Percentages as Part Whole Relationships

Paul White
ACU National
<paul.white@acu.edu.au>

Sue Wilson
ACU National
<sue.wilson@acu.edu.au>

Rhonda Faragher
ACU National
< rhonda.faragher@acu.edu.au>

Mike Mitchelmore
Macquarie University
<mike.mitchelmore@mq.edu.au>

Five practising teachers in regional NSW implemented Teaching for Abstraction for the Year 6 topic "Percentages". The authors constructed materials for a unit in which students explored familiar percentage contexts, searched for similarities in their mathematical structures and then applied their learnings to more abstract situations. Particular emphasis was given to additive versus multiplicative approaches in different percentage situations. After an introductory workshop, teachers taught the topic in eight 40 minute lessons. The results show that even though this approach is radically different from that to which students and teachers are accustomed, it has the potential to benefit student engagement, learning, and attitudes for both students and teachers. The overall conclusions have implications for how professional development for Teaching for Abstraction is addressed.

Mitchelmore and White (2004) outline an approach to teaching based on the fact that most elementary mathematical ideas are abstractions from experience. Emphasised is the importance of empirical abstraction in mathematics learning, focusing on an abstract concept as "the end-product of ... an activity by which we become aware of similarities ... among our experiences" (Skemp, 1986, p. 21). This view of abstraction leads to a theory for teaching early mathematical concepts called *Teaching for Abstraction* (Mitchelmore & White, 2000), where students engage in:

- *familiarising themselves* with the structure of a variety of relevant contexts;
- *recognising* the similarities between these different contexts;
- *reifying* the similarities to form a general concept, and then
- *applying* the concept in new situations.

Much of the theory has been developed from investigations into young children's understanding of the angle concept (Mitchelmore & White, 2000), but also from mathematical concepts involving rates of change (White & Mitchelmore, 1996), decimals (Mitchelmore, 2002), and percentages (White & Mitchelmore, 2005). Two further studies took place in 2006. The first was an extension of the earlier percentage study with Year 6, but in regional schools; the other was on rates and ratios with Year 8. The Year 6 study is reported here, the Year 8 study elsewhere.

Percentage as a Multiplicative Relation

Percentage is a multiplicative relationship that causes students particular difficulties—it forms a bridge between real-world situations and mathematical concepts of multiplicative structures (Parker & Leinhardt, 1995). The concise, abstract language of percentages often uses misleading additive terminology with a multiplicative meaning. Misailidou and Williams (2003) showed that inappropriate additive strategies were the dominant errors made by students aged 10-13 years. On the other hand, Van Dooren and De Bock (2005) claim that extensive attention to proportional reasoning in school mathematics results in the misapplication of proportional methods. Whatever the situation, a cursory look at the school mathematics curriculum shows that multiplicative relations underpin almost all number-related concepts studied in school (e.g., fractions, percentages, ratio, rates, similarity, trigonometry, rates of change). Hence, percentages and proportional reasoning in general are areas deserving research especially if a different methodology is adopted which goes beyond that in the research cited above.

Aims of the Study

The object of our research project was to build on the previous study (White & Mitchelmore, 2005) about how Year 5/6 classroom teachers adapt to using everyday situations and about how students abstract the multiplicative structure of percentages. That study developed a unit of work based on Teaching for Abstraction that emphasised underlying structure in percentage situations, including helping students to differentiate multiplicative from additive relations. The analysis showed that the approach was radically different to that which students and teachers are accustomed. Many students did learn to apply percentages even though the final level of achievement was not as high as had been expected. Two reasons for the lower than expected achievement were insufficient time to explore individual contexts in enough detail and inadequate attention to calculation skills. A new unit was developed which addressed fewer contexts and had a greater focus on calculating with percentages – using 10% as a base for calculations.

Method

Participants

Participants were students and teachers of five Year 6 classes in three regional primary schools. In each class, five students were selected as a representative "target group" for closer study.

Teaching Materials

The four phases of the theoretical framework for Teaching for Abstraction were used in planning the activities for the experimental unit as follows.

- *Familiarising*: Students explored individual, supposedly familiar contexts. Simple percentages were initially used (50%, 10%) but these increased in complexity to 25%, 75%, 20%, 30%, …, 90%, and 5%.
- *Recognising*: Activities required students to compare and contrast the use of percentages in different contexts. Calculations were based on first calculating10% and then multiplying by the appropriate factor.
- *Reifying*: Students were asked to make and explain generalisations based on the similarities found in the Recognising phase.
- *Application*: Students created their own problems.

The resulting lesson topics are shown in Table 1. The lesson titles used syllabus familiar terms, addressing the appropriate skills and outcomes. The lesson structure, however, followed the theory of abstraction: beginning with a context with embedded skills and concepts and leading on to discussion about the underlying abstract notions.

Table 1
Topics for Percentage Lessons

1. Thinking percent	Students interpret percentages in situations involving bar models. The focus is on percent as a part of 100.
2. Calculating percentages	Students extend their previous experience of percentages to simple percentages (multiples of 10%) of 200, 300 and 50 objects.
3. Calculating more percentages	Students further extend their previous experience of percentages to simple percentages (multiples of 10%) of any number of objects.
4. Discounts	Students investigate discounts and compare percentage discounts with fixed discounts.
5. How do I choose?	Students compare the appropriateness of additive versus multiplicative strategies.
6. Taxes	Students compare different ways the GST could have been charged and decide on fair ways of doing so.
7. What is the best way?	Students investigate problems involving different comparisons and decide the best way to solve these problems.
8. Summary	Students bring together the main ideas and skills learnt in this unit.

Procedure

The study took place in Term 4, 2006. A one-day orientation workshop was held, in which teachers were introduced to Teaching for Abstraction and the proposed teaching unit. They then taught the unit over a period of 2 to 3 weeks, and returned for a second workshop for an assessment of the effectiveness of the unit. The first three authors visited schools to assess students' understanding before and after the teaching, to observe lessons, and to interview teachers. Thus the following sources of data were generated.

- A written pre and post test assessment of all students on their ability to calculate with percentages.
- A 15-minute interview given before and after the teaching with the five targeted students in each class.
- Worksheets completed by the targeted students.
- Observations and subsequent interviews with the teachers. Each teacher was observed twice, once by the first author and once by the second or third author.
- Teachers' evaluations of each lesson and of the unit.

Results

Based on White and Mitchelmore (2005), the results are presented in two categories –calculating with percentages and interpretation of percentage contexts. The format centres on the pre and post quantitative data with support from qualitative data.

Calculating with Percentages

This section looks at the written pre/post test and the associated Lessons 1 – 3.

Written Test. The written test consisted of six questions requiring calculations with percentages. Question 1 asked "percent means out of ___" (this was not scored).

Question 2 involved calculating 10%, 20%, 25%, 50%, 75%, and 90% of 100 jelly beans in a jar. Question 3 asked for the same percentages of 200, and Question 4, the same percentages of 50. Question 5 required students to colour in 50% of a bar that was (a) 10 boxes, (b) 8 boxes long. Question 6 required colouring 25% of the same bars. The combined results for each question from the 5 classes are shown in Figure 1.

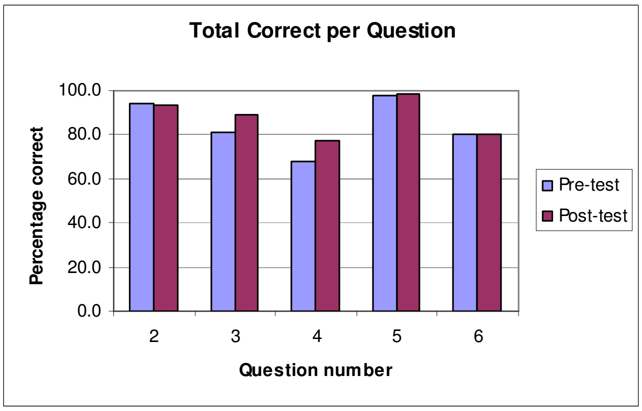

Figure 1. Aggregated percentage correct before and after teaching.

The results indicate no apparent change in Question 2 (percentage calculation out of 100) and Question 5 (colour in 50% of a bar). The scores were 94% and 98%, respectively. The consistently high scores can be attributed to the familiarity of students with calculating 50% and percentages out of 100. Question 6 (colour in 25% of a bar) also shows no change, with a pre and post result of about 80%. The lower score for Question 6 can be attributed to the less familiar 25% and the fact that in part (a) 25% of 10 required two and a half boxes to be coloured.

Questions 3 and 4 showed increases from 80% to 89% and 67% to 78% respectively. A closer look shows that the most common error in the pre test was calculating as if there were 100 jelly beans – that is, treating the percentage as always out of 100. This error did not occur in the post test. The overall lower facility of Question 4 arose because parts (c) and (e) involved fractional answers. In these calculations, only about 50% of students were able to respond correctly in the post test compared to about 43% in the post test. Also in Question 4, part (f) (find 90% of 50) correct responses rose from 60% to 81%.

In summary, the results indicate that 50%, 10%, and percentages out of 100 are familiar to students entering Year 6 and that the teaching here improved calculation facility for examples like 20%, 25%, 75% and 90% of numbers other than 100 except where fractional answers were involved.

Lesson Analysis. The first three lessons related to the written test as they focused on calculating percentages, beginning with 50% and 10% of 100 and moving on to more complex examples.

Teachers brought in food containers with percentages on them to introduce their early lessons. They discovered that students had an understanding of the difference between “percent fat” and “percent fat free” and that for any product, the two values added to 100%. They also discussed the use of these percentages as a marketing ploy. The contexts employed here clearly assisted students clarify their understanding of percentage. Similarly, little difficulty was found with Question 2 on the worksheet for Lesson 1 – colouring in 50% of a 14 cm bar with no box markings – because of the "half" connection. However, colouring in 10% and 90% proved more troublesome. A common mistake was to colour in 1 cm for 10% and 13 cm (14 cm – 1cm) for 90%.

Teachers' feedback indicated that these colouring activities and the discussion of errors like above helped students think beyond 50% and out of 100. This is consistent with the results of the quantitative analysis reported above. The observations suggest that using unmarked bars was an effective strategy and question the sense of only using marked bars in the written tests. The presence of markings goes some way to explaining why students had little difficulty with Question 5 of that test.

The 10% approach used in these lessons was given positive feedback by teachers and adopted by the majority of students. For a few students, however, the 10% approach conflicted with other rote learnt procedures. For example, although most students knew that you divide by ten to find 10% of something, one student wrote: *I take the first number if it's a 2 digit number or if it's a number greater than 100 I get rid of one zero*; *Move the decimal point forward once.* One author observed that in one class the "10% method" was effectively one recipe replacing another.

In conclusion, improvement in calculation facility where fractional answers were not involved is supported, but in some instances rote learning may have been the likely reason. Calculations resulting in fractional answers received no attention in the teaching.

Interpretation of Percentage Contexts

This section looks at the pre and post interviews and the associated Lessons 4 – 7.

Table 2
Interview Questions

Question
1. Two basketball players compare their shooting from the free throw line. The first player has scored 20 goals from 40 shots. The second player has scored 25 goals from 50 shots. Which player is the better shooter? Why?
2. Meg is 10 years old. Her little sister Lisa is 5 years old. How much older is Meg than Lisa? How old will Meg be when she is double her now? How old will Lisa be when Meg is double her age now? Explain your answer.
3. (a) Marcos purchases a new Mobile Phone. The original cost is $100. Marcos is offered a choice of the cost being reduced by a 10% discount or having $10 taken off the price. Which should Marcos choose? Explain your answer. (b) Pam purchases a TV. The original cost is $200. Pam is offered a choice of the cost being reduced by a 10% discount or having $10 taken of the price. Which should Pam choose? Explain your answer. (c) Does a 10% reduction or a $10 reduction always give more off the price? (d) Give some examples to explain your answer.
4. (a) At one store, new joggers have a price of $80, but because it is 'Cheap Tuesday', the price is reduced by 10%. How much do they cost on Cheap Tuesday? (b) At another store the same joggers have a price of $100, but the store has a sale on and the price is reduced by 20%. What is the sale price? (c) Does a bigger percentage reduction always mean the price is cheaper? (d) Explain your answer.

Assessment Interviews. The interviews contained the four questions shown in Table 2. All questions were presented orally and in writing. They were administered to 21 of the 25 target students before and after the teaching. (The others were absent on one or both occasions.)

The overall performance on these questions went from 60% correct to 90% correct. Figure 2 shows the breakdown across the questions.

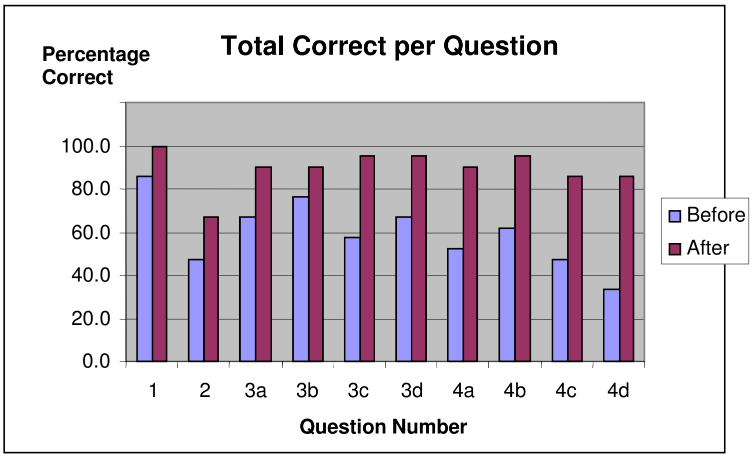

Figure 2. Aggregated percentage correct before and after teaching.

The correct responses to Question 1 included the expected assertions that each player scored 50%, but also scored as correct were arguments like: *The 25 was better because they were the same but kept it up longer.* The few errors fell into two categories: additive strategies (*First player because I took 10 away from the 50 and that equalled 40, so then I had to take 10 away from the 25 and that gave me the answer*) and incorrect multiplicative strategies (*First player because he got half. The other one got 25 out of 50 so he only got a quarter*). All errors disappeared after the teaching.

In Question 2, an additive response was required. For example: *15. She's 15 because it's only 5 years. They are 5 years apart. 20-5 would be 15.* Before the teaching, 48% used a correct strategy, rising to 67% afterwards. A few students attempted to use an additive strategy but made an arithmetical error. The major error, however, was the inappropriate use of a multiplicative strategy (43% before and 24% after) such as: *10. If I double Meg, I'll have to double Lisa because it will be the same time.*

In Question 3(a) and (b), improved arithmetic accounted for the improvement in facility. In 3(c), one third of students opted for the 10%—a response that virtually disappeared in the post test. For example: *10% gives more off the price because if the price was $100 and you take off $10 it would be $90, but if you take off 10% it would be $80; 10% gives more off the price. $10 reduction is just $10 but 10% depends on how much money you had.* In the pre interview, some reasons incorrectly relied on one example whereas others were basically sound but failed to come to the correct conclusion. In the post interview, the 95% facility for Question 3(d) shows that students' reasoning was clearer. For example: *It depends what the price is. If it is a higher number then $100 it is always a bit more than $10. If it is lower than $100, it is less.*

Like Question 3, improved facility in responses to Question 4 (a) and (b) was a result of improved arithmetic. In 4(c), the choice of the "bigger percentage reduction means a cheaper price" option fell from 52% to 14%. Like Question 3, pre interview conclusions were often based on a single example but also included some percentage misconceptions. For example: *Yes. Because the % usually means the same as the dollar amount so you take that off* and *Yes. The bigger the percent off, the less money* *you pay.* Of note is that in the pre interview only half of the 38% who said that the bigger percentage does not always give a cheaper price could give a valid reason, whereas in the post interview all could.

In summary, after the teaching, the number of students who could both calculate with percentages like 10% and 20% and use these percentages appropriately in context doubled. This included explaining why they came to the answers they did, in particular identifying percentage as a relative comparison and the need to identify "percentage of what". Question 2 seemed the most problematic. Of course, we would expect to see improvements after a teaching episode no matter what the approach was, especially when the same questions were used. But here the improvement in the students' explanation is striking and transcends what could be expected from either memory of questions or just currency of the concepts following teaching. The move from inappropriate additive strategies highlighted in the literature is particularly encouraging, whereas the issues with Question 2 could support the arguments of Van Doren and De Bock (2005) about over-use of proportional strategies or could also be attributed to the multiplicative language being misleading.

Lesson Analysis. Lessons 4 – 7 focused on using percentages in contexts like discounts, comparing discounts, and taxation, and investigations of when to use additive strategies and when to use multiplicative strategies. The overwhelming response here was that the extended discussion generated by the lesson materials was a great success and promoted student engagement and learning. For example, feedback from both teachers and students indicated that the time spent talking about what a discount is with examples from real life was particularly valuable. One teacher described Lesson 6 as "the epiphany lesson" where the students realised why they needed to be able to calculate percentages. The opportunity for students to elaborate their thinking was the main reason for the positive response to this aspect of the lessons. Students embraced the approach. Another teacher comment was: *The high point of the whole thing was that they did have to nut things out, discuss.*

In Lesson 4, students compared a fixed discount of $1 off meals deals for "math burgers" and whether it was better to buy two $5 deals (Nell) or one $10 deal (Grace). A typical answer was: *Nell, because she would get a $2 discount whereas Grace only gets a $1 discount*.

Using percentages to compare discounts was common in two classes at different schools but not mentioned in the other three classes. In one school a student came up with the idea (Nell gets 20% discount, Grace only 10%); it spread among more students and finally the teacher caught on and used it with other students.

When a comparison of a fixed tax of $10 over the 10% GST was discussed, students' reactions were mixed as to whether the GST was fairer. *Yes, because otherwise you could buy a $1 lollypop and the tax would come in and it would cost you $11 which is a rip off. No, it's not fair because if you get something that's expensive, you pay a lot of tax.*

In other questions where differing discounts of different amounts occurred, nearly everyone stated that a bigger percentage reduction does not always mean a cheaper buy because it depends on the original price—they observed that both the discount and "percent of what" were relevant. Some students gave a couple of examples to illustrate this point. However, the notion of "best" could still have different interpretations, with one student thinking the best deal was the lower cost not the bigger discount.

When asked which is a better result, 60 merit certificates in a school of 500 or 80 in a school of 800, most students compared the results of the two schools using percentages – one having 10% of students awarded certificates and the other school having a higher percentage. Only a couple of students tried to calculate what the other percentage was. One calculated it as being 15% and the other calculated it as 16% (the correct percentage being 12%), but their reasoning was correct.

Not all of the lessons were received positively. Lesson 5, which focused on the fact that additive comparisons are sometimes more appropriate (as with ages in interview Question 2), was seen as problematic. This lesson also involved answers which were value judgments (e.g., Which is worse: losing 50% of $1, $10, or $100?). Teachers reported being very uncomfortable with this lesson and, in fact, in one school the teacher handed over the teaching to one of the authors who was present.

Although the teachers agreed about the benefits of the open discussion, it was also a challenge because it went beyond what was their normal practice. Time was a factor especially when students got carried away with a digression like the size of a burger. With respect to tax, some thought GST is fair because the money comes back to you but one student was adamant that the government should not take 10% because they did not make the things.

Another aspect is that teachers differed in the way in which they marked worksheets.

- One teacher simply checked the worksheets for completeness and ticked once on the front page.
- Students marked each other's work. Every answer was ticked, even when the explanations showed that an answer was wrong.
- Students marked each other's work, but afterwards the worksheets were marked by the teacher. The teacher crossed out ticks and wrote specific comments such as "*50% of what?*" and "*It should be split into 10 parts!*"

Students' marking of each others work is a useful practice commonly followed in primary schools. However, marking an explanation is much more difficult than marking numerical answers and clearly requires a greater level of supervision by the teacher.

The other challenge brought forward by the teachers was the suggested order within the lessons. The materials began with contextual investigations without a great deal of scaffolding, and left discussion of the general principles to the end. Two teachers changed the sequence of this lesson by moving the final discussion (Step 4) to the beginning of the lesson. They then had little or no closing time in which students could discuss what they had learnt from the lesson. Another teacher agreed with these two, saying she had followed the prescribed order but in retrospect would choose to do it their way. In an observed "mathsburger" lesson, the teacher began by modeling a similar context where pets were sold for varying fixed discounts and talking extensively about what a fixed discount was. The teachers generally felt that the students needed more guidance before starting on the worksheet. There was the natural feeling, perhaps arising from traditional practice, that it is important for students to get worksheet answers correct. This is not likely to happen when worksheets are used to pose challenging problems for children to consider and learn from and to form the basis for later class discussion. Only one teacher said starting with the worksheet was a good way to proceed.

In conclusion, both qualitative and quantitative data support the claim that the extended discussion generated by the lesson materials was generally successful. The exception is Lesson 5, where the need for additive strategies and value judgements seemed too unusual for most teachers.

Conclusions

The results are consistent with White and Mitchelmore (2005) in showing that the approach taken has the potential to benefit student engagement, learning, and attitudes for both students and teachers. The regional setting did not seem to provide any different results to the previous study, except that the teachers indicated they did not normally have opportunities for such professional development. The decision to reduce the content of the unit does seem to have given sufficient time. Further development of the unit appears to be worth pursuing, with perhaps a further look at Lesson 5. What, however, do the results show about the theoretical model of *Teaching for Abstraction*?

We recall that Teaching for Abstraction consists of four stages:

- *familiarising* oneself with the structure of a variety of relevant contexts;
- *recognising* the similarities between these different contexts;
- *reifying* the similarities to form a general concept; and then
- *applying* the concept in new situations.

The *familiarizing* stage in the previous study showed a need to explore separate contexts in more detail. This aspect was successfully adopted here, with the choice of context exploration and discussion being strongly supported by teachers and students. A possibly negative aspect was that context discussion in areas like tax and discount was enriching but time consuming, and could provide different answers to the anticipated mathematical one. One need (expressed by teachers in the final workshop) was to learn more of the teaching approach adopted in the materials, especially the strategy of allowing the children to explore ideas and problems before the teacher telling them.

In the previous study, the *recognising* stage in calculation skills was identified as requiring attention. This extra attention was given here and the results indicate success apart from where fractional answers resulted. More contexts involving fractions are indicated as desirable.

The assessment of *reification* in the previous study indicated more emphasis needed to be put on explaining when and why percentages “work”. This unit actually showed explaining was a strength and the learning here is considered most valuable. The post interview analysis shows the students readily *applied* their knowledge to new situations and so, again, the discussion/investigation aspect of the unit was shown to be successful.

White and Mitchelmore (2005) emphasised that preparation for Teaching for Abstraction needs to be carefully thought out. It is again evident that this approach is radically different from that which students and teachers are accustomed to. In particular, the teachers’ inclination to reorder lessons to provide the general principle before immersion in the contexts shows a lack of comfort with or understanding of the Teaching for Abstraction approach. A possible conclusion is that the approach is too radical. It could be argued that the positive outcomes were simply the result of establishing interactive classrooms. However, we claim that the true cause was the context-based learning which is a feature of our theory. Our conclusion, therefore, is that the theoretical model (even if it was not followed rigorously) resulted in new directions for teachers and improved learning for students. The teachers involved were in fact extremely positive about the approach, and have asked for further professional development in this area. The challenges for them, though, are clear—addressing and assessing generalisations and when these are introduced in a lesson; accepting multiple answers and methods of doing calculations; and coping with a lack of confidence in working with new ideas. Teachers need more support in terms of both content and pedagogy. A project where teachers are assisted to develop their own materials following the Teaching for Abstraction model would seem an appropriate next step.

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References

Misailidou, C., & Williams, J. (2003). Diagnostic assessment of children's proportional reasoning. *The Journal of Mathematical Behaviour*, *22*, 335-368.

Mitchelmore, M. C. (2002). The role of abstraction and generalisation in the development of mathematical knowledge. In D. Edge & Y. B. Har (Eds.). *Mathematics education for a knowledge-based era* (Proceedings of the Second East Asia Regional Conference on Mathematics Education and the Ninth Southeast Asian Conference on Mathematics Education, Vol. 1, pp. 157-167). Singapore: Association of Mathematics Educators.

Mitchelmore, M. C., & White, P. (2000). Development of angle concepts by progressive abstraction and generalisation. *Educational Studies in Mathematics*, *41*, 209–238.

Mitchelmore, M. C., & White, P. (2004). Abstraction in mathematics and mathematics learning. In M. J. Høines & A. B. Fuglestad (Eds.),, *Proceedings of the 28th annual conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 329-336). Bergen, Norway: PME.

Parker, M., & Leinhardt, G. (1995). Percent: A privileged proportion. *Review of Educational Research, 65*, 421-482.

Skemp, R. (1986). *The psychology of learning mathematics* (2nd ed.). Harmondsworth, England: Penguin.

White, P., & Mitchelmore, M. C. (1996). Conceptual knowledge in introductory calculus. *Journal for Research in Mathematics Education*, *27*, 79-95.

White, P., & Mitchelmore, M. C. (2005). Teaching percentage as a multiplicative relationship. In P. Clarkson, A. Dowton, D. Gronn, A. McDonough, R. Pierce, & A. Roche (Eds.), *Building connections: Theory, research and practice* (Proceedings of the 28th annual conference of the Mathematics Education Research Group of Australasia, Melbourne, pp. 783-790). Sydney: MERGA.

Van Dooren, W., & De Bock, D. (2005). Not everything is proportional: Effects of age and problem type on propensities for overgeneralization. *Cognition and Instruction, 23, 57-86.*