A review on Poisson, Cox, Hawkes, shot-noise Poisson and dynamic contagion process and their compound processes

Jiwook Jang *and Rosy Oh †

Abstract

The Poisson process is an essential building block to move up to complicated counting processes, such as the Cox ('doubly stochastic Poisson') process, the Hawkes ('self-exciting') process, exponentially decaying shot-noise Poisson (simply 'shot-noise Poisson') process and the dynamic contagion process. The Cox process provides flexibility by letting the intensity not only depend on time but also allowing it to be a stochastic process. The Hawkes process has self-exciting property and clustering effects. Shot-noise Poisson process is an extension of the Poisson process, where it is capable of displaying the frequency, magnitude, and time period needed to determine the effect of points. The dynamic contagion process is a point process, where its intensity generalises the Hawkes process and Cox process with exponentially decaying shot noise intensity. To facilitate the usage of these processes in practice, we revisit the distributional properties of the Poisson, Cox, Hawkes, shot-noise Poisson and dynamic contagion process and their compound processes. We provide simulation algorithms for these processes, which would be useful to statistical analysis, further business applications and research. As an application of the compound processes, numerical comparisons of Value-at-Risk (VaR) and Tail Conditional Expectation (TCE or TailVaR) are made.

Keywords: Cox process; Hawkes process; Shot-noise Poisson process; Dynamic contagion process; Compound process

1. Introduction

The mathematical analysis of stochastic point processes has been an object of study for applied probabilities for many years (Cox, 1955; Bartlett, 1963; Snyder, 1975; Cox and Isham, 1980, 1986; Bremaud, 1981; Daley and Vere-Jones, 2003, 2008). In practice, to count number of points in time on which a point process is defined, counting processes such as the Poisson, Cox, Hawkes and dynamic contagion process have been used. They have been used to count the number of claims/losses in insurance modelling and the number of defaults in credit risk modelling.

The Poisson process, named after the French mathematician Siméon-Denis Poisson, is found to be an appropriate model in many applications including experiments on radioactive decay, and telephone call arrivals. It has been used in numerous disciplines such as astronomy, biology, ecology, geology, seismology, physics, economics, image processing and telecommunications (https://en.wikipedia.org/wiki/Poisson_point_process). The Poisson process is the simplest process associated with counting number of points as it has deterministic intensity and memoryless property. However even though it is the

*Department of Actuarial Studies & Business Analytics, Macquarie Business School, Macquarie University, Sydney NSW 2109, Australia., Email: jiwook.jang@mq.edu.au

†Institute of Mathematical Sciences, Ewha Womans University, Seoul, 03760, Korea., Email: rosy.oh5@gmail.com

simplest process in the counting process family, the Poisson process is important as it is an essential building block to move up to more complicated counting processes, i.e. the Cox process, the Hawkes process, shot-noise Poisson process and the dynamic contagion process.

The Cox process, named after the British statistician David Cox (1955), is a generalization of the Poisson process. Under the Cox process, its intensity function is assumed to be stochastic. The Cox process provides flexibility by letting the intensity not only depend on time but also allowing it to be a stochastic process. Hence in the Cox process, we can consider various stochastic processes for its intensity. The applications of Cox process in insurance context can be found in Grandell (1991, 1997), Rolski et al. (1998), Basu and Dassios (2002), Bening and Korolev (2002), Dassios and Jang (2003, 2005, 2008), Albrecher and Asmussen (2006), Dassios et al. (2015) and Jang et al. (2018b).

The Hawkes process, named after the British statistician Alan Hawkes (1971a, 1971b), is an extension of the Poisson process with self-exciting property, where points show clustering effects. Self-exciting (or Hawkes) processes (Hawkes, 1971a, 1971b, 1972; Hawkes and Oakes, 1974; Daley and Vere-Jones, 2003) are versatile point processes, interesting both from a theoretical as well as a practical point view. The theoretical foundation of Hawkes processes can be traced from a series of paper written by Brémaud and Massoulié (1996, 2001, 2002) and Liniger (2009). Relevant publications in seismology and the modelling of the occurrence of earthquakes are Vere-Jones (1975), Adamopoulos (1976), Vere-Jones (1978), Ozaki (1979), Vere-Jones and Ozaki (1982), and Ogata (1988).

The applications and modelling of Hawkes processes in finance can be found in Chavez-Demoulin et al. (2005), McNeil et al. (2005), Bowsher (2007), Bauwens and Hautsch (2009), Bacry et al. (2015) and Hawkes (2018). Credit default modelling using these processes can be noticed in Errais et al. (2010) and Giesecke and Kim (2011). Stabile and Torrisi (2010) applied Hawkes process in insurance context studying the asymptotic behavior of infinite and finite horizon ruin probabilities.

Embrechts et al. (2011) showed that multivariate Hawkes processes can be applied to the stock market indices. The applications and modelling of multivariate Hawkes process in high-frequency limit order book data can be found in Rombaldi et al. (2017) and Lu and Abergel (2018). Yang et al. (2018) investigated the interactions between market return events and investor sentiment using a multivariate Hawkes process. Gao et al. (2018) applied the joint Laplace transform of the classical Hawkes process and its compound process in dark pool trading, which do not display bid and ask quotes to the public. Jang and Dassios (2013) introduced a bivariate shot noise self-exciting process that can be used for the modelling of catastrophic losses. Dassios and Zhao (2013) introduced the numerical algorithm of exact simulation for Hawkes process with exponentially decaying intensity extending it to multi dimensions.

Numerous papers have also looked at the modeling of financial and insurance risks incorporating Hawkes processes into diffusion models. Aït-Sahalia et al. (2015) used a mutually-exiciting jump-diffusion process to model six stock market indices. Portfolio selections using Hawkes jump-diffusion models can be noticed in Aït-Sahalia and Hurd (2016) and Bian et al. (2019). Liu and Zhu (2019), Hainaut and Moraux (2018, 2019) and Ma et al. (2017, 2020) contain Hawkes jump-diffusion models in pricing and hedging context. Maneesoonthorn et al. (2017) modelled the price and stochastic volatility of an asset using a joint Hawkes process in conjunction with a bivariate jump diffusion for an empirical investigation on the S&P 500 market index. The applications of Hawkes jump-diffusion models have been reviewed by Hawkes (2020). Dassios and Zhao (2017a) developed exact simulation algorithms for a family of generalised self-exciting point processes with CIR-type intensities. Dassios et al. (2019) studied a generalised CIR process with externally-exciting and self-exciting jumps for insurance premium calculations and default-free zero-coupon bond pricing.

Shot noise occurs in connection with electron emissions from the cathode. Walter Schottky (1918), who studied fluctuations of current in vacuum tubes, introduced the concept of shot noise. Shot-noise Poisson process is another extension of the Poisson process, where it is capable of displaying the frequency, magnitude, and time period needed to determine the effect of points. The applications of shot noise Poisson process in insurance and credit risk context can be noticed in Klüppelberg and Mikosch (1995), Jang (2004), Jang and Krvavych (2004), Herbertsson et al. (2011) and Jang et al. (2018a).

The dynamic contagion process, which is a generalisation of the externally exciting Cox process with shot noise intensity and the self-exciting Hawkes process, was introduced by Dassios and Zhao (2011) applying to credit risk. Dassios and Zhao (2012) also examined infinite horizon ruin probability with its Monte Carlo simulation using this process as the claim arrival process. Dassios and Zhao (2017b) extended this process with diffusion component, i.e. self-exciting, externally-exciting and mean-reverting stochastic intensity to calculate the default probability and to price defaultable zero-coupon bonds.

Multivariate extension on the dynamic contagion process can be noticed in Dong (2014). She introduced the bivariate dynamic contagion process including the cross-exciting contagion effect, providing its exact simulation algorithm. The stationarity and the diffusion approximation of this process have been explored, where obtained Kalman-Bucy filter was used to calculate stop-loss reinsurance premium. She also showed that a sequence of scaled intensity process of the univariate dynamic contagion processes converges to a CIR process weakly in the path space, with which an alternative approximation simulation scheme for CIR process and the Heston model was developed. Jang and Oh (2020) introduced a bivariate compound dynamic contagion process for the modelling of aggregate losses from cyber events.

To promote the Poisson, Cox, Hawkes, shot-noise Poisson and dynamic contagion process and their compound processes widespread dissemination in academia and industry further, we revisit their distributional properties. What we present in this paper facilitate the usage of them in practice, enabling researchers to understand their probability characteristics. Given the current COVID-19 pandemic, it is envisaged that more multivariate works will be made to deal with the arrival of multiple, catastrophic and contagious losses accommodating the interdependence between risks.

This paper is structured as follows. In Section 2, we provide a mathematical definition of the dynamic contagion process. We also provide the infinitesimal generator of the dynamic contagion process, a generalised Hawkes process, the Cox process with shot-noise Poisson intensity, respectively. This section offers the joint Laplace transform - probability generating function, which is adopted from Dassios and Zhao (2011). By setting appropriate values to the relevant parameters, shot-noise self-exciting Poisson process, shot-noise self-exciting process, shot-noise Poisson process, the compound Poisson process and the Poisson process are also discussed. In Section 3, we deal with the compound dynamic contagion process, a generalised compound Hawkes process and the compound Cox process with shot-noise Poisson intensity. We analyse the compound dynamic contagion process systematically for its theoretical distributional property, based on the piecewise deterministic Markov process theory developed by Davis (1984), and the martingale methodology used by Dassios and Jang (2003). In Section 4, we present the moments and simulation algorithms for the compound processes. We also provide numerical comparisons of Value-at-Risk (VaR) and Tail Conditional Expectation (TCE or TailVaR) as an application of these processes. Section 5 concludes the paper.

2. Dynamic contagion process, Hawkes process and Cox process

2.1. Definition

In this section, we present the dynamic contagion process, based on which a generalised Hawkes process and the Cox process with shot-noise Poisson intensity are presented as its special cases. These processes are within the general framework of affine processes, for that see Duffie et al. (2000), Duffie et al. (2003) and Glasserman and Kim (2010). By setting appropriate values to the relevant parameters, we also deal with shot-noise self-exciting Poisson process, shot-noise self-exciting process, shot-noise Poisson process, the compound Poisson process and the Poisson process.

2.1.1. Dynamic contagion process

Let us start with a mathematical definition for the dynamic contagion process (DCP) in Definition 2.1 via the stochastic intensity representation. For an alternative definition for this process, we refer you Dassios and Zhao (2011), Jang and Dassios (2013) and Dong (2014), where they gave a cluster process representation for this process.

Definition 2.1. (Dynamic contagion process) Dynamic contagion process is a point process $N_t = \sum_{j\geq 1} \mathbb{I}\left(T_{2,j} \leq t\right)_{j=1,2,\cdots}$ with the non-negative $\Im_t-$stochastic intensity process λ_t, i.e.

$$\lambda_t = a + \left(\lambda_0 - a\right) e^{-\delta t} + \sum_{i\geq 1} X_i e^{-\delta\left(t-T_{1,i}\right)} \mathbb{I}\left(T_{1,i} \leq t\right) + \sum_{j\geq 1} Y_j e^{-\delta\left(t-T_{2,j}\right)} \mathbb{I}\left(T_{2,j} \leq t\right), \tag{2.1}$$

which is shot-noise self-exciting Poisson process, where

- $\{\Im_t\}_{t\geq 0}$ be a history of the process N_t, with respect to which $\{\lambda_t\}_{t\geq 0}$ is adapted;
- $\lambda_0 > 0$ is the initial intensity at time $t = 0$;
- $a \geq 0$ is the constant mean-reverting level;
- $\delta > 0$ is the rate of exponential decay;
- $\{X_i\}_{i=1,2,\cdots}$ is a sequence of *i.i.d.* positive externally-excited jumps with distribution $F(x)$, $x > 0$, at the corresponding random times $\{T_{1,i}\}_{i=1,2,\cdots}$ following a Poisson process M_t with constant rate $\rho > 0$, and $\mathbb{I}$ is the indicator function.
- $\{Y_j\}_{j=1,2,\cdots}$ is a sequence of *i.i.d.* positive self-excited jumps with distribution function $G(y)$, $y > 0$, at the corresponding random times $\{T_{2,j}\}_{j=1,2,\cdots}$ generated by the intensity process λ_t.
- $\{X_i\}_{i=1,2,\ldots}$, $\{Y_j\}_{j=1,2,\ldots}$, $\{T_{1,i}\}_{i=1,2,\cdots}$and $\{T_{2,j}\}_{j=1,2,\cdots}$ are assumed to be independent of each other.

With the aid of piecewise deterministic Markov process theory and using the results in Davis (1984), the infinitesimal generator of the dynamic contagion process (λ_t, N_t, t) acting on a function $f(\lambda, n, t)$ within its domain $\mathcal{D}(\mathcal{A})$ is given by

$$\mathcal{A}\ f(\lambda,n,t)=\frac{\partial f}{\partial t}+\delta(a-\lambda)\frac{\partial f}{\partial\lambda}+\lambda\left[\int_0^\infty f(\lambda+y,n+1,t)\,dG(y)-f(\lambda,n,t)\right]$$
$$+\rho\left[\int_0^\infty f(\lambda+x,n,t)\,dF(x)-f(\lambda,n,t)\right], \tag{2.2}$$

where $\mathcal{D}(\mathcal{A})$ is the domain of the generator $\mathcal{A}$ such that $f(\lambda,n,t)$ is differentiable with respect to λ and t for all λ and t, and

$$\left|\int_0^\infty f(\lambda+y,n+1,t)\,dG(y)-f(\lambda,n,t)\right|<\infty \quad\text{and}\quad \left|\int_0^\infty f(\lambda+x,n,t)\,dF(x)-f(\lambda,n,t)\right|<\infty.$$

If we ignore N_t in (2.2), it becomes the infinitesimal generator of shot-noise self-exciting Poisson process.

2.1.2. Generalised Hawkes process

If there are no externally-excited jumps in (2.2), i.e. $\rho=0$, we have the infinitesimal generator of a generalised Hawkes process (λ_t,N_t,t) acting on a function $f(\lambda,n,t)$ within its corresponding domain $\mathcal{D}(\mathcal{A})$, i.e.

$$\mathcal{A}\ f(\lambda,n,t)=\frac{\partial f}{\partial t}+\delta(a-\lambda)\frac{\partial f}{\partial\lambda}+\lambda\left[\int_0^\infty f(\lambda+y,n+1,t)\,dG(y)-f(\lambda,n,t)\right]. \tag{2.3}$$

If we ignore N_t in (2.3), it becomes the infinitesimal generator of shot-noise self-exciting process, that is called 'Hawkes process with exponential decay' in Dassios and Zhao (2011). All of these processes are extensions of the initial self-exciting processes proposed by Hawkes (1971a, 1971b, 1972).

2.1.3. Cox process

In this section, we start with a definition of the Cox process. Many alternative definitions of the Cox process can be given. We will offer the one adopted by Dassios and Jang (2003).

Definition 2.2. (Cox process) Let $(\Omega,\ F,\ P)$ be a probability space with information structure given by $F=\{\Im_t,\ t\in[0,T]\}$. Let N_t be a point process adapted to F. Let λ_t be a non- negative process adapted to F such that

$$\int_0^t \lambda_s ds<\infty \quad \textit{almost surely (no explosions)}.$$

If for all $0\le t_1\le t_2$ and $u\in R$

$$\mathbb{E}\left\{e^{iu\left(N_{t_2}-N_{t_1}\right)}|\Im_{t_2}\right\}=\exp\left\{\left(e^{iu}-1\right)\int_{t_1}^{t_2}\lambda_s ds\right\}, \tag{2.4}$$

then N_t is called a $\Im_t$-Cox process with intensity λ_t.

Equation (2.4) gives us

$$\Pr\left\{N_{t_2} - N_{t_1} = k | \lambda_s;\, t_1 \le s \le t_2\right\} = \frac{\exp\left(-\int\limits_{t_1}^{t_2} \lambda_s ds\right)\left(\int\limits_{t_1}^{t_2} \lambda_s ds\right)^k}{k!} \tag{2.5}$$

and consider the process $\Lambda_t = \int_0^t \lambda_s ds$, that is the aggregated intensity process (also known as the compensator of point process N_t), then we can easily find the probability generating function of N_t as

$$\mathbb{E}\left(\theta^{N_{t_2} - N_{t_1}} \mid \Im_{t_1}\right) = \mathbb{E}\left\{e^{-(1-\theta)\left(\Lambda_{t_2} - \Lambda_{t_1}\right)} \mid \Im_{t_1}\right\}, \tag{2.6}$$

where $0 \le \theta \le 1$. Equation (2.6) suggests that the problem of finding the distribution of N_t, the point process, is equivalent to the problem of finding the distribution of Λ_t, It means that we just have to find the p.g.f. (probability generating function) of N_t to retrieve the m.g.f. (moment generating function) of Λ_t and vice versa.

In the Cox process N_t, we can consider various stochastic processes for λ_t. This allows us to have flexibility in counting numbers of points including the Poisson counting when λ_t becomes deterministic.

Now if we set $a = 0$, and there are no self-excited jumps in (2.2), we have the infinitesimal generator of the Cox process with shot-noise Poisson intensity (λ_t, N_t, t) acting on a function $f(\lambda, n, t)$ within its corresponding domain $\mathcal{D}(\mathcal{A})$, i.e.

$$\begin{aligned}\mathcal{A}\ f(\lambda, n, t) &= \frac{\partial f}{\partial t} - \delta\lambda\frac{\partial f}{\partial \lambda} + \lambda\left[f(\lambda, n+1, t) - f(\lambda, n, t)\right] \\ &+\rho\left[\int\limits_0^\infty f(\lambda + x, n, t)\, dF(x) - f(\lambda, n, t)\right].\end{aligned} \tag{2.7}$$

Remark 2.1. *The dynamic contagion process N_t in Definition 2.1 is not a classical Cox process. Conditional on λ_t, N_t is not of the Poisson type and it does not satisfy* (2.6)*, i.e.*

$$\mathbb{E}\left(\theta^{N_{t_2} - N_{t_1}} \mid \Im_{t_1}\right) \neq \mathbb{E}\left\{e^{-(1-\theta)\left(\Lambda_{t_2} - \Lambda_{t_1}\right)} \mid \Im_{t_1}\right\}. \tag{2.8}$$

If we ignore N_t in (2.7), it becomes the infinitesimal generator of shot-noise Poisson process. Furthermore, if we set $\delta = 0$, we have the infinitesimal generator of the compound Poisson process, i.e.

$$\mathcal{A}\ f(\lambda, t) = \frac{\partial f}{\partial t} + \rho\left[\int\limits_0^\infty f(\lambda + x, t)\, dF(x) - f(\lambda, t)\right], \tag{2.9}$$

where it becomes the Poisson process if all the sizes of externally-excited jumps are fixed to be the same.

2.2. Distributional Properties

As the distributional property of the processes presented in Section 2.1, we offer their probability generating functions adopted from Dassios and Zhao (2011) and Dassios and Jang (2003). To do so, we start with stating the propositions adopted from Dassios and Zhao (2011), where N_t is the dynamic contagion process and λ_t is shot-noise self-exciting Poisson process. They studied the joint distributional property of the intensity process and the point process via the joint Laplace transform - probability generating function of (λ_T, N_T) for a fixed time T using the infinitesimal generator of (2.2) with the martingale methodology.

We denote the first-order moments of X and Y by

$$\mu_{1_F} = \int_0^\infty x dF(x)\,,\ \mu_{1_G} = \int_0^\infty y dG(y)$$

and their Laplace transforms by

$$\hat{f}(\eta) = \int_0^\infty e^{-\eta x} dF(x)\,,\ \hat{g}(\varphi) = \int_0^\infty e^{-\varphi y} dG(y),$$

where it is assumed that they are finite.

Proposition 2.1. *Considering the constants, $0 \le \theta \le 1$, $\upsilon \ge 0$ and time $0 \le t \le T$, we have the conditional joint Laplace transform - probability generating function of the process λ_T and the point process N_T is given by*

$$\mathbb{E}\left[\theta^{(N_T - N_t)} e^{-\upsilon \lambda_T} \mid \Im_t\right] = e^{-B(t)\lambda_t} e^{-\{C(T)-C(t)\}}, \tag{2.10}$$

where $B(t)$ is determined by the non-linear ordinary differential equation (ODE)

$$-B'(t) + \delta B(t) + \theta \hat{g}\{B(t)\} - 1 = 0, \tag{2.11}$$

with the boundary condition $B(T) = \upsilon$, and $C(t)$ is determined by

$$C(t) = \rho \int_0^t \left[1 - \hat{f}\{B(s)\}\right] ds + a\delta \int_0^t B(s)\, ds. \tag{2.12}$$

Proposition 2.2. *The conditional probability generating function of the dynamic contagion process N_T given λ_0 and $N_0 = 0$ at time $t = 0$, under the condition $\delta > \mu_{1_G}$, is given by*

$$\mathbb{E}\left[\theta^{N_T} \mid \lambda_0\right] = \exp\left\{-\mathcal{G}_{0,\theta}^{-1}(T)\ \lambda_0\right\} \times \exp\left[-\int_0^{\mathcal{G}_{0,\theta}^{-1}(T)} \left[\frac{a\delta\ u + \rho\left\{1 - \hat{f}(u)\right\}}{1 - \delta\ u - \theta \hat{g}(u)}\right] du\right], \tag{2.13}$$

where

$$\mathcal{G}_{0,\theta}(\Psi) =: \int_0^\Psi \frac{1}{1 - \delta\ u - \theta \hat{g}(u)} du.$$

If there are no externally-excited jumps in (2.13), i.e. $\rho = 0$, the conditional probability generating function of a generalised Hawkes process N_T given λ_0 and $N_0 = 0$ at time $t = 0$, under the condition $\delta > \mu_{1_G}$, is given by

$$\mathbb{E}\left[\theta^{N_T} \mid \lambda_0\right] = \exp\left\{-\mathcal{G}_{0,\theta}^{-1}(T)\ \lambda_0\right\} \times \exp\left[-\int_0^{\mathcal{G}_{0,\theta}^{-1}(T)} \left[\frac{a\delta\ u}{1-\delta\ u-\theta\hat{g}(u)}\right] du\right]. \qquad (2.14)$$

If we set $a = 0$, and there are no self-excited jumps in (2.13), the conditional probability generating function of the Cox process with shot-noise Poisson intensity N_T given λ_0 and $N_0 = 0$ at time $t = 0$, is given by

$$\mathbb{E}\left[\theta^{N_T} \mid \lambda_0\right] = \exp\left\{-\frac{1-\theta}{\delta}\left(1-e^{-\delta T}\right)\lambda_0\right\} \times \exp\left[-\rho\int_0^T \left[1-\hat{f}\left\{\frac{1-\theta}{\delta}\left(1-e^{-\delta u}\right)\right\}\right] du\right], \quad (2.15)$$

for which see Dassios and Jang (2003).

If we set $\theta = 1$ in (2.10), we can easily obtain the conditional Laplace transform of shot-noise self-exciting Poisson process λ_T given λ_0, i.e. $\mathbb{E}\left[e^{-\upsilon\lambda_T} \mid \lambda_0\right]$ as its distributional property. By setting appropriate values to the relevant parameters in this Laplace transform, the corresponding conditional Laplace transforms of shot-noise self-exciting Poisson process, shot-noise self-exciting process, shot-noise Poisson process and the compound Poisson process can be also obtained, respectively.

3. Compound Dynamic contagion process, Compound Hawkes process and Compound Cox process

3.1. Definition

In Section 2, we presented the dynamic contagion process, a generalised Hawkes process and the Cox process with shot-noise Poisson intensity. So in this section, let us study their compound processes that can be used for the modelling of aggregate claims/losses/defaults. These compound processes belong to the more general class of affine processes. The compound Poisson process was discussed in Section 2.

3.1.1. Compound dynamic contagion process

Let us start with a mathematical definition for the compound dynamic contagion process (CDCP) in Definition 3.1 via the stochastic intensity representation.

Definition 3.1. (Compound dynamic contagion process) Compound dynamic contagion process is a compound point process $L_t = \sum_{j\geq 1} \Xi_j \mathbb{I}\left(T_{2,j} \leq t\right)_{j=1,2,\cdots}$ with the non-negative $\Im_t$−stochastic intensity process λ_t which is in the form of (2.1), where $\{\Xi_j\}_{j=1,2,\cdots}$ is a sequence of *i.i.d.* positive individual claim/loss amounts with distribution function $J(\xi)$, $\xi > 0$, at the corresponding random times $\{T_{2,j}\}_{j=1,2,\cdots}$. It is assumed that $\{X_i\}_{i=1,2,\ldots}$, $\{Y_j\}_{j=1,2,\ldots}$, $\{\Xi_j\}_{j=1,2,\ldots}$, $\{T_{1,i}\}_{i=1,2,\cdots}$ and $\{T_{2,j}\}_{j=1,2,\cdots}$ are independent of each other.

The infinitesimal generator of the compound dynamic contagion process (λ_t, N_t, L_t, t) acting on a function $f(\lambda, n, l, t)$ within its domain $\mathcal{D}(\mathcal{A})$ is given by

$$\begin{aligned}
&\mathcal{A}\ f(\lambda, n, l, t) = \frac{\partial f}{\partial t} + \delta(a - \lambda)\frac{\partial f}{\partial \lambda} \\
&+\lambda\left[\int_0^\infty\int_0^\infty f(\lambda + y, n+1, l+\xi, t)\,dG(y)dJ(\xi) - f(\lambda, n, l, t)\right] \\
&+\rho\left[\int_0^\infty f(\lambda + x, n, l, t)\,dF(x) - f(\lambda, n, l, t)\right],
\end{aligned} \tag{3.1}$$

where $\mathcal{D}(\mathcal{A})$ is the domain of the generator $\mathcal{A}$ such that $f(\lambda, n, l, t)$ is differentiable with respect to λ and t for all λ and t, and

$$\left|\int_0^\infty\int_0^\infty f(\lambda + y, n+1, l+\xi, t)\,dG(y)dJ(\xi) - f(\lambda, n, l, t)\right| < \infty,$$

$$\left|\int_0^\infty f(\lambda + x, n, l, t)\,dF(x) - f(\lambda, n, l, t)\right| < \infty.$$

3.1.2. Generalised Compound Hawkes process

If there are no externally-excited jumps in (3.1), we have the infinitesimal generator of a generalised compound Hawkes process (λ_t, N_t, L_t, t) acting on a function $f(\lambda, n, l, t)$ within its corresponding domain $\mathcal{D}(\mathcal{A})$, i.e.

$$\begin{aligned}
&\mathcal{A}\ f(\lambda, n, l, t) = \frac{\partial f}{\partial t} + \delta(a - \lambda)\frac{\partial f}{\partial \lambda} \\
&+\lambda\left[\int_0^\infty\int_0^\infty f(\lambda + y, n+1, l+\xi, t)\,dG(y)dJ(\xi) - f(\lambda, n, l, t)\right].
\end{aligned} \tag{3.2}$$

3.1.3. Compound Cox process

If we set $a = 0$, and there are no self-excited jumps in (3.1), we have the infinitesimal generator of the compound Cox process with shot-noise Poisson intensity (λ_t, N_t, L_t, t) acting on a function $f(\lambda, n, l, t)$ within its corresponding domain $\mathcal{D}(\mathcal{A})$, i.e.

$$
\begin{aligned}
\mathcal{A}\ f(\lambda, n, l, t) &= \frac{\partial f}{\partial t} - \delta\lambda \frac{\partial f}{\partial \lambda} \\
&+\lambda \left[\int_0^\infty f(\lambda, n+1, l+\xi, t)\, dJ(\xi) - f(\lambda, n, l, t) \right] \\
&+\rho \left[\int_0^\infty f(\lambda + x, n, l, t)\, dF(x) - f(\lambda, n, l, t) \right].
\end{aligned} \tag{3.3}
$$

3.2. Distributional Properties

As the distributional property of the compound processes, we offer their Laplace transforms in this subsection. To do so, we start with deriving the joint distributional property of the intensity process, the point process and the compound point process via joint Laplace transform, probability generating function of (λ_t, N_t, L_t) for a fixed time T, where L_t is the compound dynamic contagion process, N_t is the dynamic contagion process and λ_t is shot-noise self-exciting Poisson process.

We denote the first-order moment of ξ by

$$
\mu_{1_J} = \int_0^\infty \xi dJ(\xi)
$$

and its Laplace transform by

$$
\hat{j}(\kappa) = \int_0^\infty e^{-\kappa\xi} dJ(\xi),
$$

where it is assumed to be finite.

Theorem 3.1. *Considering the constants,* $0 \le \theta \le 1$, $\nu \ge 0$, $\upsilon \ge 0$ *and time* $0 \le t \le T$, *the conditional joint Laplace transform, probability generating function of the process* λ_T, *the point process* N_T *and the compound point process* L_T *is given by*

$$
\mathbb{E}\left[\theta^{(N_T - N_t)} e^{-\nu(L_T - L_t)} \times e^{-\upsilon\lambda_T} \mid \Im_t \right] = e^{-B(t)\lambda_t} e^{-\{C(T) - C(t)\}}, \tag{3.4}
$$

where $B(t)$ *is determined by the non-linear ordinary differential equation (ODE)*

$$
-B'(t) + \delta B(t) + \theta\ \hat{g}\{B(t)\}\ \hat{j}(\nu) - 1 = 0 \tag{3.5}
$$

with the boundary condition $B(T) = \upsilon$, *and* $C(t)$ *is determined by*

$$
C(t) = \rho \int_0^t \left[1 - \hat{f}\{B(s)\} \right] ds + a\delta \int_0^t B(s)\, ds. \tag{3.6}
$$

Proof. See Appendix A. □

Theorem 3.2. *The conditional joint Laplace transform of the process λ_T and the compound point process L_T given λ_0 and $L_0 = 0$ at time $t = 0$ is given by*

$$\begin{aligned}
&\mathbb{E}\left[e^{-\nu L_T} e^{-\upsilon\lambda_T} \mid \lambda_0\right] \\
= \; &\exp\left\{-\mathcal{G}^{-1}_{\upsilon,\nu}(T)\ \lambda_0\right\} \\
&\times \exp\left[-\rho \int_0^T \left[1 - \hat{f}\left\{\mathcal{G}^{-1}_{\upsilon,\nu}(\tau)\right\}\right] d\tau\right] \\
&\times \exp\left[-\int_{\mathcal{G}^{-1}_{\upsilon,\nu}(T)}^{\upsilon} \left\{\frac{a\delta\ u}{\delta\ u + \hat{j}(\nu)\hat{g}(u) - 1}\right\} du\right],
\end{aligned} \tag{3.7}$$

where

$$\mathcal{G}_{\upsilon,\nu}(\Psi) = \int_{\Psi}^{\upsilon} \left[\frac{1}{\delta\ u + \hat{j}(\nu)\hat{g}(u) - 1}\right] du \quad \textit{and} \quad \delta > \hat{j}(\nu)\,\mu_{1_G}.$$

Proof. See Appendix B. □

Now let us derive the conditional Laplace transform of the process L_T for a fixed time T in Theorem 3.3, where N_t is the dynamic contagion process and λ_t is shot-noise self-exciting Poisson process.

Theorem 3.3. *The conditional Laplace transform of the compound dynamic contagion process L_T given λ_0 and $L_0 = 0$ at time $t = 0$ is given by*

$$\begin{aligned}
&\mathbb{E}\left[e^{-\nu L_T} \mid \lambda_0\right] \\
= \; &\exp\left\{-\mathcal{G}^{-1}_{0,\nu}(T)\ \lambda_0\right\} \times \exp\left[-\int_0^{\mathcal{G}^{-1}_{0,\nu}(T)} \left\{\frac{a\delta\ u + \rho\left\{1 - \hat{f}(u)\right\}}{1 - \delta\ u - \hat{j}(\nu)\hat{g}(u)}\right\} du\right].
\end{aligned} \tag{3.8}$$

Proof. See Appendix C. □

If there are no externally-excited jumps in (3.8), i.e. $\rho = 0$, the conditional Laplace transform of a generalised compound Hawkes process L_T given λ_0 and $L_0 = 0$ at time $t = 0$, under the condition $\delta > \hat{j}(\nu)\ \mu_{1_G}$, is given by

$$\begin{aligned}
&\mathbb{E}\left[e^{-\nu L_T} \mid \lambda_0\right] \\
= \; &\exp\left\{-\mathcal{G}^{-1}_{0,\nu}(T)\ \lambda_0\right\} \times \exp\left[-\int_0^{\mathcal{G}^{-1}_{0,\nu}(T)} \left\{\frac{a\delta\ u}{1 - \delta\ u - \hat{j}(\nu)\hat{g}(u)}\right\} du\right].
\end{aligned} \tag{3.9}$$

If we set $a = 0$, and there are no self-excited jumps in (3.8), the conditional Laplace transform of the compound Cox process with shot-noise Poisson intensity L_T given λ_0 and $L_0 = 0$ at time $t = 0$, is given by

$$\begin{aligned} & \mathbb{E}\left[e^{-\nu L_T} \mid \lambda_0\right] \\ = \; & \exp\left\{-\frac{1-\hat{j}(\nu)}{\delta}\left(1-e^{-\delta T}\right)\lambda_0\right\} \times \exp\left[-\rho\int_0^T\left[1-\hat{f}\left\{\frac{1-\hat{j}(\nu)}{\delta}\left(1-e^{-\delta u}\right)\right\}\right]du\right]. \end{aligned} \tag{3.10}$$

Remark 3.1. *The compound dynamic contagion process L_t in Definition 3.1 is not a classical compound Cox process. Conditional on λ_t, L_t is not of the compound Poisson type and it does not satisfy the below, i.e.*

$$\mathbb{E}\left[e^{-\nu\left(L_{t_2}-L_{t_1}\right)} \mid \Im_{t_1}\right] \neq \mathbb{E}\left[e^{-\left\{1-\hat{j}(\nu)\right\}\left(\Lambda_{t_2}-\Lambda_{t_1}\right)} \mid \Im_{t_1}\right]. \tag{3.11}$$

4. Moments, VaR and TCE

In this section, we start with presenting the conditional expectations for the compound processes studied in Section 3.

Theorem 4.1. *The conditional expectation of the compound dynamic contagion process L_T given λ_0 at time $t = 0$, is given by*

$$\mathbb{E}\left(L_t \mid \lambda_0\right) = L_0 + \mu_{1_J}\left\{\begin{array}{c}\left(\lambda_0 - \frac{\mu_{1_F}\;\rho+a\delta}{\delta-\mu_{1_G}}\right)\left(\frac{1-e^{-\left(\delta-\mu_{1_G}\right)t}}{\delta-\mu_{1_G}}\right) \\ +\left(\frac{\mu_{1_{F_1}}\;\rho+a\delta}{\delta-\mu_{1_G}}\right)t\end{array}\right\}, \; \textit{for } \delta \neq \mu_{1_G}. \tag{4.1}$$

Proof. See Appendix D. □

The conditional variance of the compound dynamic contagion process can be also obtained, however as its expression would be very long with various simple exponential functions, we omit it.

If there are no externally-excited jumps in (4.1), i.e. $\rho = 0$, the conditional expectation of a generalised compound Hawkes process given λ_0 at time $t = 0$ is given by

$$\mathbb{E}\left(L_t \mid \lambda_0\right) = L_0 + \mu_{1_J}\left\{\begin{array}{c}\left(\lambda_0 - \frac{a\delta}{\delta-\mu_{1_G}}\right)\left(\frac{1-e^{-\left(\delta-\mu_{1_G}\right)t}}{\delta-\mu_{1_G}}\right) \\ +\left(\frac{a\delta}{\delta-\mu_{1_G}}\right)t\end{array}\right\}, \; \text{for } \delta \neq \mu_{1_G}. \tag{4.2}$$

If there are no self-excited jumps in (4.1), the conditional expectation of the compound Cox process with mean-reverting shot-noise Poisson intensity given λ_0 at time $t = 0$ is given by

$$\mathbb{E}\left(L_t \mid \lambda_0\right) = L_0 + \mu_{1_J}\left\{\begin{array}{c}\left(\lambda_0 - \frac{\mu_{1_F}\;\rho+a\delta}{\delta}\right)\left(\frac{1-e^{-\delta t}}{\delta}\right) \\ +\left(\frac{\mu_{1_F}\;\rho+a\delta}{\delta}\right)t\end{array}\right\}, \; \text{for } \delta \neq 0. \tag{4.3}$$

If we set $a = 0$, and there are no self-excited jumps in (4.1), the conditional expectation of the compound Cox process with shot-noise Poisson intensity given λ_0 at time $t = 0$ is given by

$$\mathbb{E}(L_t \mid \lambda_0) = L_0 + \mu_{1_J}\left\{\left(\lambda_0 - \frac{\mu_{1_F}\,\rho}{\delta}\right)\left(\frac{1-e^{-\delta t}}{\delta}\right) + \left(\frac{\mu_{1_F}\,\rho}{\delta}\right)t\right\}, \text{ for } \delta \neq 0. \tag{4.4}$$

If we denote λ_t with L_t in (2.9) replacing μ_{1_F} with μ_{1_J}, and also ρ with η, the expectation of the compound Poisson process is given by

$$\mathbb{E}(L_t) = \mu_{1_J}\eta t. \tag{4.5}$$

For the moments of N_t and λ_t presented in Section 2, we refer to Jang and Dassios (2013), Dassios and Zhao (2011) and Dassios and Jang (2003).

The moments of L_t can be used for net actuarial premium calculations for the aggregate claim amounts up to time t. It can be also used to model for the aggregate losses from operational or credit risk, so we compare numerical values of Value-at-Risk (VaR) and Tail Conditional Expectation (TCE or TailVaR), assuming that $L_0 = 0$. The VaR of $L_t \mid \lambda_0$ at the level q is the threshold level l such that

$$\mathbb{P}\{(L_t \mid \lambda_0) > l\} = q.$$

That is to say, if $\text{VaR}_q(L_t \mid \lambda_0)$ denotes the VaR of $L_t \mid \lambda_0$ at the level q, then it satisfies

$$\mathbb{P}\{(L_t \mid \lambda_0) > VaR_q(L_t \mid \lambda_0)\} = q$$

and

$$TCE_q(L_t \mid \lambda_0) = \mathbb{E}\{(L_t \mid \lambda_0) \mid (L_t \mid \lambda_0) \geq VaR_q(L_t \mid \lambda_0)\}.$$

For this purpose, and to make it easier for statistical analysis, further business applications and research, we provide the simulation algorithm for one sample path of the compound dynamic contagion process (L_t, N_t, λ_t), with m jump times $\{T_1^*, T_2^*, \cdots, T_m^*\}$ in the process λ_t (see Figure 1). This algorithm is from Dassios and Zhao (2011) Section 5 algorithm, where they have shown how to simulate the dynamic contagion process. We also refer Ogata's thinning method (1981) and Møller and Rasmussen (2005) for Hawkes processes simulations.

We provide the simulation algorithm for one sample path of the compound Cox process with shot noise intensity over a time interval $[0, T]$, adapted the one in Čížek et al. (2011) (see Figure 2). We omit the simulation algorithm for one sample path of the compound Poisson process at it is trivial.

Algorithm 4.1. (The compound dynamic contagion process simulation algorithm)

1. Set the initial conditions $T_0^* = 0$, $\lambda_{T_0^{*+}} = \lambda_0 > a$, and $i \in \{0, 1, 2, \ldots, m-1\}$.

2. Simulate the $(i+1)^{\text{th}}$ externally excited jump waiting time E_{i+1}^* by

$$E_{i+1}^* = -\frac{1}{\rho}\ln U, \qquad U \sim \mathrm{U}(0,1).$$

3. Simulate the $(i+1)^{\text{th}}$ self-excited jump waiting time S_{i+1}^* by

$$S_{i+1}^* = \begin{cases} S_{i+1}^{*(1)} \wedge S_{i+1}^{*(2)} & (d_{i+1} > 0) \\ S_{i+1}^{*(2)} & (d_{i+1} < 0) \end{cases},$$

where

$$d_{i+1} = 1 + \frac{\delta \ln U_1}{\lambda_{T_i^{*+}} - a}, \qquad U_1 \sim \mathrm{U}(0,1)$$

and

$$S_{i+1}^{*(1)} = -\frac{1}{\delta} \ln d_{i+1}; \qquad S_{i+1}^{*(2)} = -\frac{1}{a} \ln U_2, \qquad U_2 \sim \mathrm{U}(0,1).$$

4. Simulate the $(i+1)^{\text{th}}$ overall jump time T_{i+1}^* by

$$T_{i+1}^* = T_i^* + S_{i+1}^* \wedge E_{i+1}^*.$$

5. The changes at jump time T_{i+1}^* in the intensity process λ_t is given by

$$\lambda_{T_{i+1}^{*+}} = \begin{cases} \lambda_{T_{i+1}^{*-}} + Y_{i+1}, & Y_{i+1} \sim G(y) \quad \left(S_{i+1}^* \wedge E_{i+1}^* = S_{i+1}^*\right) \\ \lambda_{T_{i+1}^{*-}} + X_{i+1}, & X_{i+1} \sim F(x) \quad \left(S_{i+1}^* \wedge E_{i+1}^* = E_{i+1}^*\right) \end{cases},$$

where

$$\lambda_{T_{i+1}^{*-}} = (\lambda_{T_i^{*+}} - a)e^{-\delta(T_{i+1}^* - T_i^*)} + a.$$

6. The change at jump time T_{i+1}^* in the point process N_t is given by

$$N_{T_{i+1}^{*+}} = \begin{cases} N_{T_{i+1}^{*-}} + 1 & \left(S_{i+1}^* \wedge E_{i+1}^* = S_{i+1}^*\right) \\ N_{T_{i+1}^{*-}} & \left(S_{i+1}^* \wedge E_{i+1}^* = E_{i+1}^*\right) \end{cases}.$$

7. The change at jump time T_{i+1}^* in the compound point process L_t is given by

$$L_{T_{i+1}^{*+}} = \begin{cases} L_{T_{i+1}^{*-}} + \xi_{i+1}, & \xi_{i+1} \sim J(\xi) \quad \left(S_{i+1}^* \wedge E_{i+1}^* = S_{i+1}^*\right) \\ L_{T_{i+1}^{*-}} & \quad \left(S_{i+1}^* \wedge E_{i+1}^* = E_{i+1}^*\right) \end{cases}.$$

Algorithm 4.2. (The compound Cox process with shot noise intensity simulation algorithm)

1. Generate the intensity process λ_t over a time interval $[0, T]$ with $H(T)$ jump times $\{t_1, t_2, \cdots, t_{H(T)}\}$:
   (i) Simulate $H(T)$ number of jumps by

   $$H(T) \sim \text{Poisson}(\rho T).$$

   (ii) Set the initial conditions $t_0 = 0$, $\lambda_0 > 0$, and $j \in \{0, 1, 2, \ldots, H(T) - 1\}$.

   (iii) Conditional on the number of jumps in a Poisson process in the time interval $[0, T]$, the arrival times have a uniform $(0, T)$ distribution. Hence, simulate the $(j+1)^{\text{th}}$ conditional jump arrival time t_{j+1} by

   $$t_{j+1} \sim \mathrm{U}(0, T).$$

(iv) The changes at jump time t_{j+1} in the intensity process λ_t is given by

$$\lambda_{t_{j+1}^+} = \lambda_{t_{j+1}^-} + X_{j+1}, \quad X_{j+1} \sim F(x),$$

where

$$\lambda_{t_{j+1}^-} = \lambda_{t_j^+} \, e^{-\delta(t_{j+1}-t_j)}.$$

2. Set the initial conditions $T_0^* = 0$, $\tau = 0$, $\lambda^\# = \max\{\lambda_t;\ t \in [0,T]\}$, and $i \in \{0,1,2,\ldots,K(T)-1\}$, where $K(T) = \max\{i;\ T_i^* < T\}$.

3. Simulate jump times $\{T_1^*, T_2^*, \cdots, T_{K(T)}^*\}$ by the following steps:

(i) Simulate the jump waiting time E^* with intensity $\lambda^\#$ by

$$E^* = -\frac{1}{\lambda^\#} \ln U_1, \qquad U_1 \sim \mathrm{U}(0,1).$$

(ii) Simulate the candidate jump time τ by

$$\tau = \tau + E^*.$$

(iii) The jump time T_{i+1}^* is given by τ if

$$U_2 \le \frac{\lambda_\tau}{\lambda^\#}, \qquad U_2 \sim \mathrm{U}(0,1),$$

where

$$\lambda_\tau = \lambda_{t^*} \, e^{-\delta(\tau - t^*)} \quad \text{and} \quad t^* = \max\{t_s;\ t_s < \tau,\ s \in \{0,1,\cdots,H(T)\}\}.$$

4. The change at T_{i+1}^* in the point process N_t is given by

$$N_{T_{i+1}^{*+}} = N_{T_{i+1}^{*-}} + 1.$$

5. The change at T_{i+1}^* in the compound point process L_t is given by

$$L_{T_{i+1}^{*+}} = L_{T_{i+1}^{*-}} + \xi_{i+1},\ \xi_{i+1} \sim J(\xi).$$

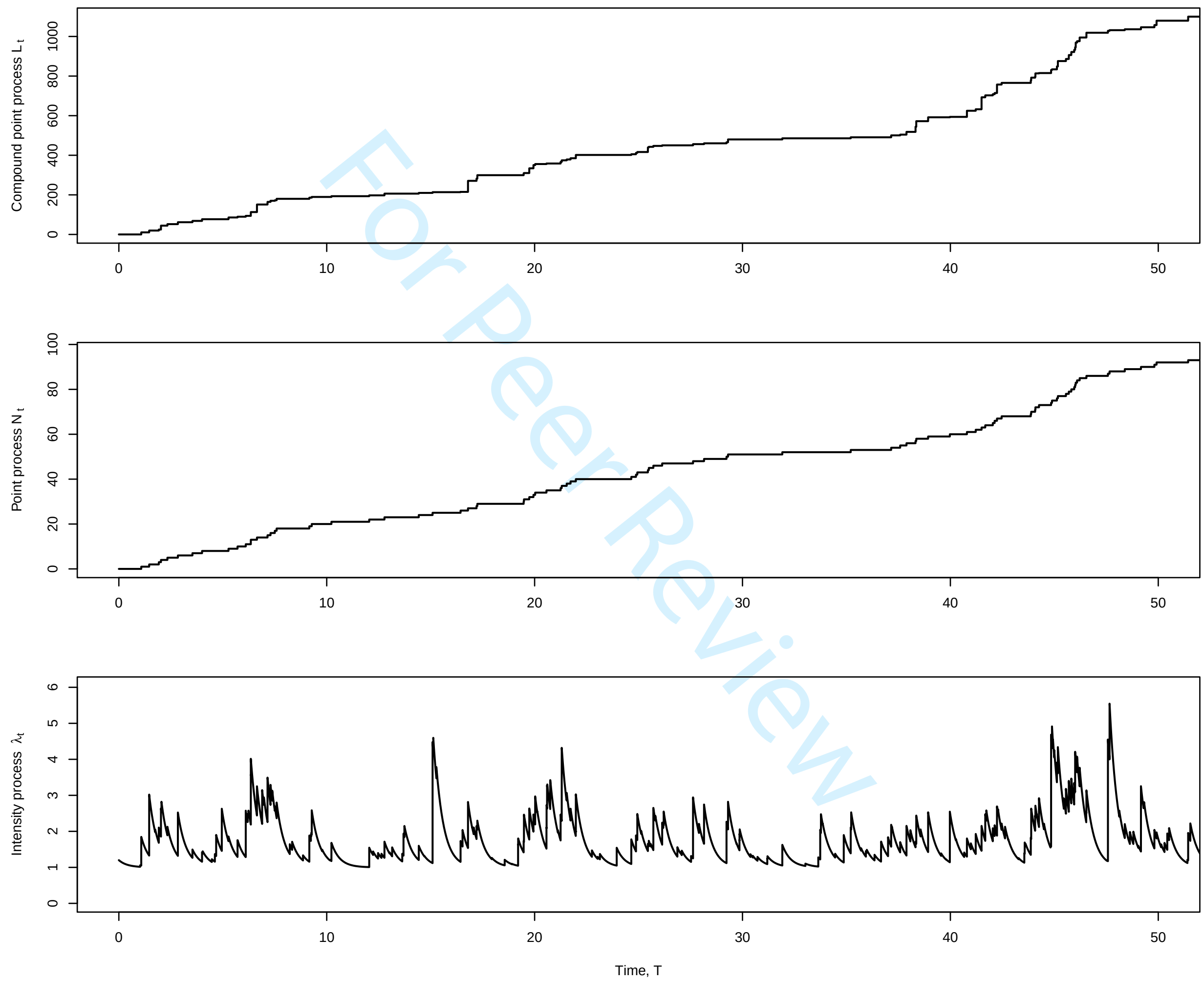


Figure 1: Simulated sample path of the compound dynamic contagion process: Intensity process λ_t, point processes N_t, and compound point processes L_t, with the parameters $(a; \rho; \delta; \alpha; \psi; \varsigma; c; \omega; \zeta; k; \lambda_0) = (1; 3; 2.5; 5; 1; 5.5; 3; 3; 4; 6; 1.2)$.

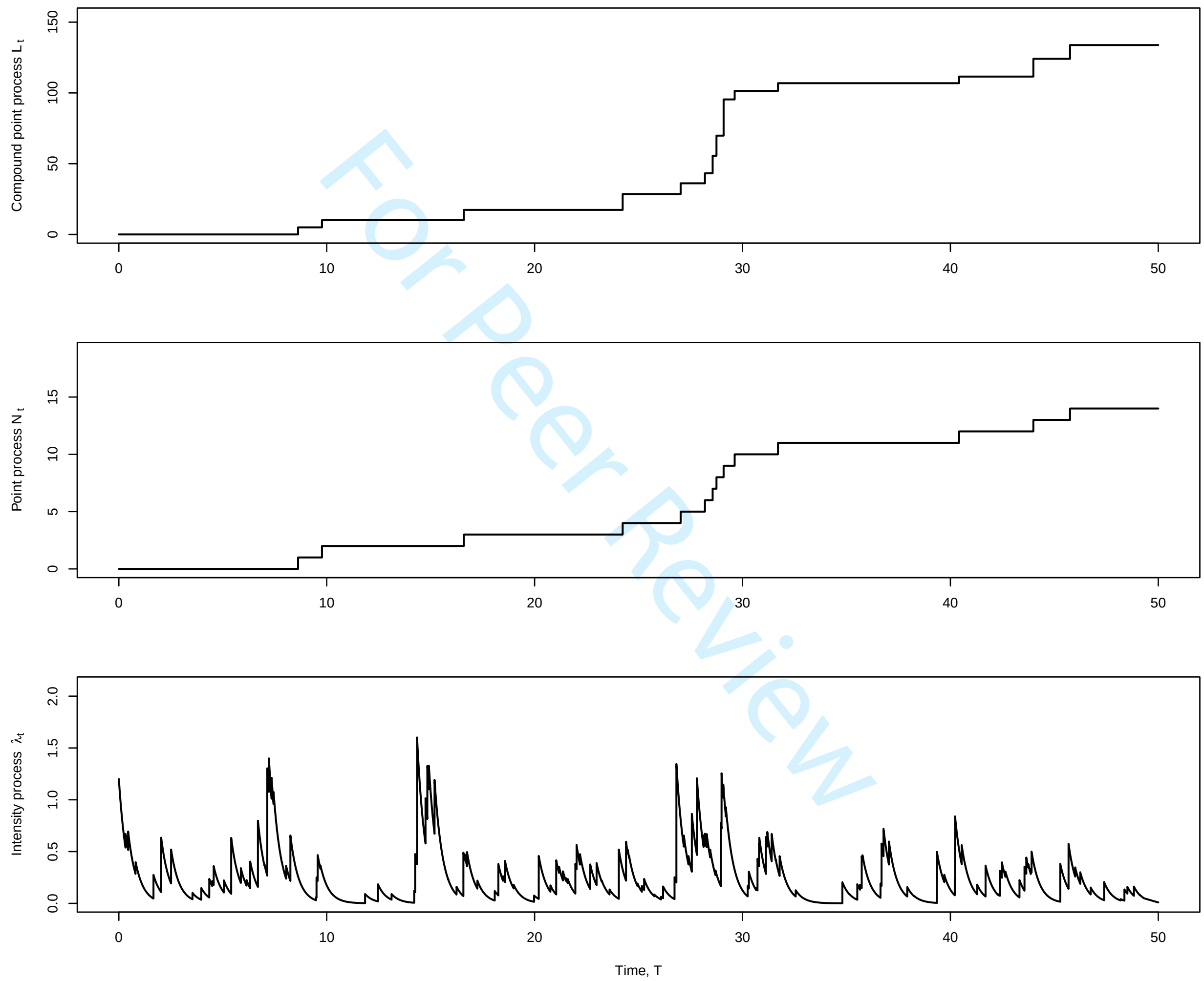


Figure 2: Simulated sample path of the compound Cox process with shot-noise intensity: Intensity process λ_t, point processes N_t, and compound point processes L_t, with the parameters $(\rho; \delta; \alpha; \omega; \zeta; k; \lambda_0) = (3; 2.5; 5; 3; 4; 6; 1.2)$.

Table 1: Value-at-Risk (VaR)

$1-q$	Poisson	Cox with shot-noise	Cox with mean-reverting shot-noise	Hawkes	DCP
0.999	242.394	254.104	556.535	644.467	776.861
0.99	125.108	141.516	344.123	447.730	521.995
0.95	78.896	88.624	262.292	337.923	397.672
0.90	62.005	69.651	228.282	290.327	345.702
0.50	22.129	26.427	140.404	162.678	204.057

Table 2: Tail Conditional Expectation (TCE or TailVaR)

$1-q$	Poisson	Cox with shot-noise	Cox with mean-reverting shot-noise	Hawkes	DCP
0.999	321.732	357.497	734.138	804.942	1036.243
0.99	172.923	190.096	435.939	542.981	638.949
0.95	116.210	129.907	336.258	432.567	506.496
0.90	100.007	112.124	312.939	401.244	472.445
0.50	95.710	109.167	393.740	486.174	587.850

Let us now illustrate the calculations of two risk measures. For $F(x)$, we use an exponential distribution, i.e.

$$1-e^{-\alpha x},\ \ \alpha>0$$

and for $G(y)$, we use a Loggamma distribution with probability density, i.e.

$$\frac{\varsigma^{c}}{\psi\Gamma(c)}\left\{\ln\left(\frac{y}{\psi}+1\right)\right\}^{c-1}\left(\frac{y}{\psi}+1\right)^{-\varsigma-1},\psi>0,\ \varsigma>0\ \text{ and }\ c>0$$

to capture the effect of sudden increases of the intensity, i.e. after-incidents/shocks driven by initial incidents/shocks. For $J(\xi)$, we use a Pareto distribution with probability density, i.e.

$$\frac{\Gamma(\omega+k)\ \ \zeta^{\omega}\ \xi^{k-1}}{\Gamma(\omega)\ \ \Gamma(k)\ \ (\zeta+\xi)^{\omega+k}},\ \omega>0,\ \zeta>0\ \text{ and }\ k>0$$

Hence the parameter values to simulate the compound processes and calculate two risk measures are

$$\begin{aligned} a &= 1,\ \rho=3,\ \delta=2.5,\ \alpha=5,\ \psi=1,\ \varsigma=5.5,\ c=3, \\ \omega &= 3,\ \zeta=4,\ k=6 \text{ and } \lambda_0=1.2. \end{aligned}$$

Simulations were run with 20,000 paths for each compound process using R. Table 1 and Table 2 show the values of Value-at-Risk (VaR) and Tail Conditional Expectation (TCE or TailVaR) at confidence level q at time $t=10$.

Remark 4.1. *Tables 1 and 2 show that two risk measures significantly increase when changing N_t from the Cox process with shot-noise Poisson intensity to a generalised Hawkes process due to self-excited jumps, and to the dynamic contagion process due to both externally-excited jumps and self-excited jumps (see Figure 3).*

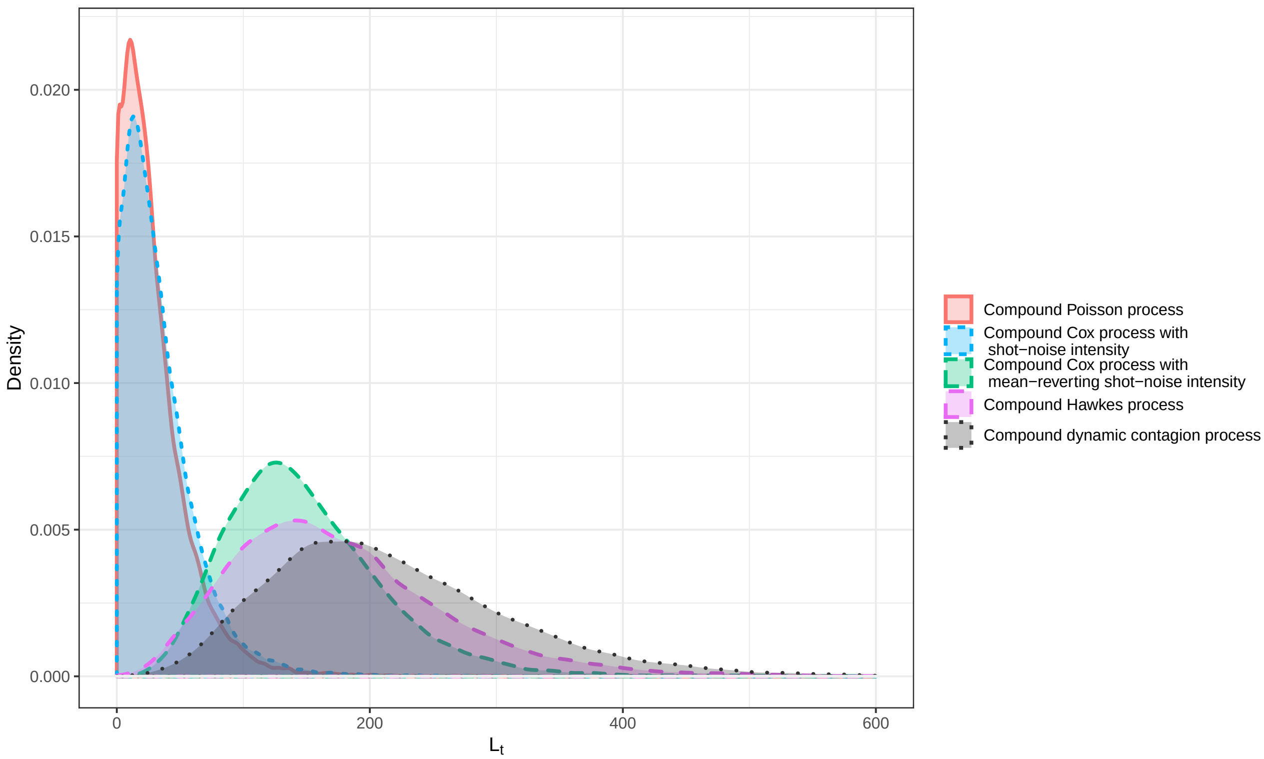


Figure 3: Density plots of the compound point processes L_t: Compound Poisson process, compound Cox process with shot-noise intensity, compound Cox process with mean-reverting shot-noise intensity, compound Hawkes process, and compound dynamic contagion process at $t = 10$ with the parameters given as $\eta = 0.24$ and $(a; \rho, \delta; \alpha; \psi; \varsigma; c; \omega; \zeta; k; \lambda_0) = (1; 3; 2.5; 5; 1; 5.5; 3; 3; 4; 6; 1.2)$.

5. Conclusion

We have reviewed and studied the compound processes of the Poisson, Cox, Hawkes, shot-noise Poisson and dynamic contagion process. The analytic expressions of the Laplace transforms of these processes and their moments have been presented, which have the potential to be applicable to a variety of problems in economics, finance and insurance. To make it easier for statistical analysis, further business applications and research, we provided the simulation algorithm for compound dynamic contagion process and compound Cox process with shot noise intensity, respectively. We also made numerical comparisons of Value-at-Risk (VaR) and Tail Conditional Expectation (TCE or TailVaR) as an application of the compound processes.

This work can be extended by incorporating interarrival jump times with renewal processes, where the moments of renewal shot-noise processes have been shown in Jang et al. (2018a) and Dassios et al. (2015) studied a risk model with renewal shot-noise Cox process.

Given the current COVID-19 pandemic, a significant challenge ahead is to model the number of losses and their magnitudes from the interruption of businesses and its economic impact. Public and private sector organisations face unexpected credit risks resulting from pandemic events. Taking into account the impact of pandemic risk in modelling the price and stochastic volatility of an asset will be the new normal post-COVID-19. Insurers will need tools to deal with the challenge of new risk dynamics arising from pandemic events. Work-from-home arrangements can expose corporate networks to the vulnerabilities found in home wi-fi routers as cyber risks. Hence when credit, cyber, economic, financial, insurance, and other risks are extreme/non-extreme losses in practice (i.e. their distributions are heavy-tailed/light-tailed), reviewed point processes can be used for these risks with flexibility.

We hope that findings of this paper provide academics/practitioners with feasible models to count numbers of claims/losses and their aggregation. We also recommend the use of more multivariate models of these processes to deal with the arrival of multiple, catastrophic and contagious losses accommodating the interdependence between risks.

A. Proof of Theorem 3.1

Consider a function $f(\lambda, n, l, t)$ with an exponential affine form

$$f(\lambda, n, l, t) = \theta^n e^{-\nu l} e^{-B(t)\lambda} e^{C(t)},$$

substitute into $\mathcal{A}\, f = 0$ in (3.1), we have

$$-\lambda B'(t) + C'(t) + \lambda \left[\theta \,\hat{g}\{B(t)\}\ \hat{j}(\nu)\right] + \delta(a-\lambda)\{-B(t)\} + \rho\left[\hat{f}\{B(t)\} - 1\right] = 0.$$

$$\left[-B'(t) + \delta B(t) + \theta\,\hat{g}\{B(t)\}\ \hat{j}(\nu) - 1\right]\lambda + \left[C'(t) + \rho\,\hat{f}\{B(t)\} - \rho - \delta a B(t)\right] = 0.$$

Since this equation holds for any l, n and λ, it is equivalent to solving two separated equations, i.e.

$$-B'(t) + \delta B(t) + \theta\,\hat{g}\{B(t)\}\ \hat{j}(\nu) - 1 = 0, \tag{A.1}$$

$$C'(t) + \rho\,\hat{f}\{B(t)\} - \rho - \delta a B(t) = 0. \tag{A.2}$$

We have two ODEs of (A.1) and (A.2) with the boundary condition $B(T) = \upsilon$. By (A.2) with boundary condition $C(0) = 0$, the integration of (3.6) follows. Since $\theta^{N_t} e^{-\nu L_t} e^{-B(t)\lambda_t} e^{C(t)}$ is a $\Im$-martingale by the property of the infinitesimal generator, we have

$$\mathbb{E}\left[\theta^{N_T} e^{-\nu L_T} e^{-B(T)\lambda_T} e^{C(T)} \mid \Im_t\right] = \theta^{N_t} e^{-\nu L_t} e^{-B(t)\lambda_t} e^{C(t)}.$$

Then, by the boundary condition $B(T) = \upsilon$, (3.4) follows.

B. Proof of Theorem 3.2

By setting $t = 0$ and $\theta = 1$ in (3.4), we have

$$\mathbb{E}\left[e^{-\nu L_T} e^{-\upsilon\lambda_T} \mid \Im_0\right] = e^{-B(0)\lambda_0} e^{-C(T)}, \tag{B.1}$$

where $B(0)$ is uniquely determined by the non-linear ordinary differential equation (ODE)

$$-B'(t) + \delta B(t) + \hat{g}\{B(t)\}\ \hat{j}(\nu) - 1 = 0 \tag{B.2}$$

with boundary condition $B(T) = \upsilon$. (B.2) can be solved, under the condition $\delta > \hat{j}(\nu)\ \mu_{1_G}$, by the following steps (1)-(7):

(1) Let us set $B(t) = \Psi(T - t) = \Psi(\tau)$. Then it becomes

$$\frac{d\Psi(\tau)}{d\tau} = 1 - \delta B(t) - \hat{g}\{B(t)\}\hat{j}(\nu) = 1 - \delta\Psi(\tau) - \hat{g}\{\Psi(\tau)\}\hat{j}(\nu) =: f(\Psi) \tag{B.3}$$

with initial condition $\Psi(0) = \upsilon$; we define the right-hand side as the function, $f(\Psi)$.

(2) Under the condition of $\delta > \hat{j}(\nu)\ \mu_{1_G}$, we have

$$\frac{\partial f(\Psi)}{\partial\Psi} = \hat{j}(\nu)\int_0^\infty ye^{-\Psi\, y}\, dG(y) - \delta\ \leq\ \hat{j}(\nu)\int_0^\infty ydG(y) - \delta =\ \hat{j}(\nu)\,\mu_{1_G} - \delta < 0,\quad \text{for } \Psi \geq 0,$$

then $f(\Psi) < 0$ for $\Psi > 0$.

(3) (B.3) can be written as

$$\frac{d\Psi(\tau)}{\delta\Psi(\tau) + \hat{j}(\nu)\hat{g}\{\Psi(\tau)\} - 1} = -d\tau.$$

Integrate both sides from time 0 to τ with initial condition $\Psi(0) = \upsilon > 0$, then we have

$$\int_\Psi^\upsilon \left[\frac{1}{\delta\ u + \hat{j}(\nu)\hat{g}(u) - 1}\right] du = \tau,$$

where $\Psi \geq 0$. Now we define the left-hand side as the function

$$\mathcal{G}_{\upsilon,\nu}(\Psi) =: \int_{\Psi}^{\upsilon} \left[\frac{1}{\delta\ u + \hat{j}(\nu)\hat{g}(u) - 1} \right] du.$$

Then we have

$$\mathcal{G}_{\upsilon,\nu}(\Psi) = \tau\ (= T - t),$$

which is the time difference between T and t and it is obvious that $\Psi \to \upsilon$ when $\tau\ (= T - t) \to 0$.

(4) As $\delta - \hat{j}(\nu)\,\mu_{1_G} > 0$ by convergence test, we have

$$\int_{0}^{\upsilon} \left[\frac{1}{\delta\ u + \hat{j}(\nu)\hat{g}(u) - 1} \right] du = \infty$$

so $\Psi \to 0$ when $\tau \to \infty$. The integrand is positive in the domain $u \in (0, \upsilon]$ and also for $\Psi \leq \upsilon$. The function $\mathcal{G}_{\upsilon,\nu}(\Psi)$ is strictly decreasing, therefore

$$\mathcal{G}_{\upsilon,\nu}(\Psi) = \tau : (0, \upsilon] \to [0, \infty)$$

is a well defined (monotone) function and its inverse function

$$\mathcal{G}_{\upsilon,\nu}^{-1}(\tau) = \Psi : [0, \infty) \to (0, \upsilon]$$

exists.

(5) The unique solution is found by

$$\Psi(\tau) = \Psi(T - t) = B(t) = \mathcal{G}_{\upsilon,\nu}^{-1}(\tau) = \mathcal{G}_{\upsilon,\nu}^{-1}(T - t)$$

and hence $B(0)$ is obtained,

$$B(0) = \Psi(T) = \mathcal{G}_{\upsilon,\nu}^{-1}(T).$$

(6) Now $C(T)$ is determined by

$$C(T) = \rho \int_{0}^{T} \left[1 - \hat{f}\left\{ \mathcal{G}_{\upsilon,\nu}^{-1}(\tau) \right\} \right] d\tau + a\delta \int_{0}^{T} \mathcal{G}_{\upsilon,\nu}^{-1}(\tau) d\tau.$$

By the change of variable $\mathcal{G}_{\upsilon,\nu}^{-1}(\tau) = u$, we have $\tau = \mathcal{G}_{\upsilon,\nu}^{-1}(u)$, and

$$\int_{0}^{T} \mathcal{G}_{\upsilon,\nu}^{-1}(\tau) d\tau = \int_{\mathcal{G}_{\upsilon,\nu}^{-1}(0)}^{\mathcal{G}_{\upsilon,\nu}^{-1}(T)} u \frac{\partial \tau}{\partial u} du = \int_{\mathcal{G}_{\upsilon,\nu}^{-1}(T)}^{\upsilon} \left\{ \frac{u}{\delta\ u + \hat{j}(\nu)\hat{g}(u) - 1} \right\} du.$$

(7) Finally, substitute $B(0)$ and $C(T)$ into (B.1) and the result follows.

C. Proof of Theorem 3.3

Set $v = 0$ in (3.7), then the result follows immediately.

D. Proof of Theorem 4.1

Setting $\mathcal{A}\ f(\lambda, n, l, t) = l$ in (3.1), we have

$$\mathcal{A}\ l = \mu_{1_J}\lambda.$$

As $L_t - L_0 - \int_0^t \mathcal{A}\ l_s ds$ is a $\Im_t$ -martingale, we have

$$\mathbb{E}\left\{ L_t - \int_0^t \mathcal{A}\ l_s ds \mid \lambda_0 \right\} = L_0.$$

Hence

$$\mathbb{E}\left(L_t \mid \lambda_0\right) = L_0 + \mathbb{E}\left\{ \int_0^t \mathcal{A}\ l_s ds \mid \lambda_0 \right\} = L_0 + \mu_{1_J} \int_0^t \mathbb{E}\left(\lambda_s \mid \lambda_0\right) ds.$$

From Proposition 3.1 in Jang and Dassios (2013), the conditional expectation of the process λ_t given λ_0 at time $t = 0$, is given by

$$\mathbb{E}\left(\lambda_t \mid \lambda_0\right) = \lambda_0 e^{-\left(\delta - \mu_{1_G}\right)t} + \frac{\mu_{1_F}\ \rho + a\delta}{\delta - \mu_{1_G}}\left(1 - e^{-\left(\delta - \mu_{1_G}\right)t}\right), \quad for\ \delta \neq \mu_{1_G},$$

and (4.1) follows.

Acknowledgements

Rosy Oh's research was supported by Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education (Grant No. 2019R1A6A1A11051177 and 2020R1I1A1A01067376).

References

Adamopoulos, L. (1976). Cluster models for earthquakes: Regional comparisons. *Journal of the International Association for Mathematical Geology*, 8(4):463–475.

Aït-Sahalia, Y., Cacho-Diaz, J., and Laeven, R. J. A. (2015). Modeling financial contagion using mutually exciting jump processes. *Journal of Financial Economics*, 117(3):585–606.

Aït-Sahalia, Y. and Hurd, T. R. (2016). Portfolio choice in markets with contagion. *Journal of Financial Econometrics*, 14(1):1–28.

Albrecher, H. and Asmussen, S. (2006). Ruin probabilities and aggregrate claims distributions for shot noise Cox processes. *Scandinavian Actuarial Journal*, (2):86–110.

Bacry, E., Mastromatteo, I., and Muzy, J.-F. (2015). Hawkes processes in finance. *Market Microstructure and Liquidity*, 1(1):1550005.

Bartlett, M. (1963). The spectral analysis of point processes. *Journal of the Royal Statistical Society: Series B (Methodological)*, 25(2):264–281.

Basu, S. and Dassios, A. (2002). A Cox process with log-normal intensity. *Insurance: mathematics and economics*, 31(2):297–302.

Bauwens, L. and Hautsch, N. (2009). Modelling financial high frequency data using point processes. In *Handbook of financial time series*, pages 953–979. Springer.

Bening, V. E. and Korolev, V. Y. (2002). *Generalized Poisson models and their applications in insurance and finance*. VSP, Utrecht.

Bian, B., Chen, X., and Zeng, X. (2019). Optimal portfolio choice in a jump-diffusion model with self-exciting. *Journal of Mathematical Finance*, 9(3):345–367.

Bowsher, C. G. (2007). Modelling security market events in continuous time: Intensity based, multivariate point process models. *Journal of Econometrics*, 141(2):876–912.

Brémaud, P. (1981). *Point processes and queues: martingale dynamics*. New York: Springer.

Brémaud, P. and Massoulié, L. (1996). Stability of nonlinear Hawkes processes. *The Annals of Probability*, 24(3):1563–1588.

Brémaud, P. and Massoulié, L. (2001). Hawkes branching point processes without ancestors. *Journal of applied probability*, 38(1):122–135.

Brémaud, P. and Massoulié, L. (2002). Power spectra of general shot noises and Hawkes point processes with a random excitation. *Advances in Applied Probability*, 34(1):205–222.

Chavez-Demoulin, V., Davison, A. C., and McNeil, A. J. (2005). Estimating value-at-risk: a point process approach. *Quantitative Finance*, 5(2):227–234.

Cizek, P., Härdle, W. K., and Weron, R. (2011). *Statistical tools for finance and insurance*. Springer.

Cox, D. R. (1955). Some statistical methods connected with series of events. *Journal of the Royal Statistical Society: Series B (Methodological)*, 17(2):129–157.

Cox, D. R. and Isham, V. (1980). *Point processes*, volume 12. Chapman & Hall, London, UK.

Cox, D. R. and Isham, V. (1986). The virtual waiting-time and related processes. *Advances in Applied Probability*, 18(2):558–573.

Daley, D. J. and Vere-Jones, D. (2003). *An introduction to the theory of point processes. Volume I: Probability and its Applications*. New York: Springer.

Daley, D. J. and Vere-Jones, D. (2008). *An introduction to the theory of point processes: Volume II: General theory and structure*. New York: Springer.

Dassios, A. and Jang, J. (2003). Pricing of catastrophe reinsurance and derivatives using the Cox process with shot noise intensity. *Finance and Stochastics*, 7(1):73–95.

Dassios, A. and Jang, J. (2005). Kalman-bucy filtering for linear systems driven by the Cox process with shot noise intensity and its application to the pricing of reinsurance contracts. *Journal of applied probability*, 42(1):93–107.

Dassios, A. and Jang, J. (2008). The distribution of the interval between events of a Cox process with shot noise intensity. *Journal of Applied Mathematics and Stochastic Analysis*, 2008.

Dassios, A., Jang, J., and Zhao, H. (2015). A risk model with renewal shot-noise Cox process. *Insurance: Mathematics and Economics*, 65:55–65.

Dassios, A., Jang, J., and Zhao, H. (2019). A generalised CIR process with externally-exciting and self-exciting jumps and its applications in insurance and finance. *Risks*, 7(4):103.

Dassios, A. and Zhao, H. (2011). A dynamic contagion process. *Advances in applied probability*, 43(3):814–846.

Dassios, A. and Zhao, H. (2012). Ruin by dynamic contagion claims. *Insurance: Mathematics and Economics*, 51(1):93–106.

Dassios, A. and Zhao, H. (2013). Exact simulation of Hawkes process with exponentially decaying intensity. *Electronic Communications in Probability*, 18(62):1–13.

Dassios, A. and Zhao, H. (2017a). Efficient simulation of clustering jumps with CIR intensity. *Operations Research*, 65(6):1494–1515.

Dassios, A. and Zhao, H. (2017b). A generalized contagion process with an application to credit risk. *International Journal of Theoretical and Applied Finance*, 20(1):1750003.

Davis, M. H. A. (1984). Piecewise-deterministic markov processes: A general class of non-diffusion stochastic models. *Journal of the Royal Statistical Society: Series B (Methodological)*, 46(3):353–376.

Dong, X. (2014). *Compensators and diffusion approximation of point processes and applications*. PhD thesis, Imperial College London.

Duffie, D., Filipović, D., Schachermayer, W., et al. (2003). Affine processes and applications in finance. *The Annals of Applied Probability*, 13(3):984–1053.

Duffie, D., Pan, J., and Singleton, K. (2000). Transform analysis and asset pricing for affine jump-diffusions. *Econometrica*, 68(6):1343–1376.

Embrechts, P., Liniger, T., and Lin, L. (2011). Multivariate Hawkes processes: an application to financial data. *Journal of Applied Probability*, 48(A):367–378.

Errais, E., Giesecke, K., and Goldberg, L. R. (2010). Affine point processes and portfolio credit risk. *SIAM Journal on Financial Mathematics*, 1(1):642–665.

Gao, X., Zhou, X., and Zhu, L. (2018). Transform analysis for Hawkes processes with applications in dark pool trading. *Quantitative Finance*, 18(2):265–282.

Giesecke, K. and Kim, B. (2011). Risk analysis of collateralized debt obligations. *Operations Research*, 59(1):32–49.

Glasserman, P. and Kim, K.-K. (2010). Moment explosions and stationary distributions in affine diffusion models. *Mathematical Finance: An International Journal of Mathematics, Statistics and Financial Economics*, 20(1):1–33.

Grandell, J. (1991). *Aspects of risk theory.* Springer-Verlag, New York.

Grandell, J. (1997). *Mixed poisson processes.* Chapman and Hall, London.

Hainaut, D. and Moraux, F. (2018). Hedging of options in the presence of jump clustering. *Journal of Computational Finance*, 22(3):1–35.

Hainaut, D. and Moraux, F. (2019). A switching self-exciting jump diffusion process for stock prices. *Annals of Finance*, 15(2):267–306.

Hawkes, A. G. (1971a). Spectra of some self-exciting and mutually exciting point processes. *Biometrika*, 58(1):83–90.

Hawkes, A. G. (1971b). Point spectra of some mutually exciting point processes. *Journal of the Royal Statistical Society: Series B (Methodological)*, 33(3):438–443.

Hawkes, A. G. (1972). Spectra of some mutually exciting point processes with associated variables. *Stochastic point processes*, pages 261–271.

Hawkes, A. G. (2018). Hawkes processes and their applications to finance: a review. *Quantitative Finance*, 18(2):193–198.

Hawkes, A. G. (2020). Hawkes jump-diffusions and finance: a brief history and review. *The European Journal of Finance*. https://doi.org/10.1080/1351847X.2020.1755712.

Hawkes, A. G. and Oakes, D. (1974). A cluster process representation of a self-exciting process. *Journal of Applied Probability*, 11(3):493–503.

Herbertsson, A., Jang, J., and Schmidt, T. (2011). Pricing basket default swaps in a tractable shot noise model. *Statistics & Probability Letters*, 81(8):1196–1207.

Jang, J. (2004). Martingale approach for moments of discounted aggregate claims. *Journal of Risk and Insurance*, 71(2):201–211.

Jang, J. and Dassios, A. (2013). A bivariate shot noise self-exciting process for insurance. *Insurance: Mathematics and Economics*, 53(3):524–532.

Jang, J., Dassios, A., and Zhao, H. (2018a). Moments of renewal shot-noise processes and their applications. *Scandinavian Actuarial Journal*, (8):727–752.

Jang, J. and Krvavych, Y. (2004). Arbitrage-free premium calculation for extreme losses using the shot noise process and the esscher transform. *Insurance: Mathematics and Economics*, 35(1):97–111.

Jang, J. and Oh, R. (2020). A bivariate compound dynamic contagion process for cyber insurance. *arXiv:2007.04758 [q-fin.RM]*.

Jang, J., Park, J. J., and Jang, H. J. (2018b). Catastrophe insurance derivatives pricing using a Cox process with jump diffusion CIR intensity. *International Journal of Theoretical and Applied Finance*, 21(7):1850041.

Klüppelberg, C. and Mikosch, T. (1995). Explosive poisson shot noise processes with applications to risk reserves. *Bernoulli*, 1(1/2):125–147.

Liniger, T. J. (2009). *Multivariate Hawkes processes.* PhD thesis, Swiss Federal Institute of Technology, Zürich.

Liu, W. and Zhu, S.-P. (2019). Pricing variance swaps under the Hawkes jump-diffusion process. *Journal of Futures Markets*, 39(6):635–655.

Lu, X. and Abergel, F. (2018). High-dimensional Hawkes processes for limit order books: modelling, empirical analysis and numerical calibration. *Quantitative Finance*, 18(2):249–264.

Ma, Y., Pan, D., and Wang, T. (2020). Exchange options under clustered jump dynamics. *Quantitative Finance*, 20(6):949–967.

Ma, Y., Shrestha, K., and Xu, W. (2017). Pricing vulnerable options with jump clustering. *Journal of Futures Markets*, 37(12):1155–1178.

Maneesoonthorn, W., Forbes, C. S., and Martin, G. M. (2017). Inference on self-exciting jumps in prices and volatility using high-frequency measures. *Journal of Applied Econometrics*, 32(3):504–532.

McNeil, A. J., Frey, R., and Embrechts, P. (2005). *Quantitative risk management: concepts, techniques and tools-revised edition.* Princeton university press, USA.

Møller, J. and Rasmussen, J. G. (2005). Perfect simulation of Hawkes processes. *Advances in applied probability*, 37(3):629–646.

Ogata, Y. (1981). On lewis' simulation method for point processes. *IEEE transactions on information theory*, 27(1):23–31.

Ogata, Y. (1988). Statistical models for earthquake occurrences and residual analysis for point processes. *Journal of the American Statistical Association*, 83(401):9–27.

Ozaki, T. (1979). Maximum likelihood estimation of Hawkes' self-exciting point processes. *Annals of the Institute of Statistical Mathematics*, 31(1):145–155.

Rambaldi, M., Bacry, E., and Lillo, F. (2017). The role of volume in order book dynamics: a multivariate Hawkes process analysis. *Quantitative Finance*, 17(7):999–1020.

Rolski, T., Schmidli, H., Schmidt, V., and Teugels, J. L. (1998). *Stochastic processes for insurance and finance.* John Wiley & Sons, UK.

Schottky, W. (1918). Über spontane stromschwankungen in verschiedenen elektrizitätsleitern. *Annalen der physik (in German)*, 362(23):541–567.

Snyder, D. L. (1975). *Random point processes.* John Wiley & Sons, USA.

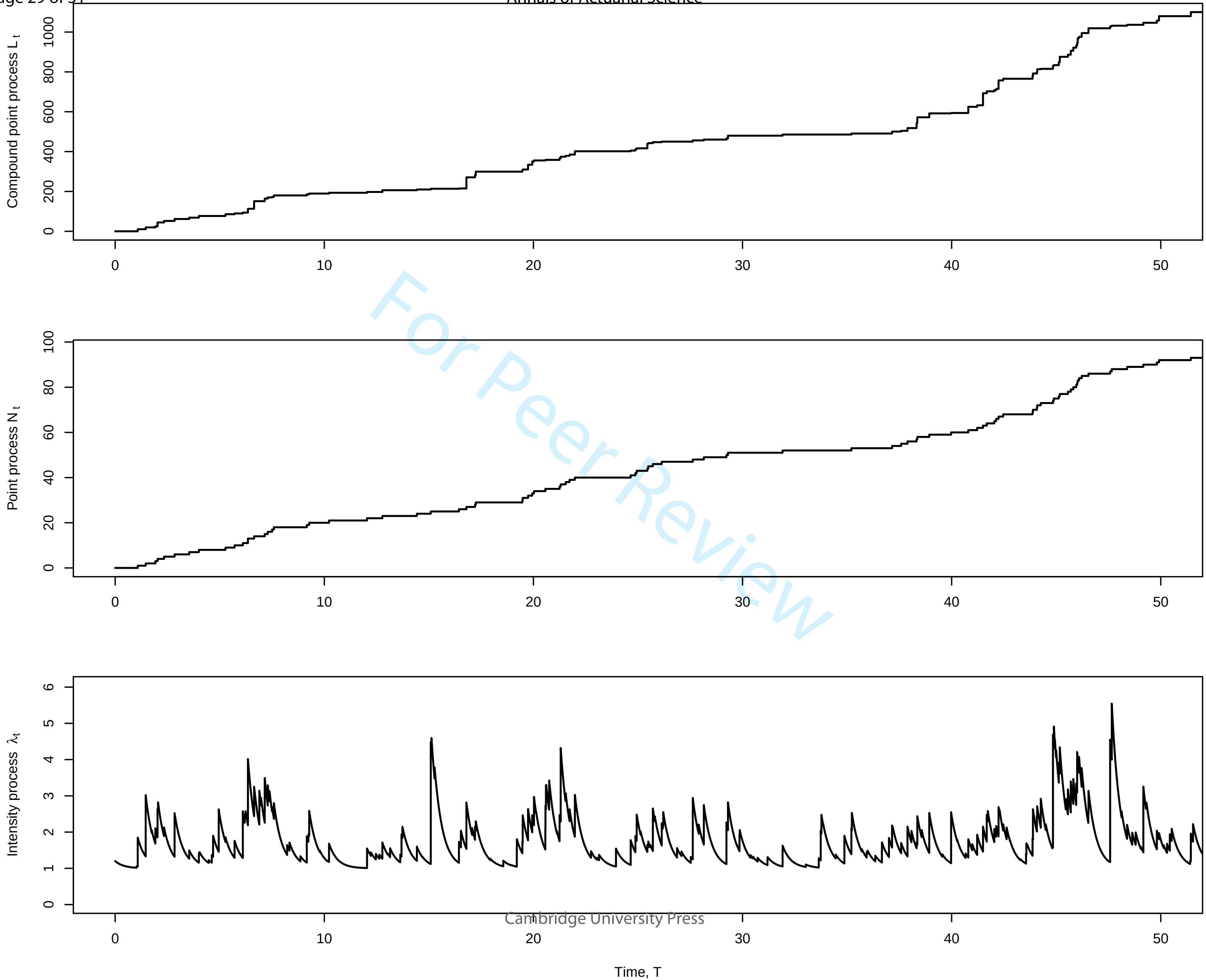
Stabile, G. and Torrisi, G. L. (2010). Risk processes with non-stationary Hawkes claims arrivals. *Methodology and Computing in Applied Probability*, 12(3):415–429.

Vere-Jones, D. (1975). Stochastic models for earthquake sequences. *Geophysical Journal International*, 42(2):811–826.

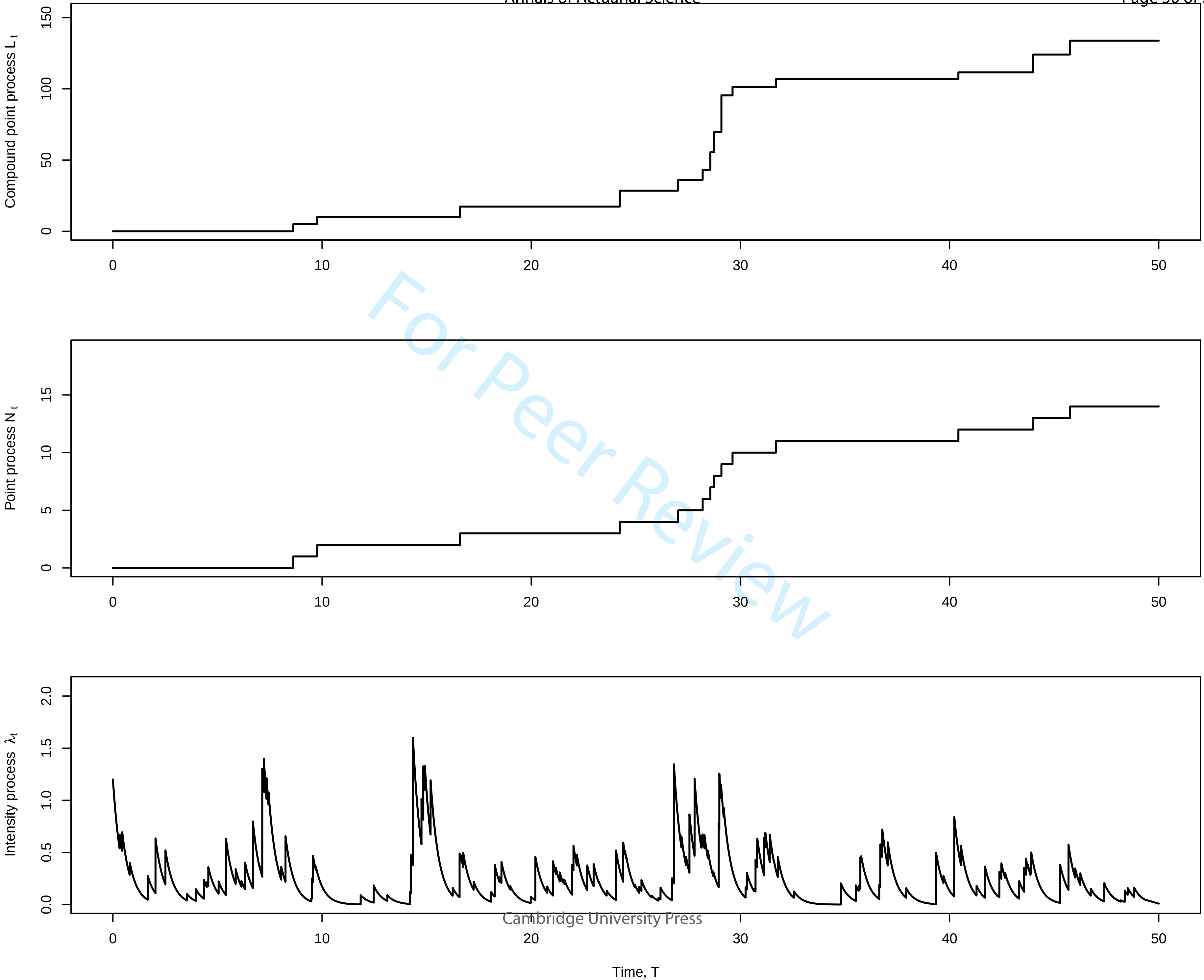
Vere-Jones, D. (1978). Earthquake prediction-a statistician's view. *Journal of Physics of the Earth*, 26(2):129–146.

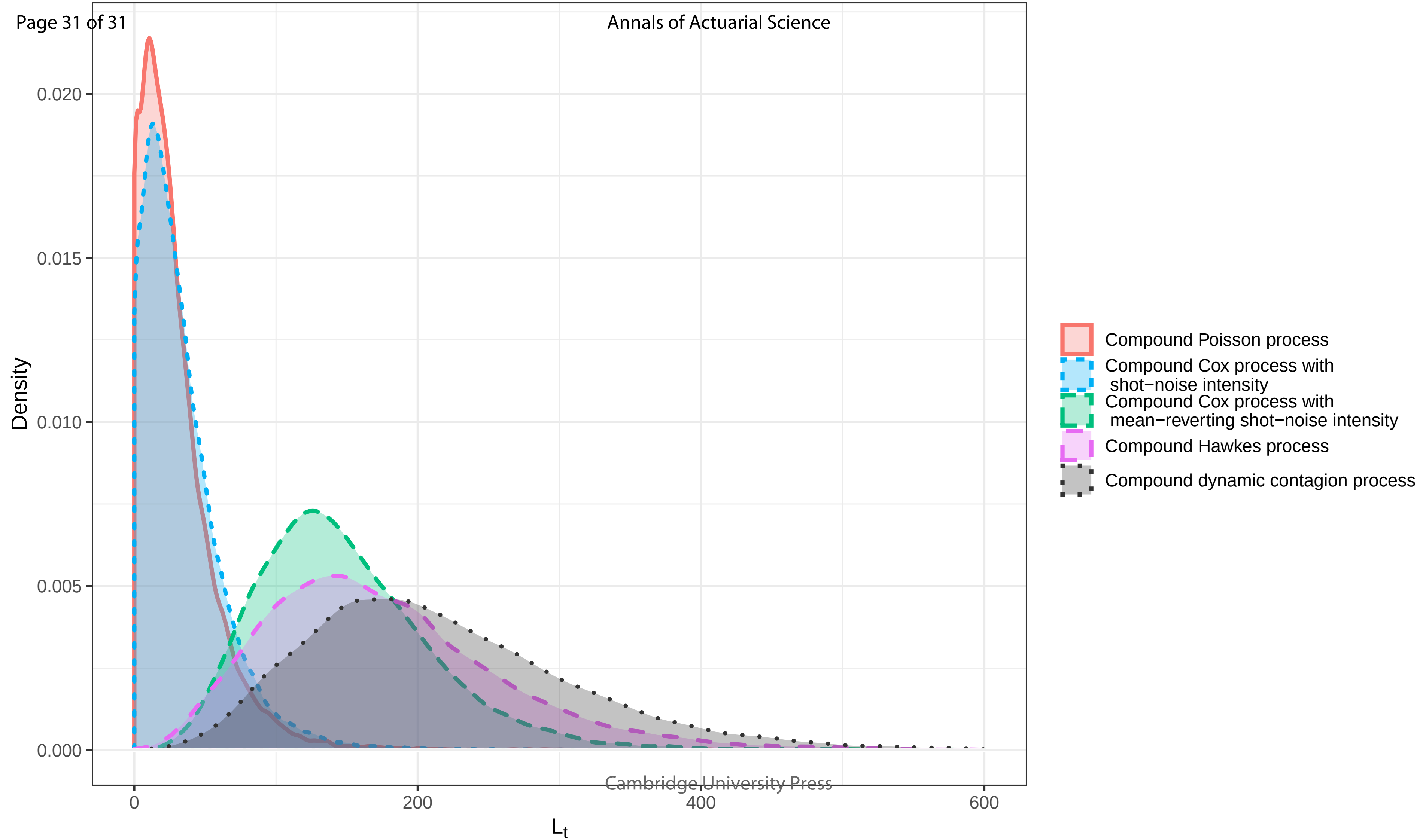
Vere-Jones, D. and Ozaki, T. (1982). Some examples of statistical estimation applied to earthquake data. *Annals of the Institute of Statistical Mathematics*, 34(1):189–207.

Yang, S. Y., Liu, A., Chen, J., and Hawkes, A. G. (2018). Applications of a multivariate Hawkes process to joint modeling of sentiment and market return events. *Quantitative finance*, 18(2):295–310.



Compound point process L t
0
200
400
600
800
1000
Point process N t
0
20
40
60
80
100
Intensity process λt
0
1
2
3
4
5
6
0
10
20
30
40
50
Time, T


Compound point process L_t
0
50
100
150
0
10
20
30
40
50
Point process N_t
0
5
10
15
Intensity process λ_t
0.0
0.5
1.0
1.5
2.0
Time, T


Density
0.020
0.015
0.010
0.005
0.000
0
200
400
600
L_t
Compound Poisson process
Compound Cox process with shot−noise intensity
Compound Cox process with mean−reverting shot−noise intensity
Compound Hawkes process
Compound dynamic contagion process