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GENERAL STRUCTURAL DYNAMIC ECONOMIC MODELING

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This paper examines an intertemporal optimizing consumer or a representative consumer-firm in a deterministic setting subject to a general (either linear or nonlinear) capital accumulation equation. Duality theory is used to recast the Hamilton–Jacobi equation for dynamic optimization in terms of an instantaneous and an intertemporal profit function. An envelope theorem allows derivation of an explicit solution for the value of the costate variable as a function of the state and other variables. The final model form only requires specification of atemporal functions that are linked into a closed-form solution for the optimal dynamic decision variables through a system of contemporaneous simultaneous equations.

Keywords: Intertemporal Optimization, Representative Agent, Duality Theory, Profit Function, Hotelling’s Theorem, Hamilton-Jacobi Equation

1. INTRODUCTION

Explicit closed-form (synthesized) solutions of structural intertemporal optimization problems are valuable for several reasons. On the one hand, they provide a stronger link between theoretical analyses and empirical specifications based on the theoretical structure. Additionally, explicit solutions provide a means of determining the exact response of optimal decision variables to some exogenous shock, thus allowing short-run comparative static analysis in the context of satisfaction of contemporaneous equilibrium conditions that are fully compatible with intertemporal optimizing behavior. For analyses supported by empirically based parameter values, general (“flexible”) functional form specifications are desirable. However, general dynamic specifications rarely admit of closed-form solutions.

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In intertemporal models that analyze individual utility maximization or that model aggregative behavior by the representative-agent paradigm, it is common to employ primal dynamic optimization techniques. If a synthesized solution is sought, the instantaneous utility function is usually specified to lie within the isoelastic or origin translated isoelastic (HARA) class, in order to enable the (in this case) known closed-form solution of the Hamilton–Jacobi equation (in deterministic models) or the Hamilton–Jacobi–Bellman equation (in stochastic models) to be employed. Where more general functional forms are considered, closed-form solutions usually are not attempted—open-loop approaches are employed. These generally require satisfaction of the transversality condition to be checked by some type of shooting method applied to candidate initial values for the costate variable. With nonlinear transition dynamics and the potential for multiple local optima, these computationally intensive methods can become burdensome.

An alternative closed-form approach to that of primal dynamic optimization was proposed by Cooper and McLaren (1980), McLaren and Cooper (1980), and Epstein (1981). This line of research, intertemporal duality theory, generated results for which the dynamic choice variables of the consumer or firm (consumption, savings, investment) were derived by intertemporal analogues of Roy’s identity, Shephard’s lemma, or Hotelling’s theorem. A steady flow of empirical work, particularly in agricultural economics, has taken place through the 1980’s and 1990’s, applying results from intertemporal duality. In the great majority of applications, the optimal value function for the intertemporal consumer or firm problem has been specified by means of a flexible functional form. Examples of this approach to the empirical implementation of intertemporal duality include those of Epstein and Denny (1983), Vasavada and Chambers (1986), Chang and Stefanou (1988), Howard and Shumway (1988), Larson (1989), Luh and Stefanou (1991), Fernandez-Cornejo et al. (1992), and Manera (1994).

However, applications of dynamic duality have generally imposed assumptions additional to the theory. Various properties of the optimal value function, because of their complexity, have not been imposed upon the functional form in empirical specifications, obviating some of the value of duality theory. An important example of this is the way in which optimal intertemporal utility depends upon the rate of return on savings. This relationship was derived by Cooper and McLaren (1980) as a differential equation relationship involving the optimal value function. Actually, the theoretical work on intertemporal duality theory treated the interest rate as constant (in models of the firm) or the rate of return on savings as constant (in models of the consumer). Empirical applications typically have avoided the issue of the specification of the dependence of the optimal value function upon this “parameter” by subsuming it into the functional form.

The current paper aims to extend synthesized dynamic economic modeling into a context in which intertemporal maximizing models are subject to nonlinear transition dynamics. This extension has several attractive features. The rate of return on savings is endogenized in this extension, being modeled as a function of the state variable. The model may be reinterpreted as that of a representative consumer-firm under conditions of nonconstant returns to scale technology. In this

context, the representative-agent assumption allows the optimization conditions to be interpreted as equilibrium conditions. Hence the model may serve to provide the microfoundations for macroeconomic models. The nonlinearity in the transition equation then represents a nonlinear restricted (capital stock conditioned) profit function for the representative firm, or equivalently a nonlinear GNP (restricted real value added) function for the economy as a whole.

As pointed out above, traditional dynamic solution approaches are more difficult in the nonlinear context. Open-loop approaches are computationally burdensome while closed-loop approaches such as intertemporal duality lead to difficulties in the specification of the regularity conditions for the optimal value function, especially in terms of parameters of economic interest. The current paper exploits other research on the relationship between utility and profit functions to sidestep these problems. In summarizing various approaches to the derivation of tightly constrained functional forms in the context of the intertemporal consumer problem, Cooper and McLaren (1993) pointed out some potentially useful alternative approaches. One of these involved exploiting the relationship between instantaneous and intertemporal functions using the concept of consumer “profit.” The concept of the profit function as a representation of consumer preferences had been considered by Gorman (1976), who attributed the idea to Frisch (1932). It was popularized in an atemporal duality theory context by Browning (1982) and its value from an intertemporal perspective was highlighted by Browning et al. (1985).

The work of Browning et al. (1985) represents something of a watershed as it makes the first explicit use of the concept of consumer profit in a dynamic optimization context. Exploiting the assumed constancy of the rate of return on savings, marginal utility is treated as a fixed effect. In the current paper, however, the intention is to model the rate of return on savings as a variable. In the profit function context, this requires treatment of marginal utility as a latent variable.

Related research—exploiting duality aspects of the profit function in an atemporal context (Cooper, 1994) and in an intertemporal context but with a constant rate of return on savings (Cooper, 1995, 1996)—is extended in the current paper to the general variable case. Specification of the optimal value function (as an explicitly synthesized function of the state variable and parameters) remains difficult in this case, for the reasons alluded to above in the commentary on the empirical applications of intertemporal duality. However, in the current paper, duality between the optimal value function and the profit function is exploited to ultimately obtain an intertemporal analogue of Hotelling’s theorem in which the relevant structural features are self-evident without the need to explicitly specify the optimal value function.

In intertemporal optimization problems of the type considered here, optimal contemporaneous consumption and saving/investment are dependent on the current value of the costate variable in the intertemporal problem, and this variable typically jumps in response to external shocks. As an alternative to the more standard procedure of determining the value of this jump by imposing the assumption that adjustment then moves along a stable path, found by linearization around the

steady state, in this paper the optimal contemporaneous value of the costate variable is derived by consideration of its relationship to the state variable and other parameters regardless of the distance from the steady state and without the need for linearization. By exhibiting the intertemporal solution in the form of contemporaneous optimization conditions, the relationships are capable of specification in econometric models and also of application in theoretical comparative economic response analyses.

2. NOTATION, ASSUMPTIONS, AND PRELIMINARY RESULTS

2.1. A Prototype Intertemporal Model

Define $U(c)$ as the representative consumer-firm's instantaneous utility function, where c is real aggregate consumption. It is assumed that $U_c > 0$, $U_{cc} < 0$. Let δ denote the consumer's subjective rate of time preference. The representative consumer-firm's optimization problem and the optimal value function implied by the optimization may be defined by

$$J(k, \delta) = \max_{\{c\}} \int_0^{\infty} e^{-\delta t} U(c(t)) dt \quad (1a)$$

subject to

$$\dot{k}(t) = F(k(t)) - c(t), \quad (1b)$$

$$k(0) = k, \quad (1c)$$

where k is wealth (or real capital stock in a consumer-firm or aggregative growth interpretation) and $F(k)$ is a restricted (i.e., capital stock conditioned) real net output or real value-added function. By implication, variable factors are optimized out as functions of their prices and other relevant parameters and these variables/parameters are subsumed within the functional form of $F(k)$ for notational convenience.

Analysis of the case in which $F(k)$ is linear in k and $U(c)$ is isoelastic is well established in both the consumer intertemporal optimization (with linear budget constraint) and the growth literatures (the "AK" model). The inherent tractability of the linear transition equation and isoelastic utility case allows closed-form solutions to be derived for optimal consumption and saving/investment and this has clearly been a major factor in its appeal. Cooper (1995) has extended this to more general functional forms for $U(c)$ but still in the context of a linear transition equation. The nonlinear $F(k)$ case requires some additional manipulation. In what follows, the linear case is carried along with the nonlinear case in order to establish the relationship between them.

It is possible to set out the restricted value-added function in more detail as $F(k) = wL(k, w) + R(k)k$, where w is the real wage, $L(k, w)$ is optimized labor demand, $R(k)$ is the rate of return on capital, and the real wage w is understood as

subsumed into the functional form of $F(k)$. The case in which $F(k)$ is linear in k corresponds to $L(k, w)$ linear in k and R independent of k . This case is consistent with linearly homogeneous technology and firm price-taking behavior. In this case, it is convenient to employ the explicit linear specification $F(k) = \rho k$ and to write the linear transition equation variant of the optimal value function as $J^L(k, \rho, \delta)$. Here, ρ may itself be interpreted as a function of further parameters (such as the real wage) that determine the levels of variable inputs, optimally chosen in the context of a linearly homogeneous technology.

For expositional purposes, it will be convenient to interpret ρ as the implicit rental rate of capital, and to treat the return to the consumer-firm's ownership of capital as the sole factor payment. With this convention, in the linear case, $\rho k \equiv F_k k$ implies that factor payments exhaust the value of output. More generally, in the nonlinear case, $F(k)$ may be interpreted as the restricted net output or value-added function resulting from optimal variable input and/or price setting in the context of a noncompetitive environment. In this case, $\rho \equiv F_k(k)$ is explicitly acknowledged to be a variable (dependent upon k) and $F(k) - \rho k$ will measure the extent of economic rents. By analogy with an approach popularized by Romer (1986) in the endogenous growth literature, it will be convenient to distinguish between a competitive and a social optimum. Within the context of the methodology to be proposed in this paper, the distinction can be characterized in the following terms:

COMPETITIVE OPTIMUM (CO). *The representative competitive agent treats ρ and its path of evolution as given for purposes of computing optimal behavior; however, in the nonlinear case in equilibrium, ρ evolves according to $F_k(k)$.*

SOCIAL OPTIMUM (SO). *The social optimizer recognizes that, in the nonlinear case, its decisions on the path of k affect the marginal productivity of capital and hence flow through to the evolution of ρ .*

It is worth emphasizing that the distinction between the CO and the SO rests on the ability of the optimizer to control the evolution of ρ . The CO, which denies any control, is therefore stronger than the competitive optimization in the Romer case. In the Romer approach, control is allowed in both the competitive and the socially optimal cases but the underlying evaluation of the capital stock differs. It is straightforward to allow for a distinction between the representative private and the average aggregate capital stock in the context of the CO in this paper, and thereby allow for effective increasing returns to scale in the equilibrium associated with the CO, in the spirit of the endogenous growth literature. However, the key feature of the currently proposed methodology, which enables solution of the CO optimization problem for general functional form specifications, is the exogeneity of ρ from the perspective of the representative competitive decision maker, and so, emphasis in this paper is placed upon the CO/SO distinction drawn earlier.

Although it will be convenient to discuss the intertemporal optimization problem in general, and hence carry optimization conditions for both the CO and the SO cases, the primary focus of this paper is a positive one and the ultimate aim is

to provide a full solution in the CO case where ρ is acknowledged to be evolving generally, in particular as a function of k by virtue of the nonlinear technology, but independently of the actions of the representative agent. In principle, if variable inputs are optimally chosen, ρ will also be dependent upon the prices of these inputs. However, for notational convenience, parameters controlling these instantaneous optimization decisions are suppressed. Although the major focus of the paper will be on the optimal solution for the CO case, it will be useful initially to relate the methodology to control theoretic and dynamic programming approaches to the solution of problem (1). In its general form, (1) is silent with respect to assumptions on ρ . In the absence of an additional Romer-type distinction between the representative private and the average aggregate capital stock, this is most readily interpretable as compatible with the SO case, where the decision maker recognizes the full effect of variations in k on $F(k)$ resulting from optimization decisions that affect the marginal productivity of capital.

In the general nonlinear case, it is assumed that $F_k > 0$ and $F_{kk} < 0$. The latter assumption is specifically exploited, so that the linear case is not a special case of the nonlinear one. Nevertheless, it is possible to set out the two cases in a similar manner and this is useful for comparison. The methodology relies upon a recasting of preliminary results, which are a composite of concepts from control theory and dynamic programming.

2.2. Control Theoretic Approach

In a control theoretic approach to problem (1), the present-value pre-Hamiltonian may be defined as $\mathcal{H}'(c, k, \lambda', \delta t) = e^{-\delta t} U(c) + \lambda'[F(k) - c]$. However, it will be more convenient to work in current-value terms, defining the current-value costate variable as $\lambda \equiv e^{\delta t} \lambda'$. The optimality conditions may then be expressed by the current-value variant of the following principle.

PONTRYAGIN'S MAXIMUM PRINCIPLE. *Define the current-value pre-Hamiltonian as*

$$\mathcal{H}(c, k, \lambda) = U(c) + \lambda[F(k) - c]. \tag{2a}$$

Define the (optimal value of the) Hamiltonian as

$$H(k, \lambda) = \max_c \mathcal{H}(c, k, \lambda). \tag{2b}$$

Then,

$$\dot{k} = H_\lambda(k, \lambda), \tag{3a}$$

$$\dot{\lambda} = \delta\lambda - H_k(k, \lambda), \tag{3b}$$

and

$$\frac{\partial \mathcal{H}}{\partial c} = 0, \tag{3c}$$

where the differential equations (3a) and (3b) are subject to the general necessary infinite horizon transversality condition

$$\lim_{t \rightarrow \infty} e^{-\delta t} H(k(t), \lambda(t)) = 0 \quad (3d)$$

and the boundary condition (1c) on the constraining initial capital stock.

Given the structure (2a) for the pre-Hamiltonian, an envelope theorem applied to (2b) shows that

$$H_\lambda = F(k) - c, \quad (4a)$$

$$H_k = \lambda F_k, \quad (4b)$$

while the first-order condition implies that

$$U_c(c) = \lambda. \quad (4c)$$

These allow the optimality conditions (3) to be rewritten in the specific form

$$\dot{k} = F(k) - c, \quad (5a)$$

$$\dot{\lambda} = [\delta - F_k(k)]\lambda, \quad (5b)$$

$$c = U_c^{-1}(\lambda), \quad (5c)$$

$$\lim_{t \rightarrow \infty} e^{-\delta t} \{U(U_c^{-1}(\lambda(t))) + \lambda(t)[F(k(t)) - U_c^{-1}(\lambda(t))]\} = 0. \quad (5d)$$

Equations (5a)–(5c) constitute the core of the model as a set of contemporaneous optimization conditions. Together, (5a) [supplemented by (5c)] and (5b), respectively, describe the evolution of the state and costate variables k and λ , but require initial values for determination of the actual paths. Clearly, an initial value for k is prescribed by (1c). However, for determination of the exact short-run solution paths, an initial value for λ is still required. Once this is established, the system evolves according to (5a) and (5b) from the initial position to a path compatible with (5d). Virtually all of the literature approaches this problem either by assuming that the economy is already in a steady state (traveling along a balanced growth equilibrium trajectory) or by linearizing the transitional dynamics around the steady state and determining the value of λ that allows the steady state to be reached by movement along the stable arm of the saddle path system. The case for this approach as put by Lucas (1988) is

[Under certain parameter restrictions imposed to ensure this,] an economy that begins on the balanced path will find it optimal to stay there. What of economies that begin *off* the balanced path—surely the normal case? Cass showed—and this is exactly why the balanced path is interesting to us—that for *any* initial capital $k(0) > 0$, the optimal capital-consumption path $\{k(t), c(t)\}$ will converge to the balanced path asymptotically. That is, the balanced path will be a good approximation to any actual path “most” of the time. [Lucas (1988, p. 11)]

Although the Cass result is useful where it applies, there is an empirically relevant case for examining models in which a balanced growth path may not exist. Even with existence, there is the issue of speed of convergence, which requires examination. However, it seems fair to summarize the approach of much of the literature as focusing on balanced paths and steady states simply because this at least is manageable. As Lucas continues later in the same paper,

Consideration of off-steady-state behaviour would open up some new possibilities, possibly bringing the theory into greater conformity with observation, but I do not view this route as at all promising. Off steady states, [simple relationships that determine growth rates] need not hold and capital and output growth rates need not be either equal or constant. . . . [Lucas (1988, p. 14)]

In the current paper, the aim is to find optimal contemporaneous consumption and investment relationships in an intertemporal optimizing or growth model context without linearization about a stationary or steady state and without assuming any particular relationship to, or even the existence of, a balanced growth equilibrium.

To obtain optimal contemporaneous solutions for c and \dot{k} , it is necessary to resolve the issue of determination of an initial value for the costate variable λ by recognizing the fundamental endogeneity of λ in an optimizing context and hence by expressing the unobservable λ as a function of givens (in particular, as a function of the predetermined level of the quasi-fixed capital stock, k). To isolate the issues involved in determining the optimal relationship between λ and k , it is useful to employ a dynamic programming formulation.

2.3. Dynamic Programming Approach in the SO Case

Recognizing that, by its structure, the optimization problem (1) lends itself to consideration of the SO, it is useful to pursue the implications of the SO by utilizing the following principle.

BELLMAN'S PRINCIPLE OF OPTIMALITY. *The optimal value function for problem (1) satisfies*

$$\delta J(k, \delta) = \max_c \{U(c) + J_k(k, \delta)[F(k) - c]\}. \quad (6)$$

Bellman's principle implies the first-order condition

$$U_c(c) = J_k(k, \delta) \quad (7)$$

and, in conjunction with the control theoretic approach (4c), this implies that the costate variable λ must satisfy

$$\lambda = J_k(k, \delta). \quad (8)$$

In the linear transition equation case, as noted following the presentation of problem (1), the optimal value function may be written as $J^L(k, \rho, \delta)$, where $\rho (\equiv \delta F / \delta k)$

may be thought of as a function of atemporal parameters but is not a function of k . In this case, (8) may be written:

$$\lambda = J_k^L(k, \rho, \delta). \quad (8^L)$$

As long as $J_{kk} < 0$, condition (8) represents a relationship of optimality between a given level of capital stock k and the optimal value of the costate variable λ . Given shocks to parameters such as δ (and ρ in the linear case) or to parameters of the utility function (not explicitly shown), then λ must jump to a value implied by equation (8).

It should be emphasized that (8) may best be regarded as representing the optimality condition for the SO in the nonlinear case, since the optimal decision rule is not conditioned on any externally imposed values of ρ . In effect, in the nonlinear technology case, the full influence of the optimization decision on the marginal productivity of capital is implicitly taken into account in (8).

Although, conceptually, (8) endogenizes λ , the difficulty in using condition (8) in this way is that the optimal value function $J(k, \delta)$, or at least the marginal optimal value function $J_k(k, \delta)$, must first be ascertained. Direct analytical construction of J from (6) by dynamic programming is limited to specific functional forms for $U(c)$, which are not empirically flexible enough to provide a realistic model. As an alternative, numerical solution of (6) may be employed. It remains the case, however, that to solve (6) for function specifications complex enough to flexibly represent empirical realities, with potentially multiple local optima for λ , it is helpful to reduce the complexity of (6) by analytical methods as much as possible.

2.4. A Preliminary Representation of the Hamilton–Jacobi Equation

In many areas of economic analysis, duality theory has been used to enable tractable but optimization-consistent (regular) response functions of agents to be specified more flexibly than is possible for those response functions that could be explicitly derived by primal techniques. In the current paper, duality theory is used for a similar purpose. By translating (6) using dual functions, a simpler analytical relationship is derived and envelope theorems are used to generate the required optimal relationship between λ and k under very general specifications of consumer preferences.

A convenient form of (6) for this analysis is obtained by a conjunction of the Pontryagin and Bellman principles, equating (2a), (2b), and (6) in view of (8). This generates the Hamilton–Jacobi (H-J) equation, which relates the optimal value function $J \equiv J(k, \delta)$ to the Hamiltonian $H \equiv H(k, \lambda)$. In both the linear case and the nonlinear SO case, this can be written

$$\delta J(k, \delta) = H(k, \lambda). \quad (9)$$

As (2) makes clear, H may be constructed purely from the specification of the atemporal functions $U(c)$ and $F(k)$. On the other hand, the use of (8) to

endogenize λ requires specification of J . Conceptually, (9) provides an important link since it relates J to H . However, a difficulty in using (9) directly to generate J from H remains to be resolved. Specifically, in the form in which J and H have been defined, by (6) and (2b), respectively, they are conditioned on different sets of variables and parameters. Additionally, formulation (9) does not adequately account for the CO in the nonlinear case. Although the optimization conditions require (4c) in all cases, (8) only applies for the linear case and the SO variant of the nonlinear case. The CO variant of the nonlinear case needs to allow for a possible divergence between the private marginal value λ and the social marginal value $J_k(k, \delta)$. To allow for this, a generalized representation of the H-J equation (9) using (2a), (2b), (6), and (5c) in place of (8) can be written as

$$\delta J(k, \delta) = H(k, \lambda) + [J_k(k, \delta) - \lambda][F(k) - U_c^{-1}(\lambda)]. \quad (9')$$

Of course, in both the linear and the nonlinear SO cases, (9') reduces to (9).

The objective of the next two sections is to introduce a series of results in duality theory that may be used both to develop the nonlinear CO case in parallel with the linear and the nonlinear SO cases and also to re-express both sides of the generalized H-J equation (9') as functions of a common set of variables and parameters. The duality reformulations generate their own envelope theorems. These then enable the optimal relationship between k and λ to be directly ascertained at any point on the optimal path, not restricted to a linearized neighborhood of a stationary state, and for both regular and flexible functions representing the underlying consumer preference ordering and firm optimal behavior.

3. REFORMULATION OF ATEMPORAL ASPECTS OF THE OPTIMIZATION PROBLEM

3.1. Consumer Profit Maximization as a Dual to Utility Maximization

Consider a reformulation of the consumer's optimal instantaneous consumption choice problem as

$$\Phi(\lambda) = \max_c \{U(c) - \lambda c\}. \quad (10)$$

The interpretation of this optimization is that the consumer recognizes that instantaneous utility achieved via additional consumption must be paid for at opportunity cost λ . Thus the Φ function represents optimized instantaneous (normalized) profit for the consumer.

Problem (10) is conceptually consistent with the intertemporal optimizing problem in that λ is the shadow value of capital, the accumulation of which is foregone if consumption is increased. By setting the given price of consumption at this value, problem (10) generates a first-order condition,

$$U_c(c) = \lambda, \quad (11)$$

and this is equivalent to (4c), the optimality condition for consumption in problem (1).

An envelope theorem (Hotelling’s lemma) implies that

$$c = -\Phi_\lambda(\lambda) \tag{12}$$

so that $-\Phi_\lambda$ is interpretable as the U_c^{-1} function employed in (5c). Rearranging (10) by utilizing (12), a marginal-utility-conditioned (Frischian) instantaneous utility function may be defined by

$$U^F(\lambda) = \Phi(\lambda) - \lambda\Phi_\lambda(\lambda). \tag{13}$$

3.2. The Hamiltonian and Instantaneous Economic Profit

Using the concept of consumer profit, the Hamiltonian (2) can be rewritten, in view of (10), as

$$H(k, \lambda) = \Phi(\lambda) + \lambda F(k). \tag{14}$$

The Hamiltonian (14) has an interesting interpretation. It is the representative agent’s (and hence the economy’s) restricted value-added function, being the sum of the “consumer-side” profit function and the “firm-side” restricted value-added function, taking the capital stock as given, with all profits priced in utility terms. In this interpretation, the representative agent receives “profits” both from excess consumer utility and from excess value added. Excess consumer utility is measured by utility greater than the opportunity cost of consumption, λc . Excess value added represents the ability of the firm to repay the agent as supplier of variable factor inputs. However, to measure the true economic value of excess value added, restricted value added needs to be adjusted for the opportunity cost of capital. In the linear transition equation case, the marginal profitability of capital has been denoted by ρ . It is convenient, therefore, to denote the implicit rental rate of capital generally by ρ . Since λk is the value of the capital stock in the eyes of the optimizing representative agent, it follows that excess value added may be measured by value added greater than the opportunity cost of capital, $\rho\lambda k$. (Restricted) economywide instantaneous economic profit therefore can be defined by

$$\mathcal{P}(k, \lambda, \rho) = \Phi(\lambda) + \lambda[F(k) - \rho k]. \tag{15}$$

In the linear technology case, firm-side economic profit is zero [$F(k) = \rho k$] and (15) simply reduces to consumer profit $\Phi(\lambda)$. In the nonlinear case, it has been assumed that $F_{kk} < 0$. In this case, the optimized instantaneous profit function may be defined as

$$\Pi(\lambda, \rho) = \max_k \mathcal{P}(k, \lambda, \rho), \tag{16}$$

which implies the first-order condition

$$\rho = F_k(k), \tag{17}$$

which can be inverted to give

$$k = F_k^{-1}(\rho). \tag{18}$$

Thus, in the nonlinear case, the capital stock can be optimized out as a function of its implicit rental rate. For convenience in what follows, reference is made to $\Pi(\lambda, \rho)$ as a general expression for economywide instantaneous economic profit, with the understanding that $\Pi(\lambda, \rho) \rightarrow \Phi(\lambda)$, independent of ρ , in the linear case. In both the linear and nonlinear cases, it is now possible, by use of the synthetic variable ρ , to write the Hamiltonian in linear-affine-in- k form,

$$\bar{H}(k, \lambda, \rho) = \Pi(\lambda, \rho) + \lambda \rho k. \tag{19}$$

It is also convenient to note that, by an envelope result from (16) and (15) in the nonlinear case and by obvious definition in the linear technology case,

$$\Pi_\lambda(\lambda, \rho) = \Phi_\lambda(\lambda) + F(k) - \rho k \tag{20}$$

and where, in the nonlinear case, k satisfies (18). Consequently, an alternative expression to (13) for Frischian utility is

$$U^F(\lambda) = \Pi(\lambda, \rho) - \lambda \Pi_\lambda(\lambda, \rho). \tag{21}$$

Results (20) and (21) will be useful later in an examination of the optimal output /consumption relationship.

4. REFORMULATION OF INTERTEMPORAL ASPECTS OF THE OPTIMIZATION PROBLEM

4.1. Alternative Representation of the Intertemporal Problem

Having introduced the synthetic variable ρ in the instantaneous component of the representative consumer-firm's problem, it is helpful to re-express the intertemporal optimization problem (1) in a form that emphasizes the role of ρ . Define average productivity, $M(\rho)$, in terms of ρ as

$$M(\rho) \equiv \begin{cases} \rho, & F_{kk} = 0 \\ F(F_k^{-1}(\rho))/F_k^{-1}(\rho), & F_{kk} \neq 0 \end{cases} \tag{22}$$

Utilizing (22), problem (1) may be recast in a form that defines the optimal value function as a function of both the original state variable k and the additional synthetically constructed state variable ρ . Consider the following intertemporal utility-maximizing problem, which is slightly more general than the prototype (1):

$$\bar{J}(k, \rho, \delta) = \max_{\{c\}} \int_0^\infty e^{-\delta t} U(c(t)) dt \tag{23a}$$

subject to

$$\dot{k}(t) = M(\rho(t))k(t) - c(t), \tag{23b}$$

$$\dot{\rho}(t) = \begin{cases} 0, & F_{kk} = 0 \\ F_{kk}(k(t))\dot{k}(t), & F_{kk} < 0 \end{cases} \tag{23c}$$

$$k(0) = k, \tag{23d}$$

$$\rho(0) = \rho. \tag{23e}$$

Problem (23) includes the synthetic variable ρ as a second state variable. Although this introduces an apparent complexity, it allows the characteristics of the optimal value function, viewed as a function of k but conditioned on ρ , to be represented more simply. In the linear case, $F_{kk} = 0$, ρ is constant and (23) reduces to (1) with $\bar{J}(k, \rho, \delta) \equiv J^L(k, \rho, \delta)$. In the nonlinear case, (23) is more general than (1), but contains (1) as the special case in which the optimization takes place in the absence of initial arbitrage opportunities—that is, where the initial value of ρ is consistent with the initial k in the sense that $\rho(0) = F_k(k(0))$ is satisfied by the initial conditions (23d) and (23e). However, an important advantage of formulation (23) over (1) is its explicit recognition of ρ as a state variable. For nonlinear $F(k)$, this allows the CO and SO cases to be distinguished.

Case CO. Although the decision maker takes account of the effect of the choice of $c(t)$ directly on $\dot{k}(t)$ through (23b), the subsequent indirect effect on $\dot{\rho}(t)$ in (23c) is not recognized to be a consequence of the optimality decision; rather, (23c) is treated as an exogenously evolving constraint.

Case SO. The decision maker takes account of the effect of the choice of $c(t)$ not only directly on $\dot{k}(t)$ through (23b) but also, subsequently, indirectly on $\dot{\rho}(t)$ through (23c).

The implications of (23) and of the capability of making these distinctions can be summarized in Lemma 1.

LEMMA 1. *An Equivalent Intertemporal Utility Maximizing Problem. In the absence of initial arbitrage opportunities, problems (1) and (23) are equivalent, the relationship between the optimal value functions is*

$$J(k, \delta) \equiv J^L(k, \rho, \delta) \equiv \bar{J}(k, \rho, \delta), \quad F_{kk} = 0, \tag{24a}$$

$$J(k, \delta) \equiv \bar{J}(k, F_k(k), \delta), \quad F_{kk} < 0, \tag{24b}$$

the costate variable may be expressed in terms of the optimal value functions for (23) and (1) as

$$\lambda = J_k^L(k, \rho, \delta) = \bar{J}_k(k, \rho, \delta) = J_k(k, \delta), \quad F_{kk} = 0, \tag{25a}$$

$$\lambda = \bar{J}_k(k, \rho, \delta) \neq J_k(k, \delta), \quad F_{kk} < 0 \text{ and CO}, \tag{25b}$$

$$\lambda = \bar{J}_k(k, \rho, \delta) + \bar{J}_\rho(k, \rho, \delta)F_{kk}(k) = J_k(k, \delta), \quad F_{kk} < 0 \text{ and SO}, \tag{25c}$$

and the Hamilton–Jacobi equation is

$$\delta \bar{J}(k, \rho, \delta) = H(k, \bar{J}_k(k, \rho, \delta)), \quad F_{kk} = 0, \tag{26a}$$

$$\delta \bar{J}(k, \rho, \delta) = H(k, \bar{J}_k(k, \rho, \delta)) + \Omega(\rho, \delta), \quad F_{kk} < 0 \text{ and CO}, \tag{26b}$$

$$\delta \bar{J}(k, \rho, \delta) = H(k, \bar{J}_k(k, \rho, \delta) + \bar{J}_\rho(k, \rho, \delta)F_{kk}(k)), \quad F_{kk} < 0 \text{ and SO}, \tag{26c}$$

where, in (26b),

$$\begin{aligned} \Omega(\rho, \delta) = & \bar{J}_\rho(F_k^{-1}(\rho), \rho, \delta)F_{kk}(F_k^{-1}(\rho)) \left[F(F_k^{-1}(\rho)) \right. \\ & \left. + \Phi_\lambda(\bar{J}_k(F_k^{-1}(\rho), \rho, \delta)) \right]. \end{aligned} \tag{27}$$

Proof. In the linear technology case, $F(k) = \rho k$, $F_{kk} = 0$, $M(\rho) \equiv \rho$, and ρ is constant and is subsumed as a parameter into the functional form of $J(k, \delta)$. Here, the equivalence of (1) and (23) is obvious and (24a) clearly holds. In the nonlinear case, the absence of initial arbitrage means $\rho(0) = F_k(k(0))$. Given this restriction on the initial conditions, the nonlinear variant of (23c) ensures that (17) continues to hold along the optimal path for k . In view of (22) and (18), $F(k(t)) = M(\rho(t))k(t)$ and problem (23) together with (17) is identical to problem (1) so that (24b) must hold. A constructive proof of (25) is available by applying Bellman’s principle of optimality to problem (23). The optimal value function for (23) satisfies

$$\begin{aligned} \delta \bar{J}(k, \rho, \delta) = & \max_c \langle U(c) + \bar{J}_k(k, \rho, \delta)[\rho k - c] \rangle, \quad F_{kk} = 0 \\ \delta \bar{J}(k, \rho, \delta) = & \max_c \langle U(c) + \bar{J}_k(k, \rho, \delta)[M(\rho)k - c] \rangle + \bar{J}_\rho(k, \rho, \delta)F_{kk}(k) \\ & \times [M(\rho)k - U_c^{-1}(\bar{J}_k(k, \rho, \delta))], \quad F_{kk} < 0 \text{ and CO} \\ \delta \bar{J}(k, \rho, \delta) = & \max_c \left\langle U(c) + \begin{bmatrix} \bar{J}_k(k, \rho, \delta) + \\ \bar{J}_\rho(k, \rho, \delta)F_{kk}(k) \end{bmatrix} [M(\rho)k - c] \right\rangle, \\ & F_{kk} < 0 \text{ and SO.} \end{aligned}$$

The first-order conditions for each of these cases in conjunction with the control theoretic optimality condition (4c) for problem (1) implies that, for equivalence of problems (1) and (23), the costate variable λ must satisfy (25a)–(25c) for the respective cases. Expressions (26a)–(26c) for the H-J equations follow, in view of definitions (2a)–(2b), by insertion of these results in the generic expression (9'). In the nonlinear CO case, the term $\bar{J}_\rho(k, \rho, \delta)\dot{\rho}$ is not considered to be under the influence of the optimizing agent in its decision making with respect to fulfillment of the first-order condition for optimality. However, the relationship of optimality between k and ρ given by (18) is maintained. Using (18), (23c), (22), (23b), (12), (5c), and (25b), this term is equivalent to $\Omega(\rho, \delta)$ in (27). ■

Note that, in all cases, λ is equated to the marginal instantaneous utility of consumption as indicated in (4c). In the nonlinear CO case, this is equated to the

perceived private marginal present value of the optimized utility of resources, \bar{J}_k , as given in (25b). In the nonlinear SO case, it is equated to the social marginal present value of optimized utility of resources, $\bar{J}_k + \bar{J}_\rho F_{kk}$, as given in (25c). Clearly, λ is different in the CO and SO cases. If \bar{J} could be ascertained, the difference could be measured as $\bar{J}_\rho(k, \rho, \delta) F_{kk}(k)$.

4.2. The Intertemporal Profit Function as a Dual to the Optimal Value Function

The overall objective of Sections 3 and 4 is to rewrite both the instantaneous (Hamiltonian) component and the intertemporal (optimal value function) component of the H-J equation in terms of common variables and parameters. To complete this task, use is made of the optimal value function, treating it as “output” and defining its dual, optimal intertemporal profit.

THEOREM 1. *A Dual Intertemporal Profit Maximizing Problem. Let $\bar{J}(k, \rho, \delta)$ denote the optimal value function for problem (23). Dual to problem (23) is the following problem:*

$$\Psi(q, \rho, \delta) \equiv \max_k \{ \bar{J}(k, \rho, \delta) - qk \} \tag{28}$$

and, in the absence of arbitrage, problem (28) is also dual to problem (1) with

$$q = \lambda, \quad F_{kk} = 0 \tag{29a}$$

$$q = \lambda, \quad F_{kk} < 0 \text{ and CO} \tag{29b}$$

$$q = \lambda - \Xi(\rho, \delta), \quad F_{kk} < 0 \text{ and SO} \tag{29c}$$

which may be written generically as

$$q = Q(\lambda, \rho, \delta), \tag{29}$$

noting that $Q_\lambda = 1$ in all cases; and where, in (29c),

$$\Xi(\rho, \delta) = \bar{J}_\rho(F_k^{-1}(\rho), \rho, \delta) F_{kk}(F_k^{-1}(\rho)). \tag{30}$$

Proof. The Legendre transformation (28) establishes standard duality between $\Psi(q, \rho, \delta)$ and $\bar{J}(k, \rho, \delta)$. The first-order condition for (28) is

$$\bar{J}_k(k, \rho, \delta) = q \tag{31}$$

and envelope results are

$$\Psi_q(q, \rho, \delta) = -k, \tag{32a}$$

$$\Psi_\rho(q, \rho, \delta) = \bar{J}_\rho(k, \rho, \delta). \tag{32b}$$

In the absence of arbitrage, Lemma 1 has established the equivalence of problems (23) and (1), and that (25) holds in the cases indicated. In the linear case, $F_{kk} = 0$. Then, the combination of (31) with (25a) immediately gives (29a). In the nonlinear

CO case, the combination of (31) and (25b) gives (29b). In the nonlinear SO case, the combination of (31) and (25c) implies that

$$q = \lambda - \bar{J}_\rho(k, \rho, \delta) F_{kk}(k). \tag{33}$$

However, (18) also holds in the nonlinear case, and the combination of (18) with (33) gives (29c). ■

The importance of Theorem 1 is that it allows the derivation of a simple and powerful envelope theorem, which may be used to generate the optimal relationship between λ and k . This important implication of the theorem is highlighted in Corollary 1.

COROLLARY 1: *An Intertemporal Analogue of Hotelling’s Theorem. Let the intertemporal profit function conditioned on λ be defined by*

$$\tilde{\Psi}(\lambda, \rho, \delta) \equiv \Psi(Q(\lambda, \rho, \delta), \rho, \delta). \tag{34}$$

Then, the optimal contemporaneous relationship between k and λ , conditional on ρ , is given by

$$k = -\tilde{\Psi}_\lambda(\lambda, \rho, \delta). \tag{35}$$

Proof. From (34), (29), and (32a), $\tilde{\Psi}_\lambda = \Psi_q Q_\lambda = \Psi_q = -k$. ■

The relevance of Corollary 1 is that, if the functional form of $\tilde{\Psi}$ can be ascertained, (35) can be used to establish the optimal λ to associate with initial values of k, ρ, δ or the optimal jump in λ following a shock to any of k, ρ, δ . It remains to establish the functional form of $\tilde{\Psi}$ in terms of the initially specified functions Φ and F .

4.3. Reformulation of the Hamilton–Jacobi Equation

Recall that the basic issue in the use of the H–J equation is the need to construct J from H . However, Corollary 1 shows that the solution to the intertemporal problem may be obtained if the functional form for $\tilde{\Psi}$ can be ascertained. On the other hand, Theorem 1 defined a relationship between \bar{J} and Ψ . In addition, the alternatively conditioned intertemporal profit functions Ψ and $\tilde{\Psi}$ are related via (34), with the conditioning variables q and λ themselves related through (29). These results are now exploited to reconstruct the H–J equation in an amenable form.

LEMMA 2. *A Profit-Function Representation of the Hamilton-Jacobi Equation. Let the instantaneous consumer-firm economic profit function $\Pi(\lambda, \rho)$ be defined by (16) in the nonlinear case and simply by $\Phi(\lambda)$ in the linear technology case. Let the intertemporal profit function $\tilde{\Psi}(\lambda, \rho, \delta)$ be defined by (34), given (28) and (23). Define $\Omega(\rho, \delta)$ by (27) and $\Xi(\rho, \delta)$ by (30). Then the Hamilton–Jacobi equation for the intertemporal problem (1) has the representation*

$$\delta \tilde{\Psi}(\lambda, \rho, \delta) + [\rho - \delta] \lambda \tilde{\Psi}_\lambda(\lambda, \rho, \delta) = \Pi(\lambda, \rho), \quad F_{kk} = 0 \tag{36a}$$

$$\delta \tilde{\Psi}(\lambda, \rho, \delta) + [\rho - \delta] \lambda \tilde{\Psi}_\lambda(\lambda, \rho, \delta) = \Pi(\lambda, \rho) + \Omega(\rho, \delta),$$

$$F_{kk} < 0 \text{ and CO} \tag{36b}$$

$$\delta\tilde{\Psi}(\lambda, \rho, \delta) + [\rho - \delta]\lambda\tilde{\Psi}_\lambda(\lambda, \rho, \delta) + \delta\Xi(\rho, \delta)\tilde{\Psi}_\lambda(\lambda, \rho, \delta) = \Pi(\lambda, \rho),$$

$$F_{kk} < 0 \text{ and SO. (36c)}$$

Proof. Let k^* be optimal for problem (28). Then, k^* satisfies (35), and using this together with (34) to rearrange (28) gives

$$\bar{J}(k^*, \rho, \delta) = \Psi(q, \rho, \delta) + qk^* = \tilde{\Psi}(\lambda, \rho, \delta) + qk^*. \quad (37)$$

Now, LHS (37) can be represented as a costate conditioned variant of the optimal value function by the following definition:

$$\text{LHS(37)} = \bar{J}(k^*, \rho, \delta) = \bar{J}(-\tilde{\Psi}_\lambda(\lambda, \rho, \delta), \rho, \delta) \equiv \tilde{J}(\lambda, \rho, \delta). \quad (38)$$

On the other hand, RHS (37) has various representations appropriate to the definition of q from (29a)–(29c) for the linear, nonlinear CO, and nonlinear SO cases, respectively. When (38) is employed with each of these in turn, (37) becomes, for the various cases,

$$\tilde{J}(\lambda, \rho, \delta) = \tilde{\Psi}(\lambda, \rho, \delta) - \lambda\tilde{\Psi}_\lambda(\lambda, \rho, \delta), \quad F_{kk} = 0 \quad (39a)$$

$$\tilde{J}(\lambda, \rho, \delta) = \tilde{\Psi}(\lambda, \rho, \delta) - \lambda\tilde{\Psi}_\lambda(\lambda, \rho, \delta), \quad F_{kk} < 0 \text{ and CO} \quad (39b)$$

$$\tilde{J}(\lambda, \rho, \delta) = \tilde{\Psi}(\lambda, \rho, \delta) - \lambda\tilde{\Psi}_\lambda(\lambda, \rho, \delta) + \Xi(\rho, \delta)\tilde{\Psi}_\lambda(\lambda, \rho, \delta),$$

$$F_{kk} < 0 \text{ and SO. (39c)}$$

A matching costate conditioned variant of the Hamiltonian can be obtained by combining (35) with (19) to obtain

$$\tilde{H}(\lambda, \rho, \delta) \equiv \bar{H}(-\tilde{\Psi}_\lambda(\lambda, \rho, \delta), \lambda, \rho) = \Pi(\lambda, \rho) - \rho\lambda\tilde{\Psi}_\lambda(\lambda, \rho, \delta). \quad (40)$$

Now, for the linear, nonlinear CO, and nonlinear SO cases, the H-J equation is given by (26a)–(26c), respectively. Using (39a)–(39c) for J and (40) for H , the H-J equations (26a)–(26c) equate to (36a)–(36c) for the respective cases. ■

5. SOLUTION OF THE INTERTEMPORAL OPTIMIZATION PROBLEM

In this section the general solution to the reformulated H-J equation is presented for the linear technology case and for the nonlinear CO case, that is for (36a) and (36b), respectively, and the intertemporal optimizing behavioral equations are derived for these cases. The primary emphasis is placed upon the nonlinear CO case (36b). Clearly, the linear case (36a) is a special case of this. The nonlinear SO case is not pursued further. Solution of (36c) is more complex than (36b), and it is also less relevant to the determination of a system of equations capable of modeling optimal intertemporal choice via the representative-agent paradigm in a typical (imperfectly) competitive situation. Theorem 2 holds the key to the solution to (36b) because, under mild regularity conditions on the underlying consumer utility function and parameter restrictions that together imply the imposition of

an equivalent of the transversality condition (3d), the H-J equation in the form (36b) has a solution for the intertemporal economic profit function $\tilde{\Psi}(\lambda, \rho, \delta)$ in terms of the instantaneous economic profit function $\Pi(\lambda, \rho)$. Moreover, solutions exist for a wide range of regular and flexible specifications. Specifically, to solve (36b), it is necessary for the integral in (41) to be finite for all $\lambda \in (0, \infty)$. It can be shown that this implies finiteness of an integral expression for the optimal value function and equivalently for the Hamiltonian and hence implies satisfaction of the transversality condition. Given the restriction as stated in Theorem 2 and noting that λ is marginal utility, which for regular specifications of a utility function may be restricted without loss of generality to range from ∞ for $c = 0$ to 0 for $c \rightarrow \infty$, equation (36b) can be integrated, yielding the following result on the relationship between the intertemporal and instantaneous profit functions:

THEOREM 2. *Structure of the Intertemporal Profit Function.* *Let preferences and technology of the consumer-firm be represented by an instantaneous economic profit function $\Pi(\lambda, \rho)$. Suppose that $\Pi(\lambda, \rho) \underset{\lambda \rightarrow 0}{\gtrsim} \lambda^{-\delta/[\rho-\delta]} \bar{\Pi}(\rho)$ as $\rho \underset{\lambda \rightarrow 0}{\gtrsim} \delta$. Then, for the intertemporal optimization problem (1) under the competitive representative-agent paradigm, there is an implied intertemporal profit function of the form*

$$\tilde{\Psi}(\lambda, \rho, \delta) = \begin{cases} \Pi(\lambda, \rho)/\delta + \Omega(\rho, \delta)/\delta, & \rho = \delta \\ \frac{\lambda^{-\frac{\delta}{\rho-\delta}}}{\rho - \delta} \int_0^\lambda \zeta^{\frac{2\delta-\rho}{\rho-\delta}} \Pi(\zeta, \rho) d\zeta + \Omega(\rho, \delta)/\delta, & \rho > \delta \\ -\frac{\lambda^{-\frac{\delta}{\rho-\delta}}}{\rho - \delta} \int_\lambda^\infty \zeta^{\frac{2\delta-\rho}{\rho-\delta}} \Pi(\zeta, \rho) d\zeta + \Omega(\rho, \delta)/\delta, & \rho < \delta \end{cases} \quad (41)$$

Proof. By Lemma 2 and concentrating on the nonlinear CO case, the reformulated H-J equation (36b) applies. For $\rho = \delta$, it is obvious that $\tilde{\Psi} = \Pi/\delta + \Omega/\delta$. The separate solutions for $\rho > \delta$ and $\rho < \delta$ are structured to ensure that the integrals converge to finite values under the lower-bound assumption on the dependence of Π on λ if $\rho > \delta$ (thrift dominates) and the upper-bound assumption on the dependence of Π on λ if $\rho < \delta$ (impatience dominates). For $\rho > \delta$, conjecture a solution of the form

$$\tilde{\Psi} = a_1 \lambda^{b_1} \int_0^\lambda \zeta^{c_1} [\Pi(\zeta, \rho) + \Omega(\rho, \delta)] d\zeta.$$

Construct LHS (36b) explicitly for this conjecture and note that equality with RHS (36b) requires evaluation of a_1, b_1 , and c_1 as $1/(\rho - \delta), -\delta/(\rho - \delta)$, and $(2\delta - \rho)/(\rho - \delta)$, respectively. For $\rho < \delta$, conjecture a solution of the form

$$\tilde{\Psi} = a_2 \lambda^{b_2} \int_\lambda^\infty \zeta^{c_1} [\Pi(\zeta, \rho) + \Omega(\rho, \delta)] d\zeta.$$

Again construct LHS (36b) for this case and note that equality with RHS (36b) requires a_2, b_2 , and c_2 to equate to $-1/(\rho - \delta), -\delta/(\rho - \delta)$, and $(2\delta - \rho)/(\rho - \delta)$, respectively. The term involving $\Omega(\rho, \delta)$ may be integrated explicitly for these values to give the form (41). ■

The relevance of Theorem 2 is that it provides an operational basis for the use of Corollary 1, in view of the fact that it defines the intertemporal profit function in terms of the instantaneous function $\Pi(\lambda, \rho)$, a function that may be specified directly from considerations of empirical applicability. Although (41) indicates that $\tilde{\Psi}$ also depends additively upon $\Omega(\rho, \delta)$, this term is independent of λ and so is irrelevant for the application of Corollary 1. There are many specifications for which the integral in (41) can be solved analytically. Implications of Theorem 2 for construction of the optimal relationship between k and λ are presented in Corollary 2.

COROLLARY 2. *The Optimal State/Costate Relationship. For the competitive representative consumer-firm intertemporal optimization problem (1), let the intertemporal profit function be defined by (41). Then k and λ are optimally related by*

$$k = \begin{cases} -\Pi_\lambda(\lambda, \rho)/\rho, & \rho = \delta \\ \frac{\delta\lambda^{\frac{-\rho}{\rho-\delta}}}{(\rho-\delta)^2} \int_0^\lambda \zeta^{\frac{2\delta-\rho}{\rho-\delta}} \Pi(\zeta, \rho) d\zeta - \frac{\Pi(\lambda, \rho)}{(\rho-\delta)\lambda}, & \rho > \delta \\ -\frac{\delta\lambda^{\frac{-\rho}{\rho-\delta}}}{(\rho-\delta)^2} \int_\lambda^\infty \zeta^{\frac{2\delta-\rho}{\rho-\delta}} \Pi(\zeta, \rho) d\zeta - \frac{\Pi(\lambda, \rho)}{(\rho-\delta)\lambda}, & \rho < \delta. \end{cases} \quad (42)$$

Proof. Relationship (42) follows by application of (35), that is, $k = -\tilde{\Psi}_\lambda(\lambda, \rho, \delta)$, to (41). ■

Relationship (42) translates the intertemporal analogue of Hotelling’s Theorem (35) from a relationship involving the intertemporal economic profit function to one involving the economywide instantaneous profit function. It serves to determine (inversely and nonlinearly) λ as a function of k and ρ . Together with relationship (17), which defines ρ as $F_k(k)$, this effectively accomplishes the task of finding the initial value of λ to complete the specification of the model equations (5) for the competitive equilibrium solution to problem (1) in exact short-run form. To incorporate (42) into a system of equations modeling the representative-agent’s optimal consumption-investment decisions, it is convenient to reformulate (42). For purposes of interpretation, there are a number of interesting ways of reformulating (42). Using (20) and (21), (42) may be integrated by parts to obtain an optimal output–consumption relationship:

$$F(k) = \begin{cases} -\Phi_\lambda(\lambda), & \rho = \delta \\ \frac{\rho\lambda^{\frac{-\rho}{\rho-\delta}}}{\rho-\delta} \int_0^\lambda \zeta^{\frac{\delta}{\rho-\delta}} [-\Phi_\lambda(\zeta)] d\zeta, & \rho > \delta \\ \frac{-\rho\lambda^{\frac{-\rho}{\rho-\delta}}}{\rho-\delta} \int_\lambda^\infty \zeta^{\frac{\delta}{\rho-\delta}} [-\Phi_\lambda(\zeta)] d\zeta, & \rho < \delta. \end{cases} \quad (43)$$

Further rearrangement of (41)–(43), using (36b), (39b), (35), (14), and (13), allows the optimal value function to be expressed in terms of the Frischian instantaneous utility function as

$$\tilde{J}(\lambda, \rho, \delta) = \begin{cases} U^F(\lambda)/\delta + \Omega(\rho, \delta)/\delta, & \rho = \delta \\ \frac{\lambda^{-\frac{\delta}{\rho-\delta}}}{\rho - \delta} \int_0^\lambda \zeta^{\frac{2\delta-\rho}{\rho-\delta}} U^F(\zeta) d\zeta + \Omega(\rho, \delta)/\delta, & \rho > \delta \\ \frac{-\lambda^{-\frac{\delta}{\rho-\delta}}}{\rho - \delta} \int_\lambda^\infty \zeta^{\frac{2\delta-\rho}{\rho-\delta}} U^F(\zeta) d\zeta + \Omega(\rho, \delta)/\delta, & \rho < \delta \end{cases} \quad (44)$$

To set out the model in an easily interpretable form, it is convenient to define three further latent variables. These are, first, f , representing the value of current output and determined by some explicit specification of technology,

$$f = F(k); \quad (45)$$

second, ϕ , representing the value of instantaneous consumer profit and implying some specification of consumer preferences,

$$\phi = \Phi(\lambda); \quad (46)$$

and, third, j , representing the value of the agent-endogenous component of the optimal objective, defined as

$$j = \tilde{J}(\lambda, \rho, \delta) - \Omega(\rho, \delta)/\delta \quad (47)$$

and derived from (44) in view of definition (47), utilizing (13), as

$$j = \begin{cases} [\Phi(\lambda) - \lambda\Phi_\lambda(\lambda)]/\delta, & \rho = \delta \\ \frac{\lambda^{-\frac{\delta}{\rho-\delta}}}{\rho - \delta} \int_0^\lambda \zeta^{\frac{2\delta-\rho}{\rho-\delta}} [\Phi(\zeta) - \zeta\Phi_\lambda(\zeta)] d\zeta, & \rho > \delta \\ \frac{-\lambda^{-\frac{\delta}{\rho-\delta}}}{\rho - \delta} \int_\lambda^\infty \zeta^{\frac{2\delta-\rho}{\rho-\delta}} [\Phi(\zeta) - \zeta\Phi_\lambda(\zeta)] d\zeta, & \rho < \delta \end{cases} \quad (48)$$

The model can then be represented by seven equations. Four of these define latent variables: ρ , ϕ , j , and λ . (The variable f may also be treated as a latent variable if necessary.) Of these, ρ is determined as a function of the predetermined variable k via equation (17), that is, $\rho = F_k(k)$, and hence is dependent upon the technology specification (45). The other three latent variables (ϕ , j , and λ) are then determined jointly through a set of three nonlinear simultaneous equations. Specifically, ϕ is determined directly from (46) but conditionally on λ , j is constructed from (48), and λ is obtained by combining (14) with a simple rearrangement of (26b), which exploits the definitions (45), (46), and (47):

$$\lambda = [\delta j - \phi]/f. \quad (49)$$

Of these equations, the only complex relationship is the equation determining j , (48), which involves an integral calculation. Finally, given the solution for

λ from these equations, there are two equations that recursively determine the ultimate economic decision variables of interest—optimal consumption (12) and the optimal savings/investment choice (1b).

Collecting (45), (17), (46), (48), (49), (12), and (1b), and using (45) to simplify (1b), the full model is set out as

$$f = F(k), \tag{50a}$$

$$\rho = F_k(k), \tag{50b}$$

$$\phi = \Phi(\lambda), \tag{50c}$$

$$j = \begin{cases} [\Phi(\lambda) - \lambda\Phi_\lambda(\lambda)]/\rho, & \rho = \delta, \\ \frac{\lambda^{-\frac{\delta}{\rho-\delta}}}{\rho - \delta} \int_0^\lambda \zeta^{\frac{2\delta-\rho}{\rho-\delta}} [\Phi(\zeta) - \zeta\Phi_\lambda(\zeta)] d\zeta, & \rho > \delta, \\ \frac{-\lambda^{-\frac{\delta}{\rho-\delta}}}{\rho - \delta} \int_\lambda^\infty \zeta^{\frac{2\delta-\rho}{\rho-\delta}} [\Phi(\zeta) - \zeta\Phi_\lambda(\zeta)] d\zeta, & \rho < \delta, \end{cases} \tag{50d}$$

$$\lambda = [\delta j - \phi]/f, \tag{50e}$$

$$c = -\Phi_\lambda(\lambda), \tag{50f}$$

$$\dot{k} = f - c. \tag{50g}$$

Since the system (50) applies in both the linear and nonlinear CO cases, the only restrictions on $F(k)$ are the sign restrictions $F_k \geq 0$, $F_{kk} \leq 0$ together with any regularity conditions implied by the interpretation of $F(k)$ as a restricted value-added function. No “tractability” restrictions (related to the complexity of the intertemporal problem) are required. The only restrictions on Φ are those required for consistency with the atemporal problem (10), which it encapsulates—that is, the regularity conditions for utility-maximizing behavior. Again, no tractability restrictions are required, with the possible exception of restrictions on the functional form if an explicit analytical solution of the integral in (50d) is desired. However, it should be noted first that functional forms for $\Phi(\lambda)$ may be employed, dual to $U(c)$, which are much more general than the isoelastic specifications that are typically used in the literature and, second, analytical solution of the integral in (50d) is not actually required to undertake comparative economic response analysis with this model. The key comparative response results are those that express the instantaneous sensitivity of the costate variable λ to exogenous shocks such as changes in ρ and δ . These may be obtained by manipulation of the total differential of the simultaneous subsystem (50c), (50d), (50e).

6. EXAMPLE

As an example, a specification that contains isoelastic utility as a special case is considered. In the isoelastic special case, the utility function could be written

$$U(c) = \frac{c^{1-1/\epsilon} - 1}{1 - 1/\epsilon}, \tag{51}$$

where ϵ is the intertemporal elasticity of substitution, $0 < \epsilon < \infty$, and the logarithmic case corresponds to $\epsilon = 1$. From (10), the consumer profit function dual to (51) is

$$\Phi(\lambda) = \frac{\epsilon - \lambda^{1-\epsilon}}{1 - \epsilon}. \tag{52}$$

Consider, therefore, the more general consumer profit specification,

$$\Phi(\lambda) = \eta \frac{\epsilon_1 - \lambda^{1-\epsilon_1}}{1 - \epsilon_1} + (1 - \eta) \frac{\epsilon_2 - \lambda^{1-\epsilon_2}}{1 - \epsilon_2}, \tag{53}$$

where $0 < \eta < 1$ and, without loss of generality, it may be assumed that $0 < \epsilon_1 \leq \epsilon_2 < \infty$. Specification (53) reduces to the isoelastic case when $\epsilon_1 = \epsilon_2$.

Let the production technology be given by

$$F(k) = \alpha(k + \gamma)^\theta, \quad \theta \leq 1, \gamma \geq 0. \tag{54}$$

Given specifications (53) and (54), the system (50) becomes

$$f = \alpha(k + \gamma)^\theta, \tag{55a}$$

$$\rho = \alpha\theta(k + \gamma)^{\theta-1}, \tag{55b}$$

$$\phi = \eta \frac{\epsilon_1 - \lambda^{1-\epsilon_1}}{1 - \epsilon_1} + (1 - \eta) \frac{\epsilon_2 - \lambda^{1-\epsilon_2}}{1 - \epsilon_2}, \tag{55c}$$

$$j = \eta \left(\frac{\epsilon_1}{1 - \epsilon_1} \right) \left[\frac{1}{\delta} - \frac{\lambda^{1-\epsilon_1}}{\epsilon_1 \delta + (1 - \epsilon_1)\rho} \right] + (1 - \eta) \left(\frac{\epsilon_2}{1 - \epsilon_2} \right) \left[\frac{1}{\delta} - \frac{\lambda^{1-\epsilon_2}}{\epsilon_2 \delta + (1 - \epsilon_2)\rho} \right], \tag{55d}$$

$$\lambda = [\delta j - \phi]/f, \tag{55e}$$

$$c = \eta \lambda^{-\epsilon_1} + (1 - \eta) \lambda^{-\epsilon_2}, \tag{55f}$$

$$\dot{k} = f - c. \tag{55g}$$

In this example, the evaluation of the integral in (50d) leading to the expression for j in (55d) requires the parameter restrictions

$$\epsilon_2 < \frac{\rho}{\rho - \delta} \quad \text{if } \rho > \delta, \tag{56a}$$

$$\epsilon_1 > \frac{\rho}{\rho - \delta} \quad \text{if } \rho < \delta. \tag{56b}$$

Since $\epsilon_1 > 0$, (56b) is not an effective restriction and only (56a) is of concern. Restriction (56a) is required to ensure that $\epsilon_2\delta + (1 - \epsilon_2)\rho > 0$ in cases where $\rho > \delta$. This is necessary to ensure convergence of the integral in (50d) for the specifications (53) and (54). Consequently, it ensures satisfaction of the transversality condition (3d). Since growth is occurring when $\rho > \delta$ and ρ approaches δ from above as k rises, the restriction becomes less demanding as k rises. At lower k (or higher ρ), the restriction is more relevant, and if k effectively starts from zero, the parameter restriction at its most binding becomes, for the specification (54),

$$\epsilon_2 < \frac{\alpha\theta\gamma^{\theta-1}}{\alpha\theta\gamma^{\theta-1} - \delta}. \tag{57}$$

Restriction (57) allows $\epsilon_2 > 1$ except in the limiting case $\gamma = 0$ when the restriction requires $\epsilon_2 < 1$.

It can be shown that the optimal growth path for the specification (53)–(54) is not balanced. In fact, given the technology assumed in (54), this statement is true even for the isoelastic special case (52), let alone the generalized specification (53). Given the complexity of the optimal growth paths, it suffices to demonstrate the unbalanced growth for the combination of (52) and (54), with the additional simplification in (54) that $\gamma = 0$. For this case, the model equations (55) simplify to

$$f = \alpha k^\theta, \tag{58a}$$

$$\rho = \alpha\theta k^{\theta-1}, \tag{58b}$$

$$\phi = \frac{\epsilon - \lambda^{1-\epsilon}}{1 - \epsilon}, \tag{58c}$$

$$j = \left(\frac{\epsilon}{1 - \epsilon} \right) \left[\frac{1}{\delta} - \frac{\lambda^{1-\epsilon}}{\epsilon\delta + (1 - \epsilon)\rho} \right], \tag{58d}$$

$$\lambda = [\delta j - \phi]/f, \tag{58e}$$

$$c = \lambda^{-\epsilon}, \tag{58f}$$

$$\dot{k} = f - c. \tag{58g}$$

With these specifications, it is possible to explicitly solve (58c)–(58e) simultaneously. The solutions for the agent-endogenous component of the optimal value function, j , and for the costate variable, λ , are

$$j \equiv \bar{J}(k, \rho, \delta) - \frac{\Omega(\rho, \delta)}{\delta} = \frac{\left[\frac{\epsilon\delta + (1 - \epsilon)\rho}{\rho} \right]^{-1/\epsilon} \frac{(\alpha k^\theta)^{1-1/\epsilon}}{\rho} - \frac{1}{\delta}}{1 - 1/\epsilon}, \tag{59}$$

$$\lambda = \left\{ \frac{[\epsilon\delta + (1 - \epsilon)\rho]}{\theta} k \right\}^{-1/\epsilon}. \tag{60}$$

Using (58f), the explicit solution for optimal consumption in this case is

$$c = \frac{\epsilon\delta + (1 - \epsilon)\rho}{\theta} k. \tag{61}$$

Given that ρ is a variable, evolving in equilibrium as a function of k according to (58b), it follows from (61) that in the competitive equilibrium c/k is not constant and hence growth cannot be balanced along the optimal path. In example (58), of course, this result is entirely due to the nonlinear technology. In general, noniso-elastic utility such as in example (55) will also generate this effect. Looking at the evolution of consumption implied by the time derivative of (58f), using (58b) and also utilizing (5b), it can be seen that, for the model described by system (58),

$$\dot{c}/c = \epsilon[\alpha\theta k^{\theta-1} - \delta]. \tag{62}$$

Additionally, constructing \dot{k}/k from (58g) using (58a) and (61), the difference in the growth rates of the capital stock and consumption can be derived as

$$\frac{\dot{k}}{k} - \frac{\dot{c}}{c} = \epsilon \left(\frac{1 - \theta}{\theta} \right) [\alpha\theta k^{\theta-1} - \delta]. \tag{63}$$

Result (63) shows that the difference between the growth rates of consumption and capital stock is dependent upon the capital stock level. It also shows that linear technology ($\theta = 1$) is sufficient for the transition path to be balanced. However, the further technology is from linear, and the further is the marginal product of capital from the subjective rate of time preference, the greater will be the optimal contemporaneous imbalance between consumption growth and capital stock growth.

7. CONCLUSION

By making use of results from duality theory, it has been demonstrated that the Hamilton–Jacobi equation for dynamic optimization may be recast in a form relating concepts of intertemporal and instantaneous profit. An intertemporal analogue of Hotelling’s theorem is then available, and this allows the derivation of an explicit solution for the value of the costate variable as a function of the state variable and parameters under general preference specifications and for either linear technology or, in the case of nonlinear technology, for a competitive equilibrium solution under the representative-agent paradigm. The form of the explicit solution, which involves an integral expression in the representative consumer’s instantaneous

profit function, admits exact analytical solutions for quite general specifications of preferences and technology.

The general dynamic model of a representative optimizing consumer-firm is set out in equations (50). In employing these results to construct a general intertemporal economic model based on system (50), only functional forms at the atemporal level need to be specified, and this is the big advantage of this approach over the traditional intertemporal duality approach of Cooper and McLaren (1980), McLaren and Cooper (1980), and Epstein (1981). The contemporaneous relationship between the costate variable λ and all other variables/parameters is set out in the completely contemporaneous simultaneous subsystem (50c), (50d), and (50e), and this indicates the advantage of the approach over open-loop techniques, which require checking of transversality by computing candidate optimal paths over time. The potential advantage of the closed-loop system (50) is enhanced in the presence of nonlinearities. Since (50) holds contemporaneously for any empirically suitable functional forms for $\Phi(\lambda)$ and $F(k)$, regardless of distance from the stationary state, this also indicates the advantage of this formulation over open-loop approaches, which require linearized steady-state analysis.

A simple example is presented to demonstrate the power of the approach in analyzing the (most likely) empirically relevant case of off-balance transition dynamics.

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