by incorporating other methods such as critical incident techniques following classes and stimulated recalls. Other researchers have chosen to examine individuals' stated and enacted beliefs in situ. Beswick (2004) argues that "more certainty can be attached to the existence of a belief that is evident in both the words and the actions of an individual" (pp. 111). Research has suggested a link between teacher beliefs and their teaching in classrooms (Staub & Stern, 2002); beliefs are crucial to a teacher's perception of teaching situations and choice of teaching methods (Leder et al., 2002). Beliefs about mathematics and mathematics teaching and learning are foundational in teaching practices (e.g., Beswick, 2005). Some researchers have shown the consistency between beliefs and practices (e.g., Stipek et al., 2001), while others show inconsistency (e.g., Shield, 1999). However, Beswick (2004) argued that the context and the level of specificity the previous researchers used to examine the two aspects could explain this inconsistency.

Research on the impacts of LS on teacher knowledge and beliefs

Research has argued that teachers improve their knowledge, practices, and beliefs when engaging in LS (Vermunt et al., 2019). For example, Corcoran and Pepperell (2011) concluded that Irish preservice teachers (PSTs) developed mathematical content knowledge for teaching when participating in LS. The PSTs developed their identity over the three cycles of LS, where they were more active in the community and transformed their participation. They embraced the roles as teachers, despite feelings of inadequacy. Similarly, Ní Shúilleabháin (2016) investigated the development of mathematics teachers' PCK when participating in LS. The researcher documented 12 teachers' development in two Irish post-primary schools. The research displayed distinct features of PCK incorporated in teachers' planning and reflections, and observed an increase of frequency in incorporating PCK over four cycles.

Very few studies pay attention to how teachers' beliefs change due to LS (Dudley, 2013; Schipper et al., 2018). These studies investigated how professional self-efficacy changes as teachers participate in LS. Using the pre-post design, Schipper et al. (2018) discovered that teacher self-efficacy improved at the end of LS. The teachers significantly improved their beliefs about engaging students in learning, including providing clear instruction, activating learning, and teaching–learning strategies. However, research seldom analyzes *how* teacher beliefs about mathematics teaching and learning change. In one study, Dudley (2013) showed that when participating in LS, a teacher changed her beliefs about the role of mathematics open-ended tasks and saw the potential of using them to enhance student reasoning.

In summary, research has investigated how LS impacts teacher knowledge and beliefs. However, they have not explicitly addressed how LS was adopted in the culture when examining the impacts. Cultural transposition allows teachers to raise the awareness of implicit assumptions about their current practice when facing adopted practices (Mellone et al., 2019). However, cultural transposition has not been used to examine how changes happen in LS. The curriculum levels used to conceptualize the cultural aspects of teacher knowledge can be applied to examine the changes in LS adopted to a different culture. In addition, based on our awareness of multiple power relationships (Saito & Atencio, 2013), we discuss how we addressed these issues in this LS to contextualize the impacts of adopting LS on Vietnamese teacher knowledge and beliefs.

Methodology

Setting and participants

An explanatory case study of three male high school teachers was conducted. This methodology is appropriate to explain how teachers change their knowledge and beliefs in LS (Mills et al., 2012). A convenience sample of experienced teachers from a school in a suburban area of Hue city, Vietnam, comprise the participant group. The researchers have built rapport with the teachers for several years, which helped alleviate any suspicion of teachers toward the research team. The researchers played the role of participant-observer in the team. The researchers introduced the LS model and challenging task approach to the teachers. Additional resources about (a) ways to conceptualize comprehensive mathematics learning and outcomes, (b) the nature of challenging tasks to help students achieve the outcomes, and (c) the roles of teachers in social constructivist classrooms were introduced and discussed with the group. The researchers regularly offered teaching resources during the LS. However, the teachers created lesson plans and taught them. During meetings and classroom observations, the researchers reflected on research goals as a participant and collected data for research purposes (observer). This role of experts as a more *knowledgeable other* is crucial in LS (Fujii, 2014).

All teachers had more than 13-years' experience teaching mathematics at the high school level. Quang[2] has taught for 16 years and has a master's degree in Algebra. He also attained a Certificate of Advanced Teacher—the highest-ranking schoolteacher in Vietnam. He has also served as the associate principal of the school. Kha has taught mathematics for 15 years and served as a mathematics coordinator for six years. Binh has taught for 14 years. Quang and Binh have been involved in another LS with the researchers focusing on using manipulatives in teaching, whereas Kha was new to LS. Experienced teachers were purposefully chosen due to their eagerness to meet the intricacy of lesson design. In addition, the choice of teachers at the same career stage addressed the issue of power relationships amongst teachers.

Teacher learning is a lifelong process in Japan (Fujii, 2014), but it is not the case for Vietnamese teachers. Therefore, we started this explanatory case with highly motivated teachers. At the time of this research, the Department of Education was calling for reform in teaching, focusing more on a student-centered approach. The Department proposed project-based learning, problem-based learning, and collaborative learning as ways to address it. The student-centered principle is mentioned but not elaborated on. Local departments of education have professional training for representative teachers at the district level. The view adopted in this LS aligns with the reform; however, we contextualize it specifically to mathematics teaching and learning.

LS and data collection

Goal setting

The team discussed the LS model and its research goals. Saito and Atencio (2013) suggest problems with observation experiences in Vietnam, such as time constraints and reservation. Therefore, significant attention was required at the beginning of the LS for teachers to undertake curriculum research and refine lesson plans. This process can assist groups in building a shared mutual respect culture to overcome the authoritarian and hierarchical conditions common in most Vietnamese schools (teacher-teacher and teacher-expert). Consequently, teachers will likely feel more comfortable learning about curriculum matters and pedagogical practices together, with less judgment.

The researchers focused on describing comprehensive mathematics learning and outcomes. It took the teachers significant time to articulate what they think about important mathematics for students and how they learn it (idealized curriculum). To challenge their idealized and received curricula, the teachers reflected on a framework drawn from a different culture discussing comprehensive mathematics learning (cultural transposition). The teachers read and deliberated what it means to learn mathematics and reconsidered ways to

[2] Pseudonyms are used for all teachers.

expose students to mathematical proficiency (Kilpatrick et al., 2001), comprehensive mathematical learning and doing, including *conceptual understanding*, *procedural fluency*, strategic *competence*, *adaptive reasoning*, and *productive disposition*. This process offered the teachers opportunities to reflect on their teaching and learning intentionality.

The researchers discussed how challenging tasks could offer comprehensive learning opportunities (e.g., Stein et al., 1996). With solid mathematics knowledge, the teachers can realize rich learning opportunities when they see such tasks. It is noted that highly mathematically technical tasks are common in Vietnamese teaching resources, which require complicated techniques and manipulations (intended curriculum). However, other aspects of challenging tasks, such as high cognitive demands, their open-ended exploration and inquiry affordances, and calling for multiple representations and contextualization, were not expected.

Two initial meetings (60 min each) set out the involvement of the teachers and discussed issues of mathematics teaching and learning and task design. Examples of high cognitive demand tasks (Stein & Smith, 1998) and how to adapt tasks to high cognitive demands were discussed. A high-level cognitive-demand task that addresses mathematical proficiency and requires complicated mathematical thinking and reasoning (e.g., conjecturing, justifying, interpreting) offers students opportunities to engage in *doing mathematics*. First, the researchers introduced the framework of the cognitive demands (Stein & Smith, 1998) using examples in solid geometry. Then, the researchers asked the teachers to find tasks that map to different levels in their teaching. In some cases, the teachers were guided to adapt tasks found in textbooks to meet different levels of cognitive demand. For example, the textbook task proving $1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$ was discussed. The team categorized the task as *procedures-with-connection* as the students followed the proving-by-induction strategy. However, it is not a narrow algorithm, and students must decide on connecting the $n + 1$ step to the previous steps. The group discussed how to adapt this task to the *doing-mathematics* level. Some considerations included finding real-world contexts to integrate the mathematics, opportunities for exploration, and reducing the givens to make the task more open-ended. In this task, the team considered how to establish the formula before proving it. The adapted task was regarded as the highest cognitive demand level, as it involved non-algorithmic thinking without a predictable, well-rehearsed approach or pathway explicitly suggested by the task, instructions, or a worked-out example. In addition, this adapted task offers potential for mathematical exploration and the use of multiple representations. The team also discussed textbook tasks (intended curriculum). Tasks found in Vietnamese textbooks are pure and abstract, and the main form of accepted communication is symbolic. Students are expected to show this type of outcome, but developmental ways to achieve them were not evident. The discussions also focused on distinguishing between transmitting knowledge and attending to student reasoning and how to know if students learn essential mathematics using the relationship between the mathematical task set up by the teacher, the task implemented by students in the classroom, and student learning (Stein et al., 2007). At the end of the discussions, the LS team agreed on the goal of helping students to experience the comprehensive learning of mathematical proficiency by using challenging tasks.

Lesson planning

The team agreed to implement challenging tasks based on the Vietnamese Department of Education pacing guide (intended curriculum). We spent about an hour discussing each

Lessons observed		10A1	10A2	11A1	11A2	12A1	12A2
Semester 1							
L_1	Graphs of trigonometric functions (12 Advanced Analysis)					**Quang**	Kha
L_2	Probability - The chance task (11 Advanced Algebra and Analysis)			**Kha**	Kha		
L_3	Probability - fair games (11 grade Advanced Algebra and Analysis)			**Binh**	Binh		
L_4	Quadratics functions (10 Advanced Algebra)	**Quang**	Quang				
L_5	Functions with the domain of different intervals (10 Advanced Algebra)	**Quang**	Quang				
Semester 2							
L_6	Mathematical induction method (11 Advanced Algebra and Analysis)			Quang	**Kha**		
L_7	Arithmetic sequence (11 Advanced Algebra and Analysis)			**Binh**	Binh		

Fig. 1 Sessions observed during LS. Note: Bold indicates the teacher who designed and implemented the first session

lesson plan before assigning the lesson to a teacher who would first teach it. The teacher had three days after the discussion to plan and send the lesson to the team for feedback before implementing it. The team discussed four lesson plans (3, 5, 6, and 7), and each remaining lesson was discussed between Quang, Kha, and the researchers (60 min each). Seven lesson plans were designed, three by Quang, two by Kha, and two by Binh. They generally comprised leading tasks and a draft plan for running the lessons. During planning meetings, teachers' anticipation of student learning and their responses were discussed. This way of planning reflects their expertise and confidence in the job.

Classroom observations, discussions, and interviews

Before commencing LS, we conducted nine classroom observations (45 min each) to gain insight into teacher knowledge and beliefs. Although we were familiar with enculturated teaching practices, these observations provided rich data as a baseline from which to investigate changes (if any) in this LS.

During LS, 14 sessions (60 min each except for L6 with 120 min) across three grade levels (10–12) were observed and videotaped. Each of the seven lessons was implemented twice (Fig. 1). These observations focused on teachers' task implementation and students' interactions. The field notes were then used for post-lesson discussions (30 min each). The discussions focused on lesson adaptation, lessons learned during implementation, beliefs, SCK, PCK, and reflection on the research goal. It is worth noting that 14 observation sessions and 14 post-classroom discussions happened in nine months of the school year, with ten sessions in Semester 1 (from September to January) and four in Semester 2 (from

Table 1 Summary of data collection and analysis

Data collection	Data analysis
Lesson plans: 7	Learning intentions and tasks (KC and beliefs): Challenging tasks Comparison of lesson tasks and sequences with those in textbooks (KC) Anticipation of student difficulties, misconceptions (KCS) Example and task selection and sequencing, evaluation of teaching approaches (KCT)
Classroom observations and field notes, and videotapes (9 + 14)	Learning intentions and tasks (KC and beliefs): Challenging tasks Comparison of lesson tasks and sequences with those in textbooks (KC) Classroom settings and routines (group work, individual work, responding) (beliefs) Responding to students (SCK) Example and task selection and sequencing (KCT)
Group meetings: Initial (2), planning (7), and post-lesson discussions (2 × 7)	Beliefs SCK, PCK
Individual interviews, before and end of LS: 3 + 3	Beliefs
Student work	Reference: Impact of LS on student learning

In addition to these formal data collected, the researchers had numerous informal conversations with the teachers about LS and mathematics teaching and learning throughout the school year, which was not documented

January to May). We prolonged this LS so that teachers had enough time to experience and reflect on this practice (cultural transposition).

Discussions, classroom observations, interviews, lesson plans, and student work[3] (secondary data source) form the data corpus (Table 1). Individual interviews (60 min) at the beginning and the end of the LS focused on beliefs, changes observed during LS, how they happened, and reflections on challenges and success when implementing LS.

Data analysis

For SCK analysis, the researchers narrowed down moments when the teachers responded to students in classrooms and discussed why they reacted the way they did and if they had planned their responses. We looked at how they attended to students' mathematics, evaluated the validity of student-generated strategies, made sense of student-generated solutions and representations, and how they linked students' solutions to learning intentions. For example, teachers considered how students used geometric figures to represent $1^2, 2^2, \cdots, n^2$ and which ones could be used to formulate the sum $1^2 + 2^2 + \cdots + n^2$.

[3] Students were informed that their work could be used for research publications.

We examined the textbooks used for KC to reference affordances and constraints (intended curriculum). First, we noted textbooks' learning intentions, tasks, and sequencing for all the lessons observed. We then analyzed how lessons deviated from the textbook learning intentions and the difference between textbook tasks and those in teaching. We finally categorized teachers as *implementer*s (received curriculum) if textbook tasks and sequence were used as-is or *transformer*s if the deviation was found in the lessons.

For KCS, we documented how teachers incorporated student learning concerning the content, difficulties and misconceptions predicted when planning and noticed when implementing lessons. For example, the teachers considered student patterns of thinking when students established the formula for $1^2 + 2^2 + \cdots + n^2$, how students may arrive at the formula, and the difficulties of assembling geometric representations of the components, $1^2, 2^2, \cdots, n^2$.

For KCT, we documented teachers' decisions in sequencing tasks and evaluations of the advantages and disadvantages of the representations used through lesson plans, teaching, and discussions. For example, the teachers sequenced finding the formulae for $1 + 2 + \cdots + n$ and $1^3 + 2^3 + \cdots + n^3$ before $1^2 + 2^2 + \cdots + n^2$ due to the more sophisticated nature of the last task.

We analyzed teacher beliefs displayed in the pre and post LS interviews and discussions. Also, within one specific context (time and school), it is reasonable to assume that the practice reflects their beliefs. For beliefs about mathematics, we coded for how the teachers perceived what important knowledge is (e.g., just focusing on manipulation of symbols (idealized curriculum) or conceptual understanding with reasoning and problem solving involved). The beliefs could also be examined through the cognitive demands as the task features indicated what important mathematics is and how it is related to mathematical proficiency. We examined the cognitive demands of task*s set up by the teacher in the classroom* in all lessons (Stein et al., 2007). For example, the task of proving $1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$ was considered as procedures-with-connection, whereas establishing a formula for the sum and proving it was considered as doing-mathematics. Other features of challenging tasks such as representations (symbolic, verbal, figurative) used and opportunities for exploration and multiple approaches and contextualized were noted.

The beliefs about mathematics teaching and learning were examined by analyzing how teachers organized their classroom's opportunities for students to engage in mathematical discourse with each other or with the teacher, how they responded to student thinking, and how they viewed their and their students' roles. This analysis helped infer teacher beliefs about learning (e.g., naïve beliefs vs. sophisticated beliefs). We applied (a) *methodological triangulation*—researchers evaluated what teachers said about their beliefs and what they did in practice, (b) *investigator triangulation*: two researchers examined the data and compared what we observed and interpreted from the interviews and video recordings, (c) *data triangulation*—we collected data of the same events at different time, place and of people (Denzin, 1989), and (c) member-checking to confirm researchers' interpretation of beliefs with teachers'.

We looked for evidence of the aspects above during nine lessons observed before LS (received curriculum) and tried to find any evidence that deviated from the patterns observed in classrooms. We also did a similar examination for 14 sessions during LS to see if the changes were simply random events. To confirm the changes were attributed to LS, we will use examples from multiple lessons during the LS. We will use contrasting events of what the teachers did/said before LS and during LS. We acknowledge that in classroom contexts, "the act of teaching is multi-dimensional… and teachers' choices simultaneously

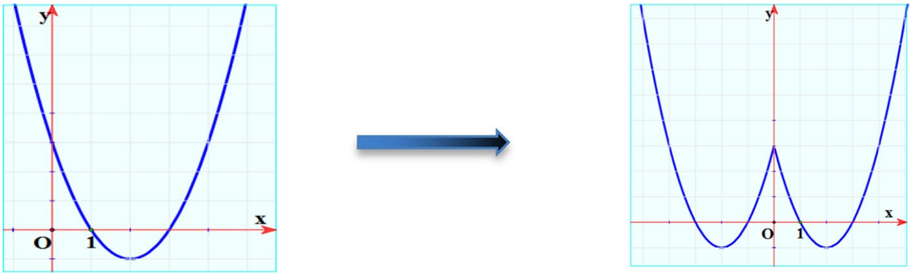


Fig. 2 Transforming a quadratic function to its absolute function

reflect mathematical and pedagogical deliberations" (Depaepe et al., 2013, p. 22). However, when possible, we point out the components of knowledge and beliefs above.

Results

Teacher knowledge changes

Responding to students' mathematics (SCK)

Before LS, the teachers often preferred and wanted all students to follow their precise solutions. When finding student misconceptions or errors, they tended to call on other students who had the correct solution to present, or the teachers presented the precise solutions themselves (received curriculum). Moreover, when a student's approach was correct but not precise, the teachers did not encourage the solution and instead asked the student to follow their way. For example, in one lesson, Quang asked students to find all values of m, so $x^2 - 4|x| + 3 - m = 0$ had four solutions. Some students solved the problem by breaking the equation into two intervals (non-negative and negative), using a determinant to make each quadratic function have two solutions, and finding the intersection of the two sets of m values. Quang stopped students from continuing their approach. He provided the students with his strategy, the intersection of the graphs of two functions $y = x^2 - 4|x| + 3$ and $y = m$. Using an even function's property; he introduced graph transformation, the quadratic function $y = x^2 - 4x + 3$ to the absolute function $y = x^2 - 4|x| + 3$ (Fig. 2). We conceived that the teachers might possess SCK, but they did not execute it except for waiting for the precise solution they wanted from the students.

In contrast, during LS, the teachers often generated a situation to create a cognitive conflict for students, encouraging them to reconsider their approach. For example, in L2, students solved the Money Distribution Task:

An and Ba take turns tossing a fair coin, and each contributes 32 dollars. An gets one point if it is a Head, whereas Ba gets one if it is a Tail. The player who gets three points is the winner. However, when An got two points, and Binh one, the game stopped unexpectedly. How should they divide the money at this stage? Explain.

Most students decided that An and Ba should get their 32 dollars back because the game had not ended. However, some students argued that the ratio of 1:1 was not fair for An and proposed the 2:1 ratio. Kha created a situation to challenge students' thinking instead of providing his precise solution. He asked what students would do if the game were:

"The player who wins 16 points is the winner. When An has 15 and Ba has 13, the game stopped". The students realized that using the ratio of points achieved was not reasonable. If the same thinking applied, they would use the ratio of 15:13 (for An: Ba) to distribute money. However, in this new scenario, An needed only one more point to win, but Ba needed to win three points in a row. This event shows his SCK as he could interpret students' solutions and build on their current reasoning to connect to the lesson intention. In the post-class discussion, Kha showed his "appreciation for students' solution and tried to work from students' solution for them to the path of solving the problem instead of presenting the correct way." He said that "creating cognitive conflicts for students will enable them to reconsider and revisit their approaches, especially their conceptions (not just mistakes). I hope they will revise their conception from the counterexample, not adopt my conception to theirs".

The initial discussions on listening to student reasoning impacted how these teachers responded to them (cultural transposition on their idealized curriculum). During the first meeting, the researchers commented on the teachers' habits of stopping students from exploring alternative solutions (received curriculum). The teachers' only concern was finishing lesson content in time with little regard for student understanding and thinking. However, during LS, Quang regarded "the power of listening to and empowering students to take ownership of their learning and how [he] saw the nuanced ways of students' reasoning help them form their agency." (Final Interview), realizing his SCK.

Curriculum implementer to transformer (KC)

Before LS, the teachers followed the pacing guide and textbooks strictly. The lessons comprised textbook tasks and examples used as-is. The teachers were regarded as *curriculum implementer*s (Stein et al., 2007), reflecting the unwritten norms of the Vietnamese educational system (received curriculum).

When participating in LS, the teachers shifted to the curriculum *transformer* (Stein et al., 2007). They still followed the pacing guide (order of and time allocated for lessons) yet constructed their lesson plans and teaching approaches quite differently. The teachers went beyond the intended curriculum, informed by their knowledge and beliefs about what was crucial for students to learn and how to make learning happen. For example, the textbook introduced the concept of arithmetic sequence, followed by pure-mathematics examples and properties, and ended with exercises for students to apply the properties. During LS, Binh chose a real-world problem as a lesson hook. First, students were engaged in arithmetic sequences without knowing their names. Next, he named and defined arithmetic sequence. He then introduced the concept embedded in the Big C context, using examples and non-examples. Finally, he helped students deduce the properties of arithmetic sequence via specific examples.

KC change was also reflected in enriching learning intentions during LS. One Year-11 learning intention was applying the mathematical induction strategy to prove symbolic rules (intended curriculum); the teacher guide stated that "students should be able to apply mathematical induction to prove mathematical statements" (Đoàn et al., 2011, pp. 130). The textbook task was "prove that $1^3 + 2^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$." During LS, Quang and Kha designed an extra learning intention for students to develop ways to establish formulas (sense-making) by connecting them to other representations (e.g., geometric representation) before applying mathematical induction. Students engaged in other mathematical thinking processes, such as specializing, generalizing, and not merely proving. The

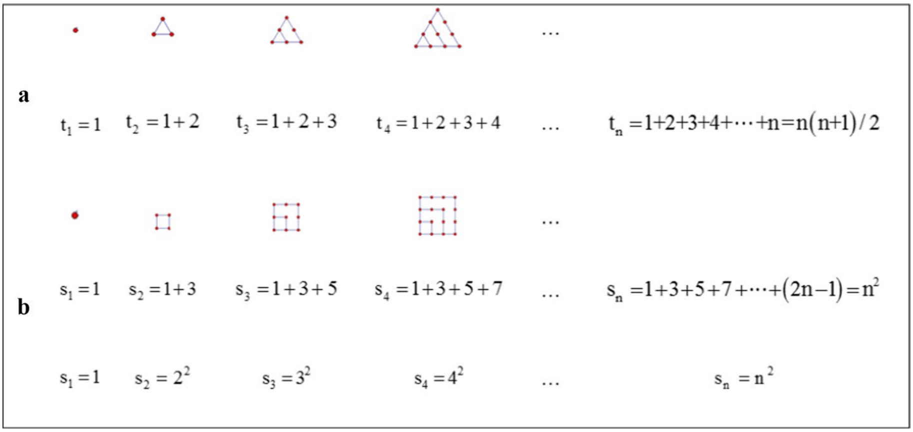


Fig. 3 Geometric representation of symbolic form

teachers found no single task in the textbook that offered students opportunities to achieve the intention. Therefore, they augmented the lesson with a new task. In L6, the teachers planned and implemented three tasks (not in textbooks) using the induction method.

First task: in 20 min, students worked in groups to find a formula for counting the number of dots in each figure and generalizing the number of dots. Next, students referred to the arithmetic sequence in the previous lesson to generalize the sums. Finally, they connected figural number patterns (e.g., triangular, squared, pentagonal numbers) to a symbolic form (Fig. 3).

For the second task, in 30 min, the teacher asked students to find a sum (a formula to simplify the sum)$S_n = 1^3 + 2^3 + 3^3 + \cdots + n^3$.. Students worked in groups to solve this problem as the teacher moved around the room to observe. Some students said they knew of the formula. However, Quang challenged them to derive a formula that makes sense without assuming it. When observing, he called students' attention to specializing by using specific n to conjecture.

Students specialized,

If $n = 1$ then $S_1 = 1^3 = 1$.

If $n = 2$ then $S_2 = 1^3 + 2^3 = 9$.

If $n = 3$ then $S_3 = 1^3 + 2^3 + 3^3 = 36$.

If $n = 4$ then $S_4 = 1^3 + 2^3 + 3^3 + 4^3 = 100$.

The students realized each was a square number (e.g., 1^2, 3^2, 6^2, and 10^2). Students' engagement in the previous task helped them connect to triangular numbers (Fig. 3). They found out that the sum was the square of triangular numbers.

If $n = 1$ then $S_1 = 1^3 = 1^2$;

If $n = 2$ then $S_2 = 1^3 + 2^3 = 9 = 3^2 = (1 + 2)^2$;

If $n = 3$ then $S_3 = 1^3 + 2^3 + 3^3 = 36 = 6^2 = (1 + 2 + 3)^2$;

If $n = 4$ then $S_4 = 1^3 + 2^3 + 3^3 + 4^3 = 100 = 10^2 = (1 + 2 + 3 + 4)^2$.

They conjectured that:

$$S_n = 1^3 + 2^3 + 3^3 + \cdots + n^3 = (1 + 2 + 3 + \cdots + n)^2 = \frac{n^2(n + 1)^2}{4}$$

and proved their conjecture by using the mathematical induction method.

The last task will be discussed in the section about changes in KCS and KCT.

Adopting the comprehensive learning views (mathematical proficiency) as a research goal and examining current curriculum resources (intended curriculum) concerning the views help teachers notice things they might have overlooked before and provide the teachers with a different view of curriculum (cultural transposition). Quang discussed that "we [Quang and Kha] needed to provide richer learning opportunities than the textbooks" (Final interview). In addition, the study of new materials provided by the researchers helped the teachers transform their curriculum.

Acknowledging students' difficulties (KCS), and selecting and evaluating the advantages and disadvantages of different approaches in teaching (KCT)

Before LS, the teachers' goal was to finish teaching the lesson content as scheduled. Therefore, their teaching approach should be logical and on time. Furthermore, it should adhere to mathematical normality when symbolic representation was the only way teachers used and expected their students to perform (received curriculum and idealized curriculum). Therefore, comparing approaches that relate to students was not necessary or irrelevant to teachers' concerns (KCT). Teachers' attention to student misconceptions, difficulties, and thinking patterns was not evident (KCS).

During LS, the teachers appreciated students' misconceptions and difficulties when learning specific topics (KCS), and then used this to inform their KCT to support student learning. They paid attention to other representations such as geometric, tabular, and graphical representations to help students make sense of mathematics. For example, after providing students with the figural number task to help them connect geometric and symbolic representations, Quang presented the following task (L6):

Calculate (find a formula to simplify the sum) $T_n = 1^2 + 2^2 + \cdots + n^2$.

When planning this task, Quang considered three approaches to solving it:

- Use of equality between expressions for similar patterns:

$$1^3 = 1^3$$
$$2^3 = (1+1)^3 = 1^3 + 3 \times 1 + 3 \times 1^2 + 1^3;$$
$$3^3 = (1+2)^3 = 1^3 + 3 \times 2 + 3 \times 2^2 + 2^3;$$
$$\ldots$$
$$(n+1)^3 = (1+n)^3 = 1^3 + 3 \times n + 3 \times n^2 + n^3.$$

  Adding side by side

$$(n+1)^3 = (n+1) + 3(1 + 2 + \cdots + n) + 3\left(1^2 + 2^2 + \cdots + n^2\right)$$
$$= (n+1) + \frac{3}{2}n(n+1) + 3T_n \Rightarrow T_n = \frac{n(n+1)(2n+1)}{6}.$$

- Functional approach

  Finding a function $F(n) = An^3 + Bn^2 + Cn + D$ that satisfies $n^2 = F(n+1) - F(n)$.

  Using the coefficient method

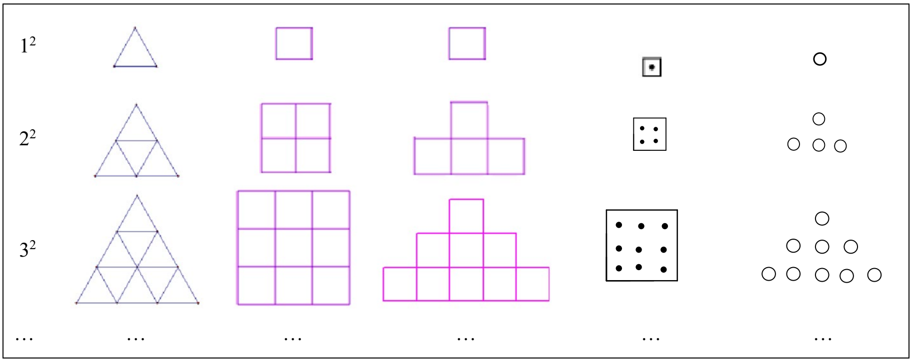


Fig. 4 Geometric representations of square numbers

$$n^2 = F(n+1) - F(n)$$
$$= 3An^2 + (3A+2B)n + (A+B+C) \Rightarrow A = \frac{1}{3},\ B = -\frac{1}{2},\ C = \frac{1}{6}.$$

$$F(0) = D$$

then choose $D = 0$, hence $F(n) = \frac{1}{3}n^3 - \frac{1}{2}n^2 + \frac{1}{6}n$.

$$T_n = 1^2 + 2^2 + \cdots + n^2 = F(2) - F(1) + F(3) - F(2) + \cdots + F(n+1) - F(n) = \frac{n(n+1)(2n+1)}{6}.$$

- Use of geometric representations
  Quang and Kha considered the third one to be most appropriate for students. The first two could be very precise in finding the formula, but students would find it challenging to construct it themselves (KCS & KCT). Instead, students must remember the procedures and apply them without explaining why (KCS). Therefore, they asked them to generate geometric representations for $1^2, 2^2, 3^2, \ldots$ (KCT). Figure 4 illustrates some representations anticipated by the teachers and generated by students.

Building on the knowledge, students tried to represent T_n by assembling multiple representations of $1^2, 2^2, 3^2, \ldots$ Unfortunately, they were unsuccessful in realizing any pattern. Quang provided the figure (second column Fig. 5) in a worksheet that asked students to connect the figure to symbolic forms (KCT). Students instantly realized the figure's left- and right-hand sides as $2\left(1^2 + 2^2 + \cdots + n^2\right)$. Some students realized the middle part of Fig. 5 related to the representation they produced in the previous figure (last column). They worked out the second row of Fig. 5. Quang suggested that students could dissect the figure to link to their previous tasks (KCT). After considerable trial and error, students realized that component was the third T_n, therefore, they matched the figure with $3 \times T_n = 3\left(1^2 + 2^2 + \cdots + n^2\right)$ (first column). Quang then suggested that students consider the figure as a whole (KCT). Students saw them as a rectangle and worked out the size of each rectangle. Students linked the figure to a rectangle size $(1 + 2 + \cdots + n)(2n+1) = \frac{n(n+1)(2n+1)}{2}$ (third column). They ended with finding the sum of $1^2 + 2^2 + \cdots + n^2$ as $\frac{n(n+1)(2n+1)}{6}$. The students then moved on to prove their formula using mathematical induction (intended curriculum).

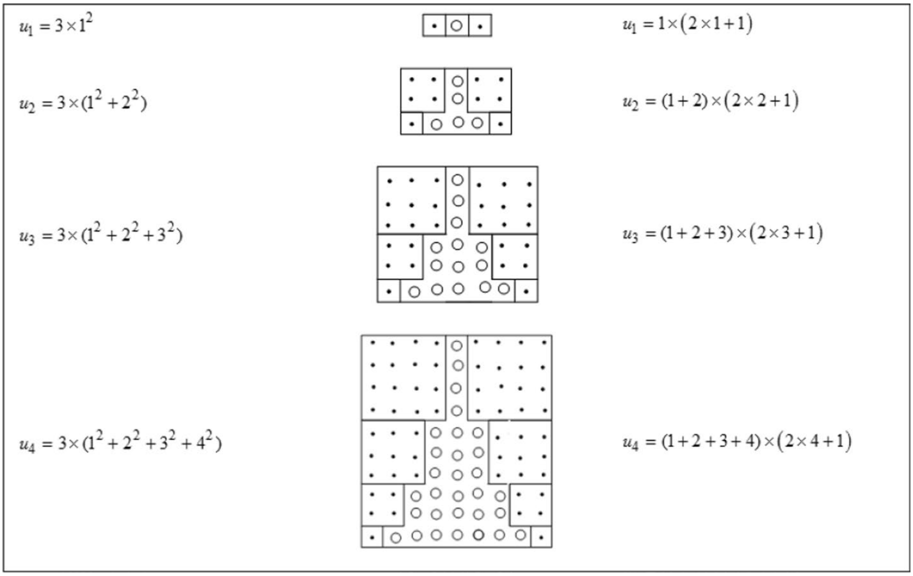


Fig. 5 Geometric representations of T_n

This example shows that KCS and KCT were evident as the teachers attended to multiple representations and approaches when designing and implementing problems, sequenced the problems to help students construct knowledge, evaluated the most appropriate approaches, and responded to students to support them in solving the task. The discussion on challenging tasks with an open-ended and explorative nature motivated them to reconsider their KCS and KCT. Kha commented in the final interview:

> When I care more about my students' voices, I need to put myself in their shoes to think about their ways of thinking and reasoning and their difficulties when learning mathematics. This motivated me to consider how I might expose students to mathematics that makes sense for them. I just cared about how I taught them before.

The cultural transposition allowed the teachers to reflect on their practices when they faced another view about learning discussed throughout the initial meetings.

Teacher beliefs changes

What important mathematics is—Using challenging tasks to expose students to comprehensive learning

The cognitive demands analysis (Table 2) shows a significant shift to higher levels during LS. For example, only 12.5% of the tasks during LS were coded at the *procedures-without-connections* level. Other tasks were at high-cognitive-demand levels, and half of them were *doing mathematics*.

Before LS, the teachers used mainly symbolic representation tasks found in textbooks (received curriculum). The tasks reflected their beliefs that "learning mathematics means mastering procedural skills, which would help students get a high-test

Table 2 Cognitive demands of tasks before and during LS

Classifying tasks	Memorization (%)	Procedures-without-connections (%)	Procedures-with-connections (%)	Doing-mathematics (%)
Before LS (n = 30)	0	63.3	33.3	3.4
During LS (n = 24)	0	12.5	37.5	50

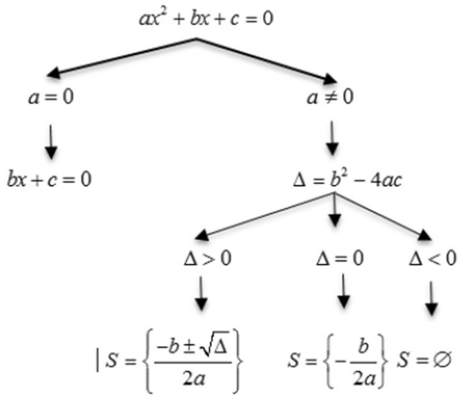


Fig. 6 General process of solving a quadratic equation presented by Quang

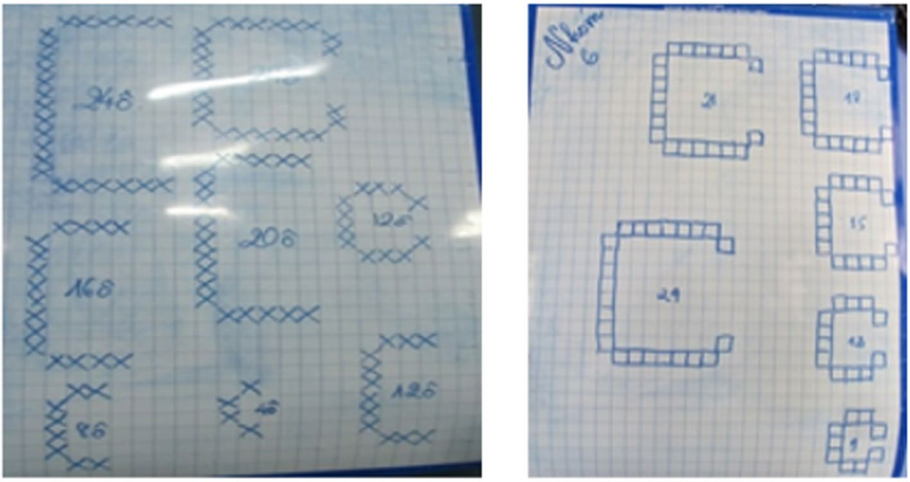
score and pass the entrance examination" (Quang, Initial Discussions) (idealized curriculum). All teachers kept referring to this goal before LS. One example was solving $mx^2 - 2(m-1)x - m - 3 = 0$. Before solving this problem, Quang presented the general process of solving a quadratic equation (Fig. 6) and applied it to solve the equation.

The focus was on following a procedure to solve quadratic equations – no other representations except for symbolic were needed.

During LS, the teachers chose challenging tasks that allowed exploration and multiple approaches and representations and are context-embedded. For example, Binh used the Big C Task in L7 to foster students' conception of arithmetic sequence.

Big C Task. Currently, Big C company has 24 branches in Vietnam. For the 20th anniversary, the executive board announced a competition to design Letter C logo using 24 square tiles representing their branches. Design such a logo with 24 tiles. What if you have different numbers of tiles to design that follows your prototype logo? How many tiles are there? What patterns about the number of tiles? What is the minimum number of tiles needed? What is the difference between two consecutive numbers in your patterns?

The students designed Letter C in multiple correct ways. Once creating their prototype, they tried out different numbers of tiles and minimized the number needed. They generated multiple valid patterns and negotiated and justified why their design was valid. Students convinced their groups and the whole class about their patterns by asking for reasoning and argumentation. Figure 7 shows two groups of students working on the task. Group 1 showed a pattern: the minimum number of tiles was 4 with an increase of 4, and Item 6 had 24 tiles. In Group 6, the minimum number of tiles was 6, increasing 3 each time, and Item 7 had 24 tiles.



a Group 1 work **b** Group 6 work

Fig. 7 Two groups of student work

Instead of just focusing on the symbolic representation, the teachers created tasks to enhance students' mathematical proficiency. In the Big C Task, Binh augmented aspects of understanding and reasoning to introduce arithmetic sequence but did not follow the fluency and symbolic approach emphasized in the textbook (intended curriculum). When the students designed the letter, counted the number of tiles, and derived a symbolic rule, they engaged in understanding as they transferred and made connections among multiple representations. In addition, the task called for generalization and justification to elicit reasoning. At the end of the lesson, Binh introduced arithmetic sequence, building on the student experience.

During the L7 post-discussion, Binh commented that when "connecting arithmetic sequence to the Big C task, I could help them understand its concept by building on specific situations to generalize and connecting with different representations instead of me giving them a definition and they apply it like a procedure." This belief is confirmed again in another lesson, "students need to understand the meaning of probability and use it, not just how to calculate probability." (Binh, Third Discussion L3). Kha also commented that to know mathematics now meant, "not only [being] able to perform symbolic manipulations but also engage in strategic competence to solve problems and use reasoning such as explanation and justification that takes into account mathematical and contextual knowledge" (Third Discussion L1). Consistent with this perspective, Quang highlighted "the use of multiple representations to help students connect with the symbolic form of the sum, $1^2 + 2^2 + \cdots + n^2$" (Second Discussion L6). These comments reflect the achievement of the research goals through the cultural transposition of deliberating their idealized curriculum and the impact of extensive time used to study the enriching teaching materials about mathematical proficiency and tasks.

Fig. 8 Classroom settings before and during LS

How learning takes place—moving toward the sophisticated beliefs

Participating in this LS allowed the teachers to shift their teaching and learning view from a teacher delivering knowledge via one-way communication to student–student interactions in small groups, with the teacher facilitating the discussion. Traditional classroom management enculturated in the Vietnamese teaching reflects in individual students sitting in rows, listening to teachers carefully, and practicing procedures introduced by the teacher as observed before LS (received curriculum). In contrast, all sessions during the LS were conducted in groups (Fig. 8). The class started with the teachers setting up a task, and students worked in groups collaboratively. The class ended with a whole-class discussion, and group work was shared and discussed. During LS, explicit teaching was reduced significantly to less than $^1/_3$ of class time. Instead, the teachers used the time to reconcile the content and connect student understanding to formal knowledge in the curriculum. About $^1/_3$ of class time was for students to work in groups, and the other $^1/_3$ for group presentations to the class to communicate outcomes to teachers and fellow students. Kha confirmed, "I believe students need opportunities to construct their knowledge in the classroom, and they deserve more time to engage in doing mathematics, especially through guided exploration" (Final Interview).

The teachers reconsidered students' roles in the class. Before LS, teachers considered that students best listen attentively (idealized curriculum). Observations during the nine lessons indicated that teachers rarely (or did not) allow students to discuss or collaborate. The teachers called on some students a few times to get an expected answer, but there was no teacher-student dialogue (received curriculum). During LS, they valued student voices by giving them opportunities to develop their knowledge, communicate with their peers, and make their reasoning explicit. Binh reflected: "I think we need to provide students with opportunities to explore mathematics themselves because it draws students' interest in solving the problems" (Planning Discussion L7), and that "group work helps students express their ideas more comfortably, and they can work from their current knowledge, instead of just listen to my way of solving mathematics problems" (Final Interview). This was also evident when Quang revealed his surprise about student engagement in a lesson. "I did not think that a task [related to quadratic functions] can engage students like that. Maybe I was not aware of student thinking before" (Second Discussion L4). Overall, Kha maintained that "I see my role as creating challenging tasks for students to engage in and actively observe and listen to students to facilitate learning" (Second Discussion L6).

The teachers changed their beliefs toward more comprehensive learning and outcomes that help students develop mathematical proficiency. As a result, students are more active in constructing their knowledge. The initial discussions on student-centered learning and how to stimulate student agency challenged their taken-for-granted approach to teaching (received curriculum). Consequently, they changed their class management and noticed new things about student learning overlooked before (cultural transposition).

Discussion

This study examined how high school teacher knowledge and beliefs changed due to the course of LS as the teachers engaged in designing and implementing challenging tasks. The results showed that the teachers displayed aspects of teacher knowledge not observed before LS. They took the time to listen to student approaches even though the approaches were not as precise and efficient as expected (SCK). SCK was shown by creating cognitive conflicts for students (e.g., responses in the Money Distribution Task) to allow them to revise their thinking. This way of responding contrasts with teachers' closing student reasoning and showing the correct approach before LS (received curriculum). The new response could potentially help students develop their agency (Tran & Diez-Palomar, 2021). in the long term and take ownership of learning.

The teachers changed from curriculum implementer (received curriculum) to curriculum transformer. Curriculum knowledge changed in two ways: (a) adapting tasks that meet curriculum intentions and (b) augmenting curriculum intentions to enrich learning opportunities for students. Curriculum development is uncommon for teachers in Vietnam, and this study shows an example of empowering teachers to take active roles in curriculum design. This aspect of curriculum knowledge is not the focus of previous studies (e.g., Corcoran & Pepperell, 2011; Ní Shúilleabháin, 2016; Ní Shúilleabháin & Clivaz, 2017); the focus on the nature of intended curriculum provides a richer account of teacher change.

The teachers changed their KCS in how they anticipated student conceptions and misconceptions. Previous studies (e.g., Pang, 2016) found that teachers feel more confident about knowing and listening to their students more attentively at the primary school level. This study confirms these findings in the high school context. The findings also confirm Ní Shúilleabháin's study (2016), showing more evidence of KCS when primary teachers engage in LS. The teachers also change KCT during LS. This transition was evident in the choice of tasks and task sequencing to help students move to more sophisticated mathematics. In addition, they acknowledged the choice of representations/approaches suitable for their students and were aware of the advantages and disadvantages of their choices. This study confirms Ní Shúilleabháin's (2016) findings, revealing that primary teachers show KCT more often when progressing through LS.

The teachers shifted towards a comprehensive view of important mathematics. They shifted from the beliefs of *teachers-as-knowledge-transmitters* (idealized curriculum) to students' active engagement in constructing their knowledge, aligning with a more social constructivist approach. This change was evident in their use of challenging tasks with higher cognitive demands and greater attention to student learning processes during LS. Their original beliefs were firmly embedded in the Vietnamese education system's culture, emphasizing high-stake exams focusing mainly on mathematical techniques with symbolic representations (intended and received curriculum). The change reflects a significant teacher-student power shift in the Vietnamese context (Saito & Atencio, 2013).

Kha's first experience with LS changed his beliefs significantly (cultural transposition). Before LS, he insisted on his confidence in teaching efficiently to help students succeed in high-stake exams by mastering procedural skills. During LS, Kha learned to listen to students instead of interrupting their thinking. One possible explanation is that Kha showed his passion for teaching and was eager to reform his practice. By building rapport with the researchers, Kha was willing to try new things, saw their effectiveness, and changed his beliefs. The two remaining teachers used to listen to students before offering their solutions in their first LS. Participating in this LS reinforced their social constructivist beliefs. They became more patient, letting students finish their responses, and created cognitive conflicts for students to help further knowledge construction. This finding confirms research about effective professional development that is sustainable and ongoing (Perry & Lewis, 2008).

Previous research estimated that teachers need to make at least 30 observations within their school to change their pedagogical practices, and it is hard to change those within the context of one or two small teams (Sato, 2006; Sato & Sato, 2003). This study does not support this finding (14 observations). A possible reason for the difference might be the opportunity for cultural transposition in this study for the experienced teachers. At the same time, the choice of participants points to the limitation of the study as we cannot argue that the case is typical of Vietnamese teachers. The teachers were committed and motivated to participate in LS, and the team spent significant time (nine months, both formal and informal data collection) with many deliberations. In addition, at that stage of their career, they were less constrained by performance pressures from the school but were more into exciting practices. Future studies could focus on how teachers with less knowledge and in early career stages change their beliefs and practices, possibly investigating starting points to motivate these teachers to adopt reformed teaching.

We argue that it was not adequate to ask how LS adopted to a different culture changes teacher knowledge and beliefs. However, researchers need to attend to essential features when adopting LS to support teacher changes. By doing this, we revisit the question raised by Stigler and Hiebert (2016) about whether LS should be changed or LS should be used as a tool for changing beliefs when adopted into a different culture. In this case, LS is not used as is but adapted, informed by researchers' considerations. It is not just adopting the form of LS stages (setting goals, planning, observing lessons, discussions), but the essence of the stages is crucial to maintain.

Essence to maintain when adopting LS

Japanese LS considers long-term goals for student learning and development. This step creates the basis for decisions in all LS stages (Fuijii, 2014; Lewis, 2002). Japan's educational goals are often developing well-rounded students, not merely an academic focus. Notwithstanding, the key feature is that the goals must be related to student thinking, not just performance. In our case, the goals related to comprehensive learning and outcomes of mathematical proficiency.

Second, LS study teams study curriculum materials collaboratively in the lesson planning phase. In the Japanese professional tradition, teachers move to work in different schools quite regularly and participate in multiple professional associations (Fernandez & Yoshida, 2004). This affords them opportunities to experience multiple approaches to teaching, which they could draw on to address the goals. Different from the Japanese context, Vietnamese teachers do not have such resources available. In this case, the researchers helped play the bridging role by introducing resources and lenses to reconsider their teaching practice. This study also supports the point raised that a successful LS requires a more knowledgeable expert (Takahashi & McDougal, 2018). Also, in the planning stage, Fujii (2018) highlighted the problem-solving structure as the second wheel of LS cart. Our study adopted the use of challenging tasks to help students achieve the comprehensive learning. Similarly, da Ponte et al. (2018) used the *explorative approach* in the Portuguese context. We argued that such a lens on student mathematical learning in planning is crucial in a successful LS.

Third, Fujii (2014) highlighted that observations should focus on teaching and students' learning, not teachers with evaluative purposes. However, as being open to classroom observation might not be a custom in other cultures, we consider making observation purposes clear crucial to successful LS. We did this by building rapport between the researchers and teachers, highlighting the purpose of observations in the discussion, and clearly articulating and actioning them throughout the LS to address the teacher-teacher and teacher-researcher power relationships.

Fourth, comprehensive details of post-lesson discussions are reported in previous LSs in Japan (Fernandez & Yoshida, 2004). In our case, we did not spend that much time on discussions—only 30 min after each lesson. Echoing previous researchers (Fujii, 2014, 2018; Takahashi & McDougal, 2018), we argue that ongoing reflection on research goals, how to achieve them, and if the lessons help achieve them is essential.

A final point to consider is who leads an LS. In Japan, the problem comes from teachers, so LS is teacher-led. However, we see several researcher-led examples (e.g., da Ponte et al., 2018), as in our case. Hence, how the problem posed by researchers and teachers intersect can be one of the criteria for successful LS, which relates to the collaborative nature of LS. In Japan, teachers are treated equally, and experts work with the teachers to solve the problem at school (Fernandez & Yoshida, 2004). Such power relationships and norms need to be addressed (Saito & Atencio, 2013).

Conclusion

This study adds to LS literature about teacher knowledge and beliefs changes with explicit deliberations of cultural factors when adopting LS. We challenged teachers' idealized curriculum, created opportunities to analyze their intended curriculum, and supported them in enhancing their received curriculum through cultural transposition. They reflected on practices (e.g., LS process and outside resources) adopted from different cultures that meet their educational intentionality. It was crucial when the researchers took significant time to discuss the view (comprehensive learning) and provided means (challenging tasks) to make the Q7research goals achievable. This process happened through the deliberations of power relationships (between researcher-teacher, teacher-teacher, and teacher-student). Formal evidence has confirmed the impacts of nine months of implementing LS on the teacher knowledge and beliefs observed in the 14 sessions. When addressing the equal relationship between the researchers and teachers, we argue that the teachers did not have to say or do things to please the researchers. However, we wonder which impacts of LS will stay with them after the life of this LS. Implementing the reformed curriculum might provide an opportunity to research the impact of LS on teacher knowledge and beliefs at the scale. Doing this would allow researchers to examine if and how the practice implemented during LS maintains in the long term.

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