Calc. Var. (2023) 62:98
https://doi.org/10.1007/s00526-023-02442-5

Calculus of Variations

Constant rank theorems for curvature problems via a viscosity approach

Paul Bryan[1] · Mohammad N. Ivaki[2] · Julian Scheuer[3]

Received: 9 March 2021 / Accepted: 12 January 2023

Abstract

An important set of theorems in geometric analysis consists of constant rank theorems for a wide variety of curvature problems. In this paper, for geometric curvature problems in compact and non-compact settings, we provide new proofs which are both elementary and short. Moreover, we employ our method to obtain constant rank theorems for homogeneous and non-homogeneous curvature equations in new geometric settings. One of the essential ingredients for our method is a generalization of a differential inequality in a viscosity sense satisfied by the smallest eigenvalue of a linear map Brendle et al. (Acta Math 219:1–16, 2017) to the one for the subtrace. The viscosity approach provides a concise way to work around the well known technical hurdle that eigenvalues are only Lipschitz in general. This paves the way for a simple induction argument.

Mathematics Subject Classification 35J60 · 35R01 · 52A05 · 58J35

1 Introduction

We introduce a viscosity approach to a broad class of constant rank theorems. Such theorems say that under suitable conditions a positive semi-definite bilinear form on a manifold, that satisfies a uniformly elliptic PDE, must have constant rank in the manifold. In this sense, constant rank theorems can be viewed as a strong maximum principle for tensors. The aim of

Communicated by A. Malchiodi.

✉ Julian Scheuer
scheuer@math.uni-frankfurt.de

Paul Bryan
paul.bryan@mq.edu.au

Mohammad N. Ivaki
mohammad.ivaki@tuwien.ac.at

1 Department of Mathematics, Macquarie University, Sydney, NSW 2109, Australia

2 Institut für Diskrete Mathematik und Geometrie, Technische Universität Wien, 1040 Wien, Austria

3 Goethe-Universität Frankfurt, Institut für Mathematik, Robert-Mayer-Str. 10, 60325 Frankfurt, Germany

Published online: 02 February 2023

this paper is two-fold. Firstly, we want to present a new approach to constant rank theorems. It is based on the idea that the subtraces of a linear map satisfy a linear differential inequality in a viscosity sense and the latter allows to use the strong maximum principle. This avoids the use of nonlinear test functions, as in [5], as well as the need for approximation by simple eigenvalues, as in [24]. Secondly, we show that the simplicity of this method allows us to obtain previously undiscovered constant rank theorems, in particular for non-homogeneous curvature type equations. To illustrate the idea, we give a new proof for the following full rank theorem for the Christoffel-Minkowski problem, a.k.a. the σ_k-equation.

Theorem 1.1 *[14, Theorem 1.2]*
Let $(\mathbb{S}^n, g, \nabla)$ be the unit sphere with standard round metric and connection. Suppose $n \geq 2$, $1 \leq k \leq n-1$ and $0 < s, \phi \in C^\infty(\mathbb{S}^n)$ satisfy

$$\nabla^2 \phi^{-\frac{1}{k}} + \phi^{-\frac{1}{k}} g \geq 0,\ 0 \leq r := \nabla^2 s + sg \in \Gamma_k,\ \sigma_k(r) = \phi,$$

where σ_k is k-th symmetric polynomial of eigenvalues of r with respect to g and Γ_k is the k-th Garding cone. Then r is positive definite.

Proof For convenience, we define

$$F = \sigma_k^{1/k},\ f = \phi^{1/k}.$$

Then $F = f$. Differentiate F and use Codazzi, where a semi-colon stands for covariant derivatives and we use the summation convention:

$$\begin{aligned} f_{;ab} = F_{;ab} &= F^{ij,kl} r_{ij;a} r_{kl;b} + F^{ij} r_{ij;ab} \\ &= F^{ij,kl} r_{ij;a} r_{kl;b} + F^{ij} r_{ab;ij} - r_{ab} F^{ij} g_{ij} + F g_{ab}. \end{aligned}$$

Hence the tensor r satisfies the elliptic equation

$$F^{ij} r_{ab;ij} = F^{ij} g_{ij} r_{ab} - f g_{ab} - F^{ij,kl} r_{ij;a} r_{kl;b} + f_{;ab}.$$

Now we deduce an inequality for the lowest eigenvalue of r, λ_1, in a viscosity sense. Let ξ be a smooth lower support at $x_0 \in \mathbb{S}^n$ for λ_1 and let $D_1 \geq 1$ denote the multiplicity of $\lambda_1(x_0)$. Denote by Λ the complement of the set $\{i, j, k, l > D_1\}$ in $\{1, \ldots, n\}^4$. We use a relation between the derivatives of ξ and r, and the inverse concavity of F (cf. [3, Lemma 5], [2]) to estimate in normal coordinates at x_0:

$$\begin{aligned} F^{ij} \xi_{;ij} &\leq F^{ij} r_{11;ij} - 2 \sum_{j > D_1} \frac{F^{ii}}{\lambda_j} (r_{ij;1})^2 \\ &= -F^{ij,kl} r_{ij;1} r_{kl;1} - 2 \sum_{j > D_1} \frac{F^{ii}}{\lambda_j} (r_{ij;1})^2 + F^{ij} g_{ij} r_{11} - (f - f_{;11}) \\ &= - \sum_{i,j,k,l > D_1} F^{ij,kl} r_{ij;1} r_{kl;1} - 2 \sum_{j > D_1} \frac{F^{ii}}{\lambda_j} (r_{ij;1})^2 - (f - f_{;11}) \\ &\quad - \sum_{(i,j,k,l) \in \Lambda} F^{ij,kl} r_{ij;1} r_{kl;1} + F^{ij} g_{ij} r_{11} \\ &\leq -\left(f + 2 f^{-1} f_{;1}^2 - f_{;11}\right) + c |\nabla \xi| + F^{ij} g_{ij} \xi \\ &\leq F^{ij} g_{ij} \xi + c |\nabla \xi|. \end{aligned}$$

Then the strong maximum principle for viscosity solutions (cf. [4]) implies that the set $\{\lambda_1 = 0\}$ is open. Hence, if λ_1 was zero somewhere, it would be zero everywhere. However, we know it is positive somewhere, since at a minimum of s we have $r > 0$. □

The proof may be summarized as follows: apply the viscosity differential inequality from [3, Lemma 5] for the minimum eigenvalue λ_1 of the spherical hessian of r. Then the strong maximum principle shows that since there is a point at which $\lambda_1 > 0$ we must have $\lambda_1 > 0$ everywhere and hence the hessian has constant, full rank. A similar argument was employed in [19] for obtaining curvature estimates along a curvature flow.

Our main approach here is to generalize the viscosity inequality to the subtrace $G_m = \lambda_1 + \cdots + \lambda_m$, the sum of the first m eigenvalues. See Lemma 3.2 below. Then by induction, we are able to show that if $\lambda_1 = \cdots = \lambda_{m-1} \equiv 0$, the strong maximum principle shows that either $G_m > 0$ or $G_m \equiv 0$ to conclude constant rank theorems (in short, CRT).

We say a symmetric 2-tensor α is Codazzi, provided $\nabla\alpha$ is totally symmetric. Here is a prototypical CRT:

Theorem 1.2 *(Homogeneous CRT) [10, Theorem 1.4] Suppose α is a Codazzi, non-negative, symmetric 2-tensor on a connected Riemannian manifold (M, g, ∇) satisfying $\Psi(\alpha, g) = f > 0$, where Ψ is one-homogeneous, inverse concave and strictly elliptic (see Definition 1.3 and Assumption 2.1), and we have $\nabla^2 f^{-1} + \tau f^{-1} g \geq 0$ with $\tau(x)$ the minimum sectional curvature at x. Then α is of constant rank,*[1]

We state a more general version of CRT that allows the curvature function to be non-homogeneous and to explicitly depend on $x \in M$ as well. To state the result, we need a few definitions.

Definition 1.3 Let $\Gamma \subset \mathbb{R}^n$ be an open, convex cone such that

$$\Gamma_+ := \{\lambda \in \mathbb{R}^n : \lambda_i > 0 \quad \forall 1 \leq i \leq n\} \subset \Gamma.$$

Suppose (M^n, g) is a smooth Riemannian manifold. A C^∞-function

$$F : \Gamma \times M \to \mathbb{R}$$

is said to be a *pointwise curvature function*, if for any $x \in M$, the map $F(\cdot, x)$ is symmetric under permutation of the λ_i. Such a map generates another map (denoted by F again) given by

$$\begin{aligned} F : \mathcal{U} \subset \mathbb{R}^{n\times n}_{\mathrm{sym}} \times \mathbb{R}^{n\times n}_{\mathrm{sym}} \times M &\to \mathbb{R} \\ (\alpha, g, x) &\mapsto F(\alpha, g, x) = F(\lambda, x), \end{aligned}$$

where $\mathcal{U}$ is a suitable open set and $\lambda = (\lambda_i)_{1\leq i\leq n}$ are the eigenvalues of α with respect to g, or equivalently, the eigenvalues of the linear map $\alpha^\sharp$ defined by $g(\alpha^\sharp(v), w) = \alpha(v, w)$. Note that F can be considered as a map on an open set of $\mathbb{R}^{n\times n}$ via $F(\alpha^\sharp, x) = F(\alpha, g, x)$; see [23].

With the convention $\alpha^i_j = g^{ik}\alpha_{kj}$, where (g^{kl}) is the inverse of (g_{kl}):

$$F^i_j := \frac{\partial F}{\partial \alpha^j_i}, \quad F^{ij} := \frac{\partial F}{\partial \alpha_{ij}}, \quad F^{ij,kl} := \frac{\partial F}{\partial \alpha_{ij}\partial \alpha_{kl}}.$$

[1] Note that in [10, Theorem 1.4] $F := -\Psi^{-1}$.

Note that $F^{ij} = F^i_k g^{kj}$. Moreover, F is said to be

(i) *Strictly elliptic*, if $F^{ij}\eta_i\eta_j > 0 \quad \forall 0 \neq \eta \in \mathbb{R}^n$,
(ii) *One-homogeneous*, if for all $x \in M$, $F(\cdot, x)$ is homogeneous of degree one, and
(iii) *Inverse concave*, if the map $\tilde{F} \in C^\infty(\Gamma_+ \times M)$ defined by

$$\tilde{F}(\lambda_i, x) = -F(\lambda_i^{-1}, x) \quad \text{is concave.}$$

We use the convention for the Riemann tensor from [11]. For a Riemannian or Lorentzian manifold (M, g, ∇),

$$\mathrm{Rm}(X, Y)Z := \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z - \nabla_{[Y,X]} Z$$

and we lower the upper index to the first slot:

$$\mathrm{Rm}(W, X, Y, Z) = g(W, \mathrm{Rm}(X, Y)Z).$$

The respective local coordinate expressions are (R^m_{jkl}) and (R_{ijkl}).

Definition 1.4

(i) A pointwise curvature function $F \in C^\infty(\Gamma \times M)$ is Φ-inverse concave for some

$$\Phi \in C^\infty(\Gamma \times M, T^{4,0}(M)),$$

provided at all $\beta > 0$ we have

$$F^{ij,kl}\eta_{ij}\eta_{kl} + 2F^{ik}\tilde{\beta}^{jl}\eta_{ij}\eta_{kl} \geq \Phi^{ij,kl}\eta_{ij}\eta_{kl},$$

where $\tilde{\beta}^{ik}\beta_{kj} = \delta^i_j$.

(ii) For $\alpha \in \Gamma$ we define a curvature-adjusted modulus of Φ-inverse concavity,

$$\omega_F(\alpha)(\eta, v) = \Phi^{ij,kl}\eta_{ij}\eta_{kl} + D^2_{xx}F(v, v) + 2D_{x^k}F^{ij}\eta_{ij}v^k + \mathrm{tr}_g\,\mathrm{Rm}(\alpha^\sharp, v, D_{\alpha^\sharp}F, v),$$

where D denotes the product connection on $\mathbb{R}^{n\times n} \times M$. Here the curvature term denotes contracting the vector parts of the $(1, 1)$ tensors $\alpha^\sharp = \alpha^i_j$, $D_{\alpha^\sharp}F = F^k_l$ with the Riemann tensor and tracing the resulting bilinear form with respect to the metric so that

$$\mathrm{tr}_g\,\mathrm{Rm}(\alpha^\sharp, e_m, D_{\alpha^\sharp}F, e_m) = g^{jl}\alpha^i_j F^k_l R_{imkm}.$$

Remark 1.5 If $(A, x) \mapsto -F(A^{-1}, x)$ is concave (i.e., F is inverse concave), then we take $\Phi = 0$ and for all (η, v) we have

$$\omega_F(\eta, v) \geq \mathrm{tr}_g\,\mathrm{Rm}(\alpha^\sharp, v, D_{\alpha^\sharp}F, v).$$

On several occasions, where there is a homogeneity condition on F, we will be able to choose a good positive Φ that allows to relax assumptions on the other variables of the operator F; see Sect. 2.

We state the main result of the paper which contains Theorem 1.2 as a special case.

Theorem 1.6 *(Non-homogeneous CRT) Let (M, g, ∇) be a connected Riemannian manifold and Γ an open, convex cone containing Γ_+. Suppose $F \in C^\infty(\Gamma \times M)$ is a Φ-inverse concave, strictly elliptic pointwise curvature function. Let α be a Codazzi, non-negative, symmetric 2-tensor with eigenvalues in Γ and*

$$F(\alpha^\sharp, \cdot) = 0 \quad \text{on } M.$$

Suppose for all $\Omega \Subset M$ there exists a positive constant $c = c(\Omega)$, such that for all eigenvectors v of $\alpha^\sharp$ there holds

$$\omega_F(\alpha)(\nabla_v \alpha, v) \geq -c(\alpha(v, v) + |\nabla \alpha(v, v)|).$$

Then α is of constant rank.

Remark 1.7 It might seem more natural to replace the condition on ω_F with the condition

$$\omega_F(\alpha)(\eta, v) \geq -c(\alpha(v, v) + |\nabla \alpha(v, v)|)$$

for every η and all v. Indeed such a condition certainly leads to constant rank theorems since taking in particular $\eta = \nabla_v \alpha$, and v and eigenvector, we may apply Theorem 1.6. However, the requirement holding for all η, v is too restrictive for applications such as in Theorem 1.2. See the proof in Sect. 2 below where the required inequality is only proved to hold for $\eta = \nabla_v \alpha$ and v an eigenvector.

An application of Theorem 1.6 to a non-homogeneous curvature problem is given in Theorem 2.4. Such a result was declared interesting in [16]. The full results are listed in Sect. 2.

CRT (also known as the microscopic convexity principle) was initially developed in [9] in two-dimensions for convex solutions of semi-linear equations, $\Delta u = f(u)$ using the maximum principle and the homotopy deformation lemma. The result was extended to higher dimensions in [20]. The continuity method combined with a CRT yields existence of strictly convex solutions to important curvature problems. For example, a CRT was an important ingredient in the study of prescribed curvature problems such as the Christoffel-Minkowski problem and prescribed Weingarten curvature problem [12, 14, 15]. Later, general theorems for fully nonlinear equations were obtained in [5, 10] under the assumption that $A \mapsto F(A^{-1})$ is locally convex. These approaches are based on the observation that a non-negative definite matrix valued function A has constant rank if and only if there is a ℓ such that the elementary symmetric functions satisfy $\sigma_\ell \equiv 0$ and $\sigma_{\ell-1} > 0$. To apply this observation requires rather delicate, long computations and the introduction of clever auxiliary functions. The difficulties are at least in part due to the non-linearity of σ_ℓ. An alternative approach was taken in [24, 25], using a linear combination of lowest m eigenvalues, which provides a linearity advantage at the expense of losing regularity compared with σ_ℓ. The authors get around this difficulty by perturbing A so that the eigenvalues are distinct (thus restoring regularity) but then using an approximation argument. Our approach based on the viscosity inequality shows that G_m enjoys sufficient regularity to apply the strong maximum principle and this suffices to obtain a self-contained proof of the CRT.

We remark here, that our method is capable of reproving the results in [5, 10], namely with the help of Theorem 3.4 it is possible to prove that any convex solution u to

$$H(\nabla^2 u, \nabla u, u, \cdot) = 0$$

has constant rank under the assumption that

$$(A, u, x) \mapsto -H(A^{-1}, p, u, x)$$

is concave for fixed p. This result does not follow from Theorem 1.6, but by using a suitably redefined ω_F in Theorem 3.4, this result follows in the same way as Theorem 1.6. Here we rather want to focus on geometric problems.

We proceed as follows: In Sect. 2 we collect and prove direct applications of Theorem 1.6. In Sect. 3 we prove the viscosity inequality satisfied by the subtrace, a result that is of interest by itself. After some further corollaries, we conclude with the proof of Theorem 1.6.

2 Applications

In this section, we collect a few applications of Theorem 1.6. We fix an assumption that we need on several occasions.

Assumption 2.1 Let Γ be as in Definition 1.3.

(i) $\Psi \in C^\infty(\Gamma)$ is a positive, strictly elliptic, homogeneous function of degree one and normalized to $\Psi(1,\ldots,1) = n$,
(ii) Ψ is inverse concave.

Recall that such a function Ψ at invertible arguments β satisfies

$$\Psi^{ij,kl}\eta_{ij}\eta_{kl} + 2\Psi^{ik}\tilde{\beta}^{jl}\eta_{ij}\eta_{kl} \geq \frac{2}{\Psi}(\Psi^{ij}\eta_{ij})^2 \tag{2.1}$$

for all symmetric (η_{ij}); see for example [2].

In order to facilitate notation, for covariant derivatives we use semi-colons, e.g., the components of the second derivative $\nabla^2 T$ of a tensor are denoted by

$$T_{;ij} = \nabla_{\partial_j}\nabla_{\partial_i} T - \nabla_{\nabla_{\partial_j}\partial_i} T.$$

First, we illustrate how Theorem 1.2 follows from Theorem 1.6.

Proof of Theorem 1.2 We define $F = \Psi - f$. In view of (2.1) and Definition 1.4, we have

$$\Phi^{ij,kl}\eta_{ij}\eta_{kl} = 2\Psi^{-1}(\Psi^{ij}\eta_{ij})^2.$$

Let $x_0 \in M$ and $(e_i)_{1\leq i\leq n}$ be an orthonormal basis of eigenvectors for $\alpha^\sharp(x_0)$. In the associated coordinates, we calculate

$$\begin{aligned}\omega_F(\alpha)(\nabla_{e_m}\alpha, e_m) &\geq 2f^{-1}f_{;m}^2 - f_{;mm} + \tau\Psi^{kr}\alpha_k^l(g_{lr} - g_{lm}g_{rm})\\ &\geq 2f^{-1}f_{;m}^2 - f_{;mm} + \tau f - c\alpha_{mm}\\ &= f^2\left((f^{-1})_{;mm} + \tau f^{-1}\right) - c\alpha_{mm},\end{aligned}$$

for some constant c. Hence the claim follows from Theorem 1.6. □

For a C^2 function ζ on a space (M, g) of constant curvature τ_M,

$$r_M[\zeta] := \tau_M\nabla^2\zeta + g\zeta.$$

The next theorem contains the full rank theorems from [14, 15, 17] as special cases.

Theorem 2.2 *(L_p-Christoffel-Minkowski Type Equations) Suppose (M, g, ∇) is either the hyperbolic space $\mathbb{H}^n$ or the sphere $\mathbb{S}^n$ equipped with their standard metrics and connections. Let Ψ satisfy Assumption 2.1, $k \geq 1$, $p \neq 0$ and $0 < \phi, s \in C^\infty(M)$ satisfy*

$$r_M[s] \geq 0,\ s^{1-p}\Psi^k(r_M[s]) = \phi.$$

If either

$$\begin{cases} r_{\mathbb{H}^n}[\phi^{-\frac{1}{p+k-1}}] \geq 0,\ p + k - 1 < 0,\\ or\\ r_{\mathbb{S}^n}[\phi^{-\frac{1}{p+k-1}}] \geq 0,\ \ p \geq 1,\end{cases}$$

then $r_M[s]$ is of constant rank. In particular, if $M = \mathbb{S}^n$, then we have

$$r_{\mathbb{S}^n}[s] > 0.$$

Proof Note that $\alpha = r_M[s]$ is a Codazzi tensor. We define

$$F = \Psi - \left(\phi s^{p-1}\right)^{\frac{1}{k}} = \Psi - f.$$

For simplicity, we rewrite $f = us^{q-1}$, where $u = \phi^{\frac{1}{k}}$ and $q = \frac{p+k-1}{k}$.

As in the proof of Theorem 1.2, we have

$$\omega_F(\alpha)(\nabla_{e_m}\alpha, e_m) \geq 2f^{-1}f_{;m}^2 - f_{;mm} + \tau_M f - c\alpha_{mm}.$$

Now we calculate

$$\begin{aligned} f_{;mm} - 2f^{-1}f_{;m}^2 - \tau_M f = &-\left(\tau_M qu + \frac{q+1}{q}\frac{(u_{;m})^2}{u} - u_{;mm}\right)s^{q-1} \\ &- \frac{q-1}{q}\left(\frac{u_{;m}}{u} + q\frac{s_{;m}}{s}\right)^2 f \\ &+ \tau_M(q-1)fs^{-1}r_M[s]_{mm}. \end{aligned}$$

Therefore, if either $r_{\mathbb{H}^n}[u^{-\frac{1}{q}}] \geq 0,\ q < 0$ or $r_{\mathbb{S}^n}[u^{-\frac{1}{q}}] \geq 0,\ q \geq 1$, then

$$f_{;mm} - 2f^{-1}f_{;m}^2 - \tau_M f \leq c\alpha_{mm},$$

for some $c \geq 0$. The result follows from Theorem 1.6. Since $\mathbb{S}^n$ is compact, at some point y we must have $r_{\mathbb{S}^n}[s](y) > 0$. Hence $r_{\mathbb{S}^n}[s] > 0$ on M. □

Remark 2.3 Let $M = x(\Omega), x : \Omega \hookrightarrow \mathbb{R}^{n,1}$ be a co-compact, convex, spacelike hypersurface. The support function of M, $s : \mathbb{H}^n \to \mathbb{R}$, is defined by $s(z) = \inf\{-\langle z, p\rangle;\ p \in M\}$, and $r_{\mathbb{H}^n}[s]$ is non-negative definite. Moreover, if $r > 0$, then the eigenvalues of r with respect to g are the principal radii of curvature; e.g., [1]. Therefore, the curvature problem stated in the previous theorem can be considered as an L_p-Christoffel-Minkowski type problem in the Minkowski space.

In [16] the authors asked the validity of CRT for non-homogeneous curvature problems. In this respect we have the following theorem. First we have to recall the definition of the *Garding cones*:

$$\Gamma_\ell = \{\lambda \in \mathbb{R}^n : \sigma_1(\lambda) > 0, \ldots, \sigma_\ell(\lambda) > 0\},$$

where σ_k is the k-th elementary symmetric polynomial of the λ_i. In Γ_ℓ, all σ_k, $1 \leq k \leq \ell$, are strictly elliptic and the $\sigma_k^{1/k}$ are inverse concave, see [18]. For a cone $\Gamma \subset \mathbb{R}^n$, on a Riemannian manifold (M, g) a bilinear form α is called Γ*-admissible*, if its eigenvalues with respect to g are in Γ.

Theorem 2.4 *(A non-homogeneous curvature problem) Let $\phi > 0$ be a smooth function on $(\mathbb{S}^n, g, \nabla)$ with*

$$\phi g - \nabla^2\phi \geq 0,$$

$\psi_\ell \equiv 1$ and $0 < \psi_k \in C^\infty(\mathbb{S}^n)$ for $1 \leq k \leq \ell - 1$ satisfy[2]

$$\nabla^2\psi_k - \frac{k}{k+1}\frac{\nabla\psi_k \otimes \nabla\psi_k}{\psi_k} + (k-1)\psi_k \geq 0.$$

[2] Note this forces ψ_1 to be constant.

Let α be a Γ_ℓ-admissible, Codazzi, non-negative, symmetric 2-tensor, such that

$$\sum_{k=1}^{\ell} \psi_k(x)\sigma_k(\alpha, g) = \phi(x).$$

Then α is of constant rank. In particular, when $\alpha = r_{\mathbb{S}^n}[s] \geq 0$ for some positive function $s \in C^\infty(\mathbb{S}^n)$, then in fact we have $\alpha > 0$.

Proof The result follows quickly from Theorem 1.6. We define

$$F(\alpha, g, x) = \sum_{k=1}^{\ell} \psi_k(x)\sigma_k(\alpha, g) - \phi(x).$$

Since $\sigma_k^{1/k}$ is inverse concave and 1-homogeneous, F is Φ-inverse concave with

$$\Phi^{pq,rs}\eta_{pq}\eta_{rs} := \sum_{k=1}^{\ell} \frac{k+1}{k}\psi_k \frac{\sigma_k^{pq}\sigma_k^{rs}}{\sigma_k}\eta_{pq}\eta_{rs}.$$

Let $x_0 \in M$ and $(e_i)_{1\leq i\leq n}$ be an orthonormal basis of eigenvectors for $\alpha^\sharp(x_0)$. Now using

$$F^{kr}\alpha_k^l R_{liri} = F^{kr}\alpha_k^l(g_{lr}g_{ii} - g_{li}g_{ri}) = \sum_{k=1}^{\ell} k\psi_k\sigma_k - F^{ii}\alpha^{ii},$$

we deduce

$$\begin{aligned}
&\omega_F(\alpha)(\nabla_{e_i}\alpha, e_i) + \phi_{;ii} \\
&\geq \sum_{k=1}^{\ell}\left(\sigma_k\psi_{k;ii} + 2\psi_{k;i}\sigma_{k;i} + \frac{k+1}{k}\frac{\psi_k}{\sigma_k}(\sigma_{k;i})^2 + k\psi_k\sigma_k\right) - c\alpha_{ii} \\
&\geq \sum_{k=1}^{\ell}\left(\psi_{k;ii} - \frac{k}{k+1}\frac{(\psi_{k;i})^2}{\psi_k} + (k-1)\psi_k + \psi_k\right)\sigma_k - c\alpha_{ii} \\
&= \sum_{k=1}^{\ell-1}\left(\psi_{k;ii} - \frac{k}{k+1}\frac{(\psi_{k;i})^2}{\psi_k} + (k-1)\psi_k\right)\sigma_k + \phi + (\ell-1)\sigma_\ell - c\alpha_{ii}.
\end{aligned}$$

Therefore, $\omega_F(\alpha)(\nabla_{e_i}\alpha, e_i) + c\alpha_{ii}$ is non-negative for some constant c. □

Let $(N, \bar{g}, \bar{D})$ be a simply connected Riemannian or Lorentzian spaceform of constant sectional curvature τ_N. That is, N is either the Euclidean space $\mathbb{R}^{n+1}$, the sphere $\mathbb{S}^{n+1}$, the hyperbolic space $\mathbb{H}^{n+1}$ with respective sectional curvature $0, 1, -1$ or the $(n+1)$-dimensional Lorentzian de Sitter space $\mathbb{S}^{n,1}$ with sectional curvature 1.

Assume $M = x(\Omega)$ given by $x\colon \Omega \hookrightarrow N$ is a connected, spacelike, locally convex hypersurface of N and

$$f \in C^\infty(M \times \mathbb{R}_+ \times \tilde{N}),$$

where $\tilde{N}$ denotes the *dual manifold* of N, i.e.,

$$\tilde{\mathbb{R}}^{n+1} = \mathbb{S}^n, \quad \tilde{\mathbb{S}}^{n+1} = \mathbb{S}^{n+1}, \quad \tilde{\mathbb{H}}^{n+1} = \mathbb{S}^{n,1}, \quad \tilde{\mathbb{S}}^{n,1} = \mathbb{H}^{n+1}.$$

Here f is extended as a zero homogeneous function to the ambient space. We write ν, h, s for the future directed (timelike) normal, the second fundamental form and the support function of M, respectively (cf. [7, 8]). The eigenvalues of h with respect to the induced metric on Σ are ordered as $\kappa_1 \leq \cdots \leq \kappa_n$ and we write in short

$$\kappa = (\kappa_1, \ldots, \kappa_n).$$

The Gauss equation (cf. [11, (1.1.37)]) relates extrinsic and intrinsic curvatures,

$$\begin{aligned} R_{ijkl} &= \sigma(h_{ik}h_{jl} - h_{il}h_{jk}) + \overline{\mathrm{Rm}}(x_{;i}, x_{;j}, x_{;k}, x_{;l}) \\ &= \sigma(h_{ik}h_{jl} - h_{il}h_{jk}) + \tau_N(\bar{g}_{ik}\bar{g}_{jl} - \bar{g}_{il}\bar{g}_{jk}), \end{aligned} \tag{2.2}$$

where $\sigma = \bar{g}(\nu, \nu)$ and the second fundamental form is defined by

$$\bar{D}_X Y = \nabla_X Y - \sigma h(X, Y)\nu.$$

Theorem 2.5 *Let $(N, \bar{g}, \bar{D})$ be one of the spaces above and let Ψ satisfy Assumption 2.1. Let M be a connected, spacelike, locally convex and Γ-admissible hypersurface such that*

$$\Psi(\kappa) = f(x, s, \nu),$$

where $0 < f \in C^\infty(M \times \mathbb{R}_+ \times \tilde{N})$ and

$$\bar{D}^2_{xx} f^{-1} + \tau_N f^{-1}\bar{g} \geq 0.$$

Then the second fundamental form of M is of constant rank.

Proof Define $F(h, g, x) = \Psi(h^\sharp) - f(x, s(x), \nu(x))$. Let $x_0 \in M$ and $(e_i)_{1\leq i\leq n}$ be an orthonormal basis of eigenvectors for $h^\sharp(x_0)$. Now in view of Theorem 1.6, the claim follows from [8, p. 15] and a computation using the Gauss equation (2.2):

$$\begin{aligned} &\omega_F(h)(\nabla_{e_m} h, e_m) \\ &\geq 2\Psi^{-1}(\Psi_{;m})^2 - \bar{D}^2_{xx} f(e_m, e_m) + F^{ik} h^l_i R_{kmlm} - c(h_{mm} + |\nabla h_{mm}|) \\ &\geq 2\Psi^{-1}(\Psi_{;m})^2 - \bar{D}^2_{xx} f(e_m, e_m) + \Psi^{ik} h^l_i \bar{R}_{kmlm} - c(h_{mm} + |\nabla h_{mm}|) \\ &\geq 2 f^{-1}(\bar{D}_x f(e_m))^2 - \bar{D}^2_{xx} f(e_m, e_m) + \tau_N \Psi^{ik} h^l_i (g_{kl} - g_{km} g_{lm}) \\ &\quad - c(h_{mm} + |\nabla h_{mm}|) \\ &\geq 2 f^{-1}(\bar{D}_x f(e_m))^2 - \bar{D}^2_{xx} f(e_m, e_m) + \tau_N f - c(h_{mm} + |\nabla h_{mm}|) \\ &\geq -c(h_{mm} + |\nabla h_{mm}|). \end{aligned}$$

□

The following corollary contains the CRT from [12, 13] as special cases.

Corollary 2.6 *(Curvature Measures Type Equations) Suppose the curvature function Ψ satisfies Assumption 2.1, $1 \leq k \leq n-1$, $p \in \mathbb{R}$ and $0 < \phi \in C^\infty(\mathbb{S}^n)$. Let M be a Γ-admissible convex hypersurface of $\mathbb{R}^{n+1}$ which encloses the origin in its interior and suppose*

$$\Psi(\kappa) = \langle x, \nu\rangle^p |x|^{-\frac{n+1}{k}} \phi\left(\frac{x}{|x|}\right)^{\frac{1}{k}}.$$

If

$$|x|^{\frac{n+1}{k}} \phi\left(\frac{x}{|x|}\right)^{-\frac{1}{k}} \quad \text{is convex on } \mathbb{R}^{n+1} \setminus \{0\},$$

then M is strictly convex.

3 A viscosity approach

The following lemma served as the main motivation for us to study the constant rank theorems with a viscosity approach. It shows that the smallest eigenvalue of a bilinear form satisfies a viscosity inequality. In the context of extrinsic curvature flows a similar approach was taken to prove preservation of convex cones; see [21, 22]. There it was shown that the distance of the vector of eigenvalues to the boundary of a convex cone satisfies a viscosity inequality.

Lemma 3.1 *[3, Lemma 5] Let the eigenvalues of a symmetric 2-tensor α with respect to a metric (g, ∇) at x_0 be ordered via*

$$\lambda_1 = \cdots = \lambda_{D_1} < \lambda_{D_1+1} \le \cdots \le \lambda_n,$$

for some $D_1 \ge 1$. Let ξ be a lower support for λ_1 at x_0. That is, ξ is a smooth function such that in an open neighborhood of x_0,

$$\xi \le \lambda_1$$

and $\xi(x_0) = \lambda_1(x_0)$. Choose an orthonormal frame for $T_{x_0}M$ such that

$$\alpha_{ij} = \delta_{ij}\lambda_i, \quad g_{ij} = \delta_{ij}.$$

Then at x_0 we have for $1 \le k \le n$,

(1)

$$\alpha_{ij;k} = \delta_{ij}\xi_{;k} \quad 1 \le i, j \le D_1,$$

(2)

$$\xi_{;kk} \le \alpha_{11;kk} - 2\sum_{j>D_1} \frac{(\alpha_{1j;k})^2}{\lambda_j - \lambda_1}.$$

While the previous lemma is sufficient for full rank theorems (i.e., when the respective linear map is non-negative, and positive definite at least at one point), we need to generalize [3, Lemma 5] from the smallest eigenvalue to an arbitrary subtrace of a matrix to treat constant rank theorems.

To formulate the following lemma, we introduce some notation. For a symmetric 2-tensor α on a vector space V with inner product g, let $\alpha^\sharp$ be the metric raised endomorphism defined by $g(\alpha^\sharp(X), Y) = \alpha(X, Y)$. Then $\alpha^\sharp$ is diagonalizable and we write

$$\lambda_1 \le \cdots \le \lambda_n$$

for the eigenvalues with distinct eigenspaces E_k of dimension $d_k = \dim E_k$, $1 \le k \le N$. For convenience, let $E_0 = \{0\}$ and $d_0 = 0$. Define

$$\bar{E}_j = \bigoplus_{k=0}^{j} E_k, \quad \bar{d}_j = \dim \bar{E}_j$$

for $0 \le j \le N$ so that

$$\{0\} = \bar{E}_0 \subsetneq \bar{E}_1 \subsetneq \cdots \subsetneq \bar{E}_N = V, \quad \bar{E}_k = \bar{E}_{k-1} \oplus E_k.$$

Let $(e_j)_{1\le j\le n}$ be an orthonormal basis of eigenvectors corresponding to the eigenvalues $(\lambda_j)_{1\le j\le n}$ giving $E_k = \operatorname{span}\{e_{\bar{d}_{k-1}+1}, \dots, e_{\bar{d}_k}\}$ and $\bar{E}_k = \operatorname{span}\{e_1, \dots, e_{\bar{d}_k}\}$. For each $1 \le m \le n$, there is a unique $j(m)$ such that

$$\bar{E}_{j(m)-1} \subsetneq V_m := \operatorname{span}\{e_1, \dots, e_m\} \subseteq \bar{E}_{j(m)}.$$

Then $\bar{d}_{j(m)-1} < m \le \bar{d}_{j(m)}$. For convenience, we write

$$D_m := \bar{d}_{j(m)}.$$

Note that D_m is the largest number such that

$$\lambda_1 \le \dots \le \lambda_m = \dots = \lambda_{D_m} < \lambda_{D_m+1} \le \dots \le \lambda_n,$$

and hence

$$\bar{E}_{j(m)} = \operatorname{span}\{e_1, \dots, e_{D_m}\}.$$

The subspace V_m is invariant under $\alpha^\sharp$ and the trace of $\alpha^\sharp$ restricted to V_m is the subtrace,

$$G_m := \sum_{k=1}^m \lambda_k.$$

This subtrace is characterized by Ky Fan's maximum principle (cf. [6, Theorem 6.5]), taking the infimum with respect to all traces of $\pi_P \circ \alpha^\sharp|_P$ over m-planes of the tangent spaces where π_P is orthogonal projection onto an m-plane P:

$$\begin{aligned} G_m &= \inf_P \{\operatorname{tr} \pi_P \circ \alpha^\sharp|_P \colon P = m\text{-plane}\} \\ &= \inf_{(w_k)_{1\le k\le m}} \left\{ \sum_{k,l=1}^m g^{kl} \alpha(w_k, w_l) \colon (g(w_k, w_l))_{1\le k,l\le m} > 0 \right\}, \end{aligned}$$

where (g^{kl}) is the inverse of $g_{kl} = g(w_k, w_l)$. Now suppose α is a bilinear form on a Riemannian manifold (M, g), $x_0 \in M$ and $(e_i)_{1\le i\le n}$ is an orthonormal basis of eigenvectors at x_0 with eigenvalues

$$\lambda_1(x_0) \le \dots \le \lambda_n(x_0).$$

Letting $w_i(x)$, $1 \le i \le m$, be any set of linearly independent local vector fields around x_0 with $w_i(x_0) = e_i$, then we have a smooth upper support function for G_m at x_0:

$$\Theta(x) := \sum_{k,l=1}^m g^{kl} \alpha_{kl} \ge G_m(x), \quad \Theta(x_0) = G_m(x_0),$$

where $\alpha_{kl} = \alpha(w_k(x), w_l(x))$. We make use of Θ to prove the next lemma generalizing Lemma 3.1.

Lemma 3.2 *Let (M, g) be a Riemannian manifold and let α be a symmetric 2-tensor on TM. Suppose $1 \le m \le n$ and ξ is a (local) lower support at x_0 for the subtrace $G_m(\alpha^\sharp)$. Then at x_0 we have*

(1)

$$\xi_{;i} = \operatorname{tr}_{V_m} \alpha_{;i} = \sum_{k=1}^m \alpha_{kk;i},$$

(2)

$$\xi_{;ii} \leq \sum_{k=1}^{m} \alpha_{kk;ii} - 2\sum_{k=1}^{m} \sum_{r>D_m} \frac{(\alpha_{kr;i})^2}{\lambda_r - \lambda_k},$$

where $V_m = \mathrm{span}\{e_1(x_0), \ldots, e_m(x_0)\}$ *for any choice of* m *orthonormal eigenvectors* e_k *with corresponding eigenvalues* $\lambda_1, \ldots, \lambda_m$ *satisfying*

$$\lambda_1 \leq \cdots \leq \lambda_m = \cdots = \lambda_{D_m} < \lambda_{D_m+1} \leq \cdots \leq \lambda_n.$$

Proof For this proof we use the summation convention for indices ranging between 1 and m. Let ξ be a lower support for G_m at x_0. Fix an index $1 \leq i \leq n$ and let $\gamma(s)$ be a geodesic with $\gamma(0) = x_0$ and $\dot{\gamma}(0) = e_i(x_0)$. Let $(v_k)_{1\leq k\leq m}$ be any basis (not necessarily orthonormal) for V_m as in the statement of the lemma. As mentioned above, for any m linearly independent vector fields $(w_k(s))_{1\leq k\leq m}$ along γ with $w_k(0) = v_k(x_0)$, $\alpha_{kl} = \alpha(w_k, w_l)$ and $(g^{kl}) = (g(w_k, w_l))^{-1}$, the function

$$\Theta(s) := g^{kl}\alpha_{kl} - \xi(\gamma(s))$$

satisfies

$$\Theta(s) \geq 0, \;\; \Theta(0) = 0$$

and hence

$$\dot{\Theta}(0) = 0, \;\; \ddot{\Theta}(0) \geq 0.$$

Since $V_m \subseteq \bar{E}_{j(m)}$, choosing w_k such that $\dot{w}_k(0) \perp \bar{E}_{j(m)}(x_0)$ gives

$$\dot{g}_{kl}(0) = g(\dot{w}_k(0), v_l) + g(v_k, \dot{w}_l(0)) = 0$$

and hence also

$$\dot{g}^{kl}(0) = -g^{ka}(0)\dot{g}_{ab}(0)g^{bl}(0) = 0.$$

Then we compute

$$0 = \dot{\Theta}(0) = \left(g^{kl}\alpha_{kl;i} - \xi_{;i}\right)|_{x_0}$$

giving the first part.

Now we move on to the second derivatives. For this we make the additional assumptions, $v_k = e_k$ and $\ddot{w}_k(0) = 0$. We first calculate

$$\begin{aligned}\ddot{g}^{kl}(0) &= g^{km}\dot{g}_{mr}g^{ra}\dot{g}_{ab}g^{bl} - g^{ka}\ddot{g}_{ab}g^{bl} + g^{ka}\dot{g}_{ab}g^{bm}\dot{g}_{mr}g^{rl} \\ &= -\delta^{ka}\ddot{g}_{ab}(0)\delta^{bl},\end{aligned}$$

since $\dot{g}_{kl}(0) = 0$ and $g^{kl}(0) = \delta^{kl}$. Then from $\ddot{w}_k(0) = 0$ we obtain

$$\begin{aligned}\ddot{g}^{kl}(0) &= -\left[g(\ddot{w}_k, w_l) + g(w_k, \ddot{w}_l) + 2g(\dot{w}_k, \dot{w}_l)\right](0) \\ &= -2\delta^{ka}g(\dot{w}_a(0), \dot{w}_b(0))\delta^{bl}.\end{aligned}$$

From the local minimum property,

$$
\begin{aligned}
0 &\le \ddot{\Theta}(0)\\
&= \ddot{g}^{kl}(0)\alpha_{kl} + \delta^{kl}\frac{d^2}{ds^2}_{|s=0}\alpha_{kl}(s) - \xi_{;ii}(x_0)\\
&= -2g(\dot{w}_k(0), \dot{w}_l(0))\alpha^{kl} + \delta^{kl}\alpha_{kl;ii}\\
&\quad + 4\delta^{kl}\nabla_i\alpha(\dot{w}_k(0), w_l(0)) + 2\delta^{kl}\alpha(\dot{w}_k(0), \dot{w}_l(0)) - \xi_{;ii}(x_0)\\
&= \sum_{k=1}^{m}\alpha_{kk;ii} - \xi_{;ii}(x_0)\\
&\quad + 2\sum_{k=1}^{m}\left(2\nabla_i\alpha(\dot{w}_k(0), e_k) + \alpha(\dot{w}_k(0), \dot{w}_k(0)) - g(\dot{w}_k(0), \dot{w}_k(0))\lambda_k\right).
\end{aligned}
$$

From $\dot{w}_k(0) \perp \bar{E}_{j(m)}$, we may write $\dot{w}_k(0) = \sum_{r>D_m} c_k^r e_r$ giving

$$
\begin{aligned}
\xi_{;ii}(x_0) - \sum_{k=1}^{m}\alpha_{kk;ii} &\le 2\sum_{k=1}^{m}\sum_{r>D_m}\left(2c_k^r\alpha_{kr;i} + (c_k^r)^2\lambda_r - (c_k^r)^2\lambda_k\right)\\
&= 2\sum_{k=1}^{m}\sum_{r>D_m} c_k^r\left(2\alpha_{kr;i} + c_k^r(\lambda_r - \lambda_k)\right).
\end{aligned}
$$

Optimizing yields the specific choice

$$
\dot{w}_k(0) = -\sum_{r>D_m}\frac{\alpha_{kr;i}}{\lambda_r - \lambda_k}e_r.
$$

From this we obtain

$$
\begin{aligned}
\xi_{;ii}(x_0) - \sum_{k=1}^{m}\alpha_{kk;ii} &\le -2\sum_{k=1}^{m}\sum_{r>D_m}\frac{\alpha_{kr;i}}{\lambda_r - \lambda_k}\left(2\alpha_{kr;i} - \alpha_{kr;i}\right)\\
&= -2\sum_{k=1}^{m}\sum_{r>D_m}\frac{(\alpha_{kr;i})^2}{\lambda_r - \lambda_k}.
\end{aligned}
$$

□

Corollary 3.3 *Let α be a non-negative, symmetric 2-tensor on TM. Suppose for some $1 \le m \le n$ that $\dim\ker\alpha^\sharp \ge m-1$ or equivalently that the eigenvalues of $\alpha^\sharp$ satisfy $\lambda_1 \equiv \cdots \equiv \lambda_{m-1} \equiv 0$. Then for all x_0 and any lower support ξ for G_m at x_0 and all $1 \le i \le n$ we have*

(1) $(\nabla_i\alpha(x_0))_{|\ker\alpha^\sharp\times\ker\alpha^\sharp} = 0$,
(2) $(\nabla_i\alpha(x_0))_{|E_{j(m)}\times E_{j(m)}} = g\nabla_i\xi(x_0)$, *if* $\lambda_m(x_0) > 0$.

Proof We use a basis (e_i) as in Lemma 3.2. To prove (1) we may assume $\lambda_1(x_0) = 0$, and hence the zero function is a lower support for λ_1. By Lemma 3.1, we have $\nabla\alpha_{kl} = 0$ for all $1 \le k, l \le d_1$ proving the first equation.

Now we prove (2). For $m = 1$ the claim follows from Lemma 3.2-(1). Suppose $m > 1$. If $d_1 \ge m$ at x_0 then $\lambda_m(x_0) = 0$ which violates our assumption. Hence $d_1 = m-1$ and

$E_1(x_0) = \operatorname{span}\{e_1, \dots, e_{m-1}\}$. Taking any unit vector $v \in E_2(x_0) = \operatorname{span}\{e_m, \dots, e_{D_m}\}$ and applying Lemma 3.2-(1) with $V_m = \{e_1, \dots, e_{m-1}, v\}$ gives

$$\nabla_i \alpha(v, v) = \operatorname{tr}_{V_m} \nabla_i \alpha = \nabla_i \xi \quad \forall 1 \le i \le n.$$

Polarizing the quadratic form $v \mapsto \nabla_i \alpha(v, v)$ over $E_2(x_0)$ then shows

$$\nabla_i \alpha_{kl} = \delta_{kl} \nabla_i \xi \quad \forall m \le k, l \le D_m.$$

□

Now we state the key outcome of the results in this section. We want to acknowledge that the following proof is inspired by the beautiful paper [24] and their sophisticated test function

$$Q = \sum_{q=1}^{m} G_q.$$

Theorem 3.4 *Under the assumptions of Theorem 1.6, if* $\dim \ker \alpha^\sharp \ge m-1$, *for all* $\Omega \Subset M$ *there exists a constant* $c = c(\Omega)$, *such that for all* $x_0 \in \Omega$ *and any lower support function* ξ *for* $G_m(\alpha^\sharp)$ *at* x_0 *we have*

$$F^{ij} \xi_{;ij} \le c(\xi + |\nabla \xi|).$$

Proof In view of our assumption $\lambda_{m-1} \equiv 0$. Hence the zero function is a smooth lower support at x_0 for every subtrace G_q with $1 \le q \le m-1$. Therefore by Lemma 3.2, for every $1 \le q \le m-1$ and every $1 \le i \le n$ we obtain

$$0 \le \sum_{k=1}^{q} \alpha_{kk;ii} - 2 \sum_{k=1}^{q} \sum_{j > D_q} \frac{(\alpha_{kj;i})^2}{\lambda_j - \lambda_k}. \tag{3.1}$$

Due to the Ricci identity, we have the commutation formula

$$\begin{aligned} \alpha_{ij;kl} = \alpha_{ki;jl} &= \alpha_{ki;lj} + R^p_{kjl} \alpha_{pi} + R^p_{ijl} \alpha_{pk} \\ &= \alpha_{kl;ij} + R^p_{kjl} \alpha_{pi} + R^p_{ijl} \alpha_{pk}. \end{aligned}$$

Taking into account Lemma 3.2 and adding the inequalities (3.1) for $1 \le q \le m-1$, we have at x_0,

$$\begin{aligned} F^{ij} \xi_{;ij} &\le \sum_{q=1}^{m} \sum_{k=1}^{q} F^{ij} \alpha_{kk;ij} - 2 \sum_{q=1}^{m} \sum_{k=1}^{q} \sum_{j > D_q} \frac{F^{ii} (\alpha_{kj;i})^2}{\lambda_j - \lambda_k} \\ &\le \sum_{q=1}^{m} \sum_{k=1}^{q} F^{ij} \left(\alpha_{ij;kk} - R^p_{kjk} \alpha_{pi} - R^p_{ijk} \alpha_{pk} \right) \\ &\quad - 2 \sum_{q=1}^{m} \sum_{k=1}^{q} \sum_{j > D_q} \frac{F^{ii} (\alpha_{kj;i})^2}{\lambda_j - \lambda_k}. \end{aligned}$$

Now differentiating the equation $F(\alpha^\sharp, x) = 0$ yields

$$\begin{aligned} 0 &= F^{ij} \alpha_{ij;k} + D_{x^k} F, \\ 0 &= F^{ij,rs} \alpha_{ij;k} \alpha_{rs;l} + D_{x^l} F^{ij} \alpha_{ij;k} + F^{ij} \alpha_{ij;kl} + D_{x^k} F^{rs} \alpha_{rs;l} + D^2_{x^k x^l} F. \end{aligned}$$

Then substituting above gives

$$
\begin{aligned}
F^{ij}\xi_{;ij} \le& -2\sum_{q=1}^{m}\sum_{k=1}^{q}\sum_{j>D_q}\frac{F^{ii}(\alpha_{kj;i})^2}{\lambda_j-\lambda_k}-\sum_{q=1}^{m}\sum_{k=1}^{q}F^{ij,rs}\alpha_{ij;k}\alpha_{rs;k}\\
&-\sum_{q=1}^{m}\sum_{k=1}^{q}\left(D^2_{x^kx^k}F+2D_{x^k}F^{ij}\alpha_{ij;k}+F^{ij}\left(R^p_{kjk}\alpha_{pi}+R^p_{ijk}\alpha_{pk}\right)\right)\\
\le& -2\sum_{q=1}^{m}\sum_{k=1}^{q}\sum_{j>D_m}\frac{F^{ii}(\alpha_{ij;k})^2}{\lambda_j}-\sum_{q=1}^{m}\sum_{k=1}^{q}\sum_{i,j,r,s>D_m}F^{ij,rs}\alpha_{ij;k}\alpha_{rs;k}\\
&-\sum_{q=1}^{m}\sum_{k=1}^{q}\left(D^2_{x^kx^k}F+2D_{x^k}F^{ij}\alpha_{ij;k}+F^{ij}R^p_{kjk}\alpha_{pi}\right)+c\xi\\
&+C\sum_{i=1}^{n}\sum_{j,k\le D_m}|\alpha_{jk;i}|-2\sum_{q=1}^{m}\sum_{k=1}^{q}\sum_{j=D_q+1}^{D_m}\frac{F^{ii}(\alpha_{ij;k})^2}{\lambda_j},
\end{aligned}
$$

where we have used that α is Codazzi and the fact that $1\le k\le m\le D_m$ in splitting the sum involving $F^{ij,rs}$ into terms where at least two indices are at most D_m and the remaining indices $i,j,r,s>D_m$. We have also used $\lambda_j-\lambda_k\ge\lambda_j$, and that for some constant c,

$$
F^{ij}R^p_{ijm}\alpha_{pm}\ge -c\xi.
$$

Now for every $1\le k\le m$ define

$$
\eta_k=(\eta_{ijk})=\begin{cases}\alpha_{ij;k}, & i,j>D_m\\ 0, & i\le D_m \text{ or } j\le D_m.\end{cases}
$$

Then

$$
\begin{aligned}
F^{ij}\xi_{;ij} \le& -2\sum_{q=1}^{m}\sum_{k=1}^{q}\sum_{j>D_m}\frac{F^{ii}(\eta_{ijk})^2}{\lambda_j}-\sum_{q=1}^{m}\sum_{k=1}^{q}F^{ij,rs}\eta_{ijk}\eta_{rsk}\\
&-\sum_{q=1}^{m}\sum_{k=1}^{q}D^2_{x^kx^k}F-2\sum_{q=1}^{m}\sum_{k=1}^{q}D_{x^k}F^{ij}\eta_{ijk}-\sum_{q=1}^{m}\sum_{k=1}^{q}F^{ij}R^p_{kjk}\alpha_{pi}\\
&+C\sum_{i=1}^{n}\sum_{j,k\le D_m}|\alpha_{jk;i}|-2\sum_{q=1}^{m}\sum_{k=1}^{q}\sum_{j=D_q+1}^{D_m}\frac{F^{ii}(\alpha_{ij;k})^2}{\lambda_j}+c\xi.
\end{aligned}
$$

In addition we define $\alpha^\sharp_\varepsilon=\alpha^\sharp+\varepsilon\,\mathrm{id}$, which has positive eigenvalues for $\varepsilon>0$. In the sequel, a subscript ε denotes evaluation of a quantity at $\alpha^\sharp_\varepsilon$, e.g., we put $F^{ij}_\varepsilon=F^{ij}(\alpha^\sharp_\varepsilon)$. We have

$$
\begin{aligned}
F^{ij}\xi_{;ij} \leq \sum_{q=1}^{m} \lim_{\varepsilon\to 0}\Bigg(&-2\sum_{k=1}^{q}\sum_{j=1}^{n}\frac{F_\varepsilon^{ii}(\eta_{ijk})^2}{\lambda_j+\varepsilon}-\sum_{k=1}^{q}F_\varepsilon^{ij,rs}\eta_{ijk}\eta_{rsk}\\
&-\sum_{k=1}^{q}(D^2_{x^kx^k}F)_\varepsilon-2\sum_{k=1}^{q}(D_{x^k}F^{ij})_\varepsilon\eta_{ijk}-\sum_{k=1}^{q}F_\varepsilon^{ij}R^p_{kjk}(\alpha_\varepsilon)_{pi}\Bigg)\\
&+C\sum_{i=1}^{n}\sum_{j,k\leq D_m}|\alpha_{jk;i}|-2\sum_{q=1}^{m}\sum_{k=1}^{q}\sum_{j=D_q+1}^{D_m}\frac{F^{ii}(\alpha_{ij;k})^2}{\lambda_j}+c\xi.
\end{aligned}
$$

In view of Definition 1.4, and the definition of ω_F,

$$
\begin{aligned}
F^{ij}\xi_{;ij} &\leq \sum_{q=1}^{m}\lim_{\varepsilon\to 0}\Bigg(-\sum_{k=1}^{q}\Phi_\varepsilon^{ij,rs}\eta_{ijk}\eta_{rsk}-\sum_{k=1}^{q}(D^2_{x^kx^k}F)_\varepsilon\\
&\quad -2\sum_{k=1}^{q}(D_{x^k}F^{ij})_\varepsilon\eta_{ijk}-\sum_{k=1}^{q}F_\varepsilon^{ij}R^p_{kjk}(\alpha_\varepsilon)_{pi}\Bigg)\\
&\quad +C\sum_{i=1}^{n}\sum_{j,k\leq D_m}|\alpha_{jk;i}|-2\sum_{q=1}^{m}\sum_{k=1}^{q}\sum_{j=D_q+1}^{D_m}\frac{F^{ii}(\alpha_{ij;k})^2}{\lambda_j}+c\xi\\
&\leq -\sum_{q=1}^{m}\sum_{k=1}^{q}\omega_F(\alpha)(\eta_k,e_k)+C\sum_{i=1}^{n}\sum_{j,k\leq D_m}|\alpha_{jk;i}|\\
&\quad -2\sum_{q=1}^{m}\sum_{k=1}^{q}\sum_{j=D_q+1}^{D_m}\frac{F^{ii}(\alpha_{ij;k})^2}{\lambda_j}+c\xi.
\end{aligned}
$$

Adding and subtracting some terms gives

$$
\begin{aligned}
F^{ij}\xi_{;ij} \leq &-\sum_{q=1}^{m}\sum_{k=1}^{q}\omega_F(\alpha)(\nabla_{e_k}\alpha,e_k)+c\xi\\
&+\sum_{q=1}^{m}\sum_{k=1}^{q}\omega_F(\alpha)(\nabla_{e_k}\alpha,e_k)-\sum_{q=1}^{m}\sum_{k=1}^{q}\omega_F(\alpha)(\eta_k,e_k)\\
&+C\sum_{i=1}^{n}\sum_{k,j\leq D_m}|\alpha_{jk;i}|-2\sum_{q=1}^{m}\sum_{k=1}^{q}\sum_{j=D_q+1}^{D_m}\frac{F^{ii}(\alpha_{ij;k})^2}{\lambda_j}.
\end{aligned}
\tag{3.2}
$$

Next we estimate the last two lines of (3.2). We have

$$
\sum_{q=1}^{m}\sum_{k=1}^{q}\omega_F(\alpha)(\nabla_{e_k}\alpha,e_k)-\sum_{q=1}^{m}\sum_{k=1}^{q}\omega_F(\alpha)(\eta_k,e_k)\leq C\sum_{i=1}^{n}\sum_{j,k\leq D_m}|\alpha_{jk;i}|,
$$

$$
C\sum_{i=1}^{n}\sum_{j,k\leq D_m}|\alpha_{jk;i}|\leq C\sum_{i=1}^{n}\sum_{k=1}^{D_1}\sum_{j=D_1+1}^{D_m}|\alpha_{jk;i}|+c|\nabla\xi|,
$$

where for the last inequality we used Corollary 3.3. Let us define

$$\begin{aligned}\mathcal{R} &= C\sum_{i=1}^{n}\sum_{k=1}^{D_1}\sum_{j=D_1+1}^{D_m}|\alpha_{jk;i}| - 2\sum_{q=1}^{m}\sum_{k=1}^{q}\sum_{j=D_q+1}^{D_m}\frac{F^{ii}(\alpha_{ij;k})^2}{\lambda_j-\lambda_k}\\ &= C\sum_{i=1}^{n}\sum_{k=1}^{D_1}\sum_{j=D_1+1}^{D_m}|\alpha_{jk;i}| - 2\sum_{q=1}^{m-1}\sum_{k=1}^{q}\sum_{j=D_q+1}^{D_m}\frac{F^{ii}(\alpha_{ij;k})^2}{\lambda_j-\lambda_k}.\end{aligned}$$

Note that if $\lambda_m(x_0) = 0$, then $D_q = D_m$ for all $q \le m$ and hence $\mathcal{R} = 0$. If $\lambda_m(x_0) > 0$, then we have $D_q = m-1$ for all $q \le m-1$ and

$$\mathcal{R} = C\sum_{i=1}^{n}\sum_{k=1}^{m-1}\sum_{j=m}^{D_m}|\alpha_{jk;i}| - 2\sum_{k=1}^{m-1}(m-k)\sum_{j=m}^{D_m}\frac{F^{ii}(\alpha_{ij;k})^2}{\lambda_j-\lambda_k}.$$

Therefore, due to uniform ellipticity, we can use

$$C\sum_{i=1}^{n}|\alpha_{jk;i}| \le 2(m-k)\frac{F^{ii}(\alpha_{jk;i})^2}{\lambda_j-\lambda_k} + c\xi$$

to show that $\mathcal{R} \le c'\xi$. Then by the assumptions on ω_F, the right hand side of (3.2) is bounded by $c(\xi + |\nabla\xi|)$ completing the proof. □

Remark 3.5 Here we crucially used that F is Φ-inverse concave, then we took the limit $\varepsilon \to 0$ and finally swapped η_k with $\nabla_{e_k}\alpha$ absorbing the extra terms. If on the other hand we tried to swap first without using Φ-inverse concavity, the extra terms would involve $\sum_{r=1}^{n}\frac{F_\varepsilon^{ii}(\nabla_{e_k}(\alpha_\varepsilon)_{ir})^2}{\lambda_r+\varepsilon}$. Since $\lambda_r = 0$ for $1 \le r \le m-1$ this blows up in the limit $\varepsilon \to 0$ and cannot be absorbed.

Proof of Theorem 1.6 Let $k := \max_{x\in M}\dim\ker\alpha^\sharp(x)$. If $k = 0$, we are done. By induction we show that for all $1 \le m \le k$ we have $\lambda_m \equiv 0$. For $m = 1$, clearly we have $\dim\ker\alpha^\sharp \ge m-1$ and hence by Theorem 3.4 a lower support ξ for $G_1 = \lambda_1$ locally satisfies

$$F^{ij}\xi_{;ij} \le c(\xi + |\nabla\xi|).$$

By the strong maximum principle [4], $\lambda_1 \equiv 0$.

Now suppose the claim holds true for $m-1$, i.e.,

$$\lambda_1 \equiv \cdots \equiv \lambda_{m-1} \equiv 0.$$

Then a lower support ξ for G_m satisfies

$$F^{ij}\xi_{;ij} \le c(\xi + |\nabla\xi|).$$

Hence $G_m \equiv 0$ for all $m \le k$. Since k indicates the maximum dimension of the kernel, we must have $\lambda_{k+1} > 0$ and the rank is always $n-k$. □

Author Contributions Not applicable

Funding Open Access funding enabled and organized by Projekt DEAL. PB was supported by the ARC within the research grant "Analysis of fully non-linear geometric problems and differential equations", number DE180100110. MI was supported by a Jerrold E. Marsden postdoctoral fellowship from the Fields Institute. JS was supported by the "Deutsche Forschungsgemeinschaft" (DFG, German research foundation) within the research scholarship "Quermassintegral preserving local curvature flows", grant number SCHE 1879/3-1.

Data Availability Not applicable.

Code Availability Not applicable.

Declarations

Conflict of interests There are no conflicts of interest.

Ethical approval Not applicable.

Consent to participate Not applicable.

Consent for publication Not applicable.

References

1. Andrews, B., Chen, X., Fang, H., McCoy, J.: Expansion of Co-compact convex spacelike hypersurfaces in Minkowski space by their curvature. Indiana Univ. Math. J. **64**(2), 635–662 (2015)
2. Andrews, B.: Pinching estimates and motion of hypersurfaces by curvature functions. J. für die Reine und Angewandte Math. **608**, 17–33 (2007)
3. Brendle, S., Choi, K., Daskalopoulos, P.: Asymptotic behavior of flows by powers of the Gaussian curvature. Acta Math. **219**(1), 1–16 (2017)
4. Bardi, M., Da Lio, F.: On the strong maximum principle for fully nonlinear degenerate elliptic equations. Arch. Math. **73**(4), 276–285 (1999)
5. Bian, B., Guan, P.: A microscopic convexity principle for nonlinear partial differential equations. Invent. Math. **177**(2), 307–335 (2009)
6. Bhatia, R.: Perturbation bounds for matrix eigenvalues, Classics in Applied Mathematics, vol. **53**, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, (2007), Reprint of the 1987 originald
7. Bryan, P., Ivaki, M.N., Scheuer, J.: Harnack inequalities for curvature flows in Riemannian and Lorentzian manifolds. J. für die Reine und Angewandte Math. **2020**(764), 71–109 (2019)
8. Bryan, P., Ivaki, M.N., Scheuer, J.: Parabolic approaches to curvature equations. Nonlinear Analysis **203**, 112174 (2021)
9. Luis, A.: Caffarelli and Avner Friedman, Convexity of solutions of semilinear elliptic equations. Duke Math. J. **52**(2), 431–456 (1985)
10. Caffarelli, L., Guan, P., Ma, X.-N.: A constant rank theorem for solutions of fully nonlinear elliptic equations. Commun. Pure Appl. Math. **60**(12), 1769–1791 (2007)
11. Gerhardt, C.: Curvature problems, Series in Geometry and Topology, vol. 39. International Press of Boston Inc., Sommerville (2006)
12. Guan, P., Lin, C., Ma, X.-N.: The Christoffel-Minkowski problem II: Weingarten curvature equations. Chin. Ann Math. Series B **27**(6), 595–614 (2006)
13. Guan, P., Lin, C., Ma, X.-N.: The existence of convex body with prescribed curvature measures. Int. Math. Res. Not. **2009**(11), 1947–1975 (2009)
14. Guan, P., Ma, X.-N.: The Christoffel-Minkowski problem I: Convexity of solutions of a Hessian equation. Invent. Math. **151**(3), 553–577 (2003)
15. Guan, P., Ma, X.-N., Zhou, F.: The Christofel-Minkowski problem III: existence and convexity of admissible solutions. Commun. Pure Appl. Math. **59**(9), 1352–1376 (2006)
16. Guan, P., Zhang, X.: A class of curvature type equations. Pure Appl. Math. Quarterly **17**(3), 865–907 (2021)

17. Changqing, H., Ma, X.-N., Shen, C.: On the Christoffel-Minkowski problem of Firey's p-sum. Calc. Var. Partial. Differ. Equ. **21**(2), 137–155 (2004)
18. Huisken, G., Sinestrari, C.: Convexity estimates for mean curvature flow and singularities of mean convex surfaces. Acta Math. **183**(1), 45–70 (1999)
19. Ivaki, M.N.: Deforming a hypersurface by principal radii of curvature and support function. Calculus of Vari. Partial Diff. Equ. **58**(1), 1–18 (2019)
20. Korevaar, N.J., Lewis, J.L.: Convex solutions of certain elliptic equations have constant rank Hessians. Arch. Rational Mech. Anal. **97**(1), 19–32 (1987)
21. Langford, M.: Motion of hypersurfaces by curvature. Australian National University, Australia (2014)
22. Langford, M.: A general pinching principle for mean curvature flow and applications. Calculus of Variat. Partial Diff. Equ. **56**(4), 107 (2017)
23. Scheuer, J.: Isotropic functions revisited. Arch. Math. **110**(6), 591–604 (2018)
24. Székelyhidi, G., Weinkove, B.: On a constant rank theorem for nonlinear elliptic PDEs. Discrete Contin. Dyn. Syst. Series A **36**(11), 6523–6532 (2016)
25. Székelyhidi, G., Weinkove, B.: Weak Harnack inequalities for eigenvalues and constant rank theorems. Comm. Partial Diff. Equ. **46**(8), 1585–1600 (2021)