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This is the accepted author manuscript version of an article published as:

Sikora, A., & Zienkiewicz, J. (2021). Probabilistic approach to the quantum separation effect for the Feynman-Kac semigroup. *Studia Mathematica*, *257*(1), 1-24.

Access to the published version: https://doi.org/10.4064/SM181016-12-3

PROBABILISTIC APPROACH TO QUANTUM SEPARATION EFFECT FOR FEYNMAN-KAC SEMIGROUP

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Abstract. The quantum tunnelling phenomenon allows a particle in Schrödinger mechanics to tunnel through a barrier that it classically could not overcome. Even infinite potentials do not always form impenetrable barriers. We discuss an answer to the following question: What is a critical magnitude of potential, which creates an impenetrable barrier and for which the corresponding Schrödinger evolution system separates? In addition we describe some quantitative estimates for the separating effect in terms of cut-off potentials.

In memoriam of our high school mathematics teacher Augustyn Kałuża

1. Introduction

The main motivation for our study comes from the notion of the quantum tunnelling effect, a phenomenon which illuminates a striking difference between classical and quantum mechanics. It allows a microscopic particle to pass through the classically forbidden potential barrier, even if its height is infinite. It is easily predicted and explained by Schrödinger mechanics - the eigenstates of the Hamiltonian of the system cannot be localised. In this work we consider a quantum well and investigate the possibility that a particle trapped in a well cannot escape, that is the possibility that the barrier separates two regions. To be more precise, we consider the domain D and its boundary $K = \partial D$ that separates D and its complement D^c. Then we fix the special class of potentials V singular near K and consider the Hammiltonian of the system, that is the operator

$$L_V = \Delta - V$$

initially defined for a function belonging to $C_c^\infty(\mathbb{R}^d \setminus K)$. Here Δ is the positive standard Laplace operator. Then we consider the Feynman-Kac semigroup $\exp(-tL_V)$ generated by the Hamiltonian L_V and we denote by $p_t^V(x,y)$ the corresponding heat kernel. We address the question: *when does* $\exp(-tL_V)$ *separate* D *and* D^c*, that is when is* $p_t^V(x,y) = 0$ *for all* $x \in D$ *and* $y \in D^c$*?*

When the domain D has a smooth boundary, the problem considered by us has been satisfactory resolved by Wu in [22]. In our work we generalise the results obtained by Wu. The essential difference compared to [22] is that we do not require smoothness

1991 *Mathematics Subject Classification.* 60J35, 35J10.

Key words and phrases. Schrödinger operators, Brownian motion, Feynman-Kac semigroup, Quantum tunnelling and separation.

Adam Sikora was partially supported by an Australian Research Council (ARC) Discovery Grant DP130101302. Jacek Zienkiewicz was partially supported by NCN grant UMO-2014/15/B/ST1/00060 .

of the considered domain D. The domains we consider are irregular fractals of some special but still general type including the Koch snowflake domain. In addition, we study quantitative estimates of the tunnelling effect in our setting. Namely, we consider cut-off potentials and we estimate the rate at which they suppress the semigroup kernel $p_t(x, y)$ when x, y are separated by the boundary $K = \partial D$, see Section 5 below.

In order to deal with irregular domains, we develop a new approach, different from one developed by Wu. For the case of the separation problem, it is still an elementary and simple probabilistic argument based on the Paley-Zygmund inequality and Blumenthal's zero-one law. It becomes more involved Brownian paths analysis for the quantitative description of the tunnelling.

The assumptions we impose on the potential V, are optimal within the classes we consider. The estimates which we discuss in our note are strictly connected to the boundary behaviour of the Brownian motion. We mention [4], [5] as papers studying diffusion in this direction. Our motivation for the techniques we use partially comes from the analysis in [4, 5] and [20].

We would like to add that the questions concerning separation can be posed for any semigroup of operators, even without direct relations to Schrödinger mechanics. We mention work [8] where the authors study similar phenomena for certain types of divergence form elliptic operators. The separation phenomenon for semigroups is also related to regularity theory of the solutions of Partial Differential Equations which was investigated in [9] and [19]. Interestingly in [9] and [19] sufficient and often necessary conditions for the regularity of the system (which contradicts the separation) are formulated in terms of integrability of the coefficients of the corresponding operators whereas we consider an assumption which can be essentially formulated in term of integrability of the potential V.

2. Preliminaries

We adopt some notation from [13, Chapter 1]. We will denote by Ω the space of all continuous functions $X : \mathbb{R}^0_+ \to \mathbb{R}^d$. We will denote by X_t its value at $t \in \mathbb{R}^0_+ = [0, \infty)$.

Let $\mathbb{F}$ be the smallest σ-algebra containing all cylinders

$$C_{t_1,t_2,\ldots,t_n,A_1,\ldots,A_1} = \{X \in \Omega : X_{t_1} \in A_1, \ldots, X_{t_n} \in A_n, 0 \le t_1 < t_2 < \ldots < t_n\}$$

where $A_1, \ldots, A_n$ are Borel sets on $\mathbb{R}^d$ and n is any integer. We define $\mathbb{F}_t$ as the smallest σ-subalgebra of $\mathbb{F}$ containing all cylinders $C_{t_1,t_2,\ldots,t_n,A_1,\ldots,A_1}$ with $t_n \le t$ and we set

$$\mathbb{F}_{t^+} = \bigcap_{s>t} \mathbb{F}_s\,. \tag{1}$$

An $\mathbb{F}$-measurable function $\tau : \Omega \to \mathbb{R}^+ \cup \{\infty\}$ will be called a Markov time if $\{m < \tau\} \in \mathbb{F}_t$ for all $0 \le t \le \infty$. For a Markov time τ we define $\mathbb{F}_{\tau^+}$ as the σ-algebra of all $A \in \mathbb{F}$ such that $A \cap \{m < \tau\} \in \mathbb{F}_t$. For given sample paths X we define translation $\Theta_t X = Y$ putting $Y_s = X_{t+s}$. For Markov time τ we put $\Theta_\tau X = Y$ where $Y_s = X_{s+\tau(X)}$.

Let $\Omega_x = \{X \in \Omega : X(0) = x\}$. Denote by $P_x(dX)$ the Wiener measure on Ω_x with the density transition corresponding to Δ, see for example [13, Section 1.4]. Note that we do not utilize common probabilistic convention to use the operator $\Delta/2$ as the semigroup generator. Here, we rather use the standard Laplacian because we believe it is a more natural and convenient definition in the context of Schrödinger semigroups studied in our note. We will call $(\Omega_x, \mathbb{F}, P_x)$ the Brownian motion starting at x. For basic properties of the Brownian motion we refer readers to [3, 13, 15]. Next, for σ-subalgebra $\mathbb{M}$ of $\mathbb{F}$ we denote by $\mathrm{E}_x(f|\mathbb{M})$ the conditional expectation of f on the probability space $(\Omega_x, \mathbb{F}, P_x)$.

We will use the following version of classical strong Markov property of the Brownian motion $(\Omega_x, \mathbb{F}, P_x)$: For $A \in \mathbb{F}_{\tau^+}$ and $B \in \mathbb{F}$ we have

$$P_x\{X : X \in A \wedge \Theta_\tau X \in B \wedge X \in \{\tau < \infty\}\} = \int_A P_{X_\tau}(B) P_x(dX) \tag{2}$$

We can extend the above formula to any cylinder

$$B = C_{\eta_1(X), \eta_1(X)+\eta_2(X), \ldots, \eta_1(X)+\ldots+\eta_n(X), A_1, \ldots A_n}$$

where $\eta_1, \ldots, \eta_n$ are nonegative $\mathbb{F}_{\tau^+}$ measurable functions. This extended form can be easily obtained from (2) for simple functions $\eta_1, \cdots, \eta_n$ and general case by passing to the limit. We refer the reader to [13, p. 23, 5b)] see also [21, p. 169, (8.69)].

In the sequel we will use the following observation. Consider any set $A \in \mathbb{F}_{\tau^+}$. For a bounded continuous function $f \colon [0, \infty) \times \mathbb{R}^n \to \mathbb{R}$ the expression

$$\int_A f(\tau, X_\tau) P_x(dX)$$

defines positive linear functional. So for certain nonegative finite Borel measure $d\mu_A(\tau, w)$ we have

$$\int_A f(\tau, X_\tau) P_x(dX) = \int f(\tau, w) d\mu_A(\tau, w) \tag{3}$$

In what follow it is always clear which set A we consider so we skip the subscript A in μ_A.

Let $D \subset \mathbb{R}^d$ be open, $D^c = \mathbb{R}^d \setminus D$ be its complement. Let $V \geq 0$ be locally bounded on $D \cup \mathrm{int}(D^c) \subset \mathbb{R}^d$. Following [6], we define the Feynman-Kac functional by the formula

$$e_V(t) = \exp(-A_V(t)) \tag{4}$$

where

$$A_V(t) = \int_0^t V(X_s) ds.$$

Then the one parameter family of operators $\{T_t, t \geq 0\}$

$$T_t f(x) = \mathrm{E}_x\{e_V(t) f(X_t)\}, \tag{5}$$

where by E_x we denote expected value over the Brownian motion starting at point $x \in \mathbb{R}^d$, is called Feynman-Kac semigroup, see [6, (26) p. 76]. It is well known that the operators $\{T_t, t \geq 0\}$ form the one parameter strongly continuous symmetric semigroup of contractions on $L^p(\mathbb{R}^d)$ for all $1 \leq p < \infty$. Moreover $C_c^\infty(D \cup \mathrm{int}(D^c))$

is contained in the domain of its infinitesimal generator $-L_V$, and for all functions $\phi \in C_c^\infty(D \cup \mathrm{int}(D^c))$ we have

$$L_V \phi(x) = (\Delta - V(x))\phi(x).$$

We say that the semigroup $\{T_t, t \geq 0\}$ separates the sets D and D^c if all operators T_t preserve the subspace $L^2(D) \subset L^2(\mathbb{R}^d)$, that is

$$T_t(L^2(D)) \subset L^2(D). \tag{6}$$

Note that the operators $\{T_t, t \geq 0\}$ are symmetric so (6) implies that the subspace $L^2(D^c)$ is also preserved.

In a part of our discussion in what follows it is convenient to use a different version of Feynmann-Kac formula based on the notion of the Brownian bridge. Let $\{Y_s, t \geq s \geq 0\}$ be the Brownian bridge stochastic process connecting points $x, y \in \mathbb{R}^d$ for a definition, see for example [17, Example 3, p. 243]. Using the above notion the heat kernel corresponding to the Feynmann-Kac semigroup can be written as

$$p_t^V(x,y) = \int \exp\left(-\int_0^t V(Y_s)ds\right) d\mu_{x,y}(Y),$$

where $d\mu_{x,y}$ is the Brownian bridge measure defined on the set $\Omega^t_{x,y}$ of continuous sample paths connecting x and y, normalised in such way that $\int d\mu_{x,y}(\omega) = p_t(x,y)$, see [18, Theorem 6.6]. Note that in our notation $\frac{\mu_{x,y}(\Omega^t_{x,y})}{p_t(x,y)}$ is the Brownian bridge probability measure on the set $\Omega^t_{x,y}$. We will call $d\mu_{x,y}$ the Brownian bridge measure.

In our approach the Paley-Zygmund inequality plays a crucial role. It bounds the probability that a positive random variable is small, in terms of its mean and variance. Let us recall the statement of this result.

Proposition 2.1. *Suppose that $Z \geq 0$ is a positive random variable with finite variance and that $0 < \theta < 1$. Then*

$$P\big(Z \geq \theta\, \mathrm{E}(Z)\big) \geq (1-\theta)^2 \frac{\mathrm{E}(Z)^2}{\mathrm{E}(Z^2)}.$$

Proof. Note that

$$\mathrm{E}(Z) = \mathrm{E}(Z\,\chi_{Z<\theta\,\mathrm{E}(Z)}) + \mathrm{E}(Z\,\chi_{Z\geq\theta\,\mathrm{E}(Z)}).$$

Obviously, the first addend is at most $\theta\,\mathrm{E}(Z)$. By the Cauchy-Schwarz inequality the second one is at most

$$\mathrm{E}(Z^2)^{1/2}\,\mathrm{E}(\chi_{Z\geq\theta\,\mathrm{E}(Z)})^{1/2} = \mathrm{E}(Z^2)^{1/2} P\big(Z \geq \theta\, \mathrm{E}(Z)\big)^{1/2}.$$

This proves the required estimate. □

3. Quantum separation for Feynman-Kac semigroups

Consider a closed subset $K \subset \mathbb{R}^d$ with dimension $d \geq 2$. We will impose fractal like type regularity requirements for K. Namely for any $\gamma > 0$ we define the γ neighbourhood K_γ of K by the formula

$$K_\gamma = \{x \in \mathbb{R}^d \colon \inf_{y \in K} |x - y| \leq \gamma\}. \tag{7}$$

In what follows we will always assume that there exist an exponent $0 < \alpha < d$ and a positive constant C_1 such that

$$|K_\gamma \cap B(x,r)| \le C_1 r^\alpha \gamma^{d-\alpha} \tag{8}$$

for all $x \in \mathbb{R}^d$ and $1 \ge r \ge \gamma > 0$. We also assume that there exists a positive constant C_2 such that

$$|K_\gamma \cap B(x,r)| \ge C_2 r^\alpha \gamma^{d-\alpha} \tag{9}$$

for all $x \in K_{r/2}$ and all $1 \ge r \ge \gamma > 0$.

The above regularity conditions are frequently considered in the literature and motivated by the notion of Minkowski dimension (which is also called box dimension), see for example §3.1 and Proposition 3.2 of [10]. These conditions are also closely related to the notion of Ahlfors regularity, which is often used in the context of analysis on metric spaces, see for example [7] and [12]. Using the standard techniques, one can check that these conditions are satisfied for most of the standard fractal constructions including the classical van Koch snowflake curve, again see for example [10]. If the boundary of the region D is regular enough, for example if $K = \partial D$ is an immersed C^1 manifold then it is immediate that the estimates (8) and (9) hold with α equals to topological dimension of K.

We define the distance from K by the formula $d_K(x) = \inf\{d(x,y)\colon y \in K\}$ and then we set

$$V_\beta = C_V d_K^{-\beta}. \tag{10}$$

The precise value of the constant is irrelevant for our analysis, so in what follows we fix $C_V = 1$.

Before we state our first result, Theorem 3.2 below, we would like make the following observation which explains why we consider only the range of exponents $\alpha > d-2$ in what follows.

Remark 3.1. It is not difficult to show that if X is the Brownian motion in the Euclidean space $\mathbb{R}^d$ starting at the origin then

$$P_0\left(\int_0^\delta |X_s|^{-2} ds = \infty,\, \forall \delta > 0\right) = 1.$$

Hence there is no point to study the case $\beta \ge 2$ and we can assume that $\alpha > d-2$ in the following statement.

Theorem 3.2. *Assume that a closed subset $K \subset \mathbb{R}^d$, for some $d \ge 2$ satisfies conditions (8) and (9) with some $d > \alpha > d-2$. Suppose next that X is the Brownian motion starting at point x contained in K that is such that $X_0 = x \in K$. Then*

$$P_x\left(\int_0^\delta V_\beta(X_s) ds = \infty,\, \forall \delta > 0\right) = 1$$

for every β such that $\beta + \alpha \ge d$.

Proof. Note that by taking intersection of K with a closed ball $\overline{B(x,1)}$ we can assume without loss of generality that the set K is compact. Next note that it follows from (8) and (9) that if one takes $1 > a > 0$ such that $C_2/2 \geq a^{d-\alpha}C_1$ then for $C_3 = C_2/2$

$$|(K_\gamma \setminus K_{a\gamma}) \cap B(x,r)| \geq C_3 r^\alpha \gamma^{d-\alpha} \quad \forall_{x\in K_{r/2}}. \tag{11}$$

Now for any $n \in \mathbb{N}$ we set

$$K_n' = K_{a^n} \setminus K_{a^{n+1}},$$

where a is the constant from estimate (11). We define a sequence of random variables Z_n by the following formula.

$$Z_n(X) = \int_0^{\delta^2} \chi_{K_n'}(X_s)ds$$

where $\chi_{K_n'}$ is the characteristic function of the set K_n' described above and X is the Brownian motion process starting at some fixed point $x \in K$.

Following the idea of [20], we shall verify assumptions of the Paley-Zygmund inequality for each random variable Z_n. To that end take some $\delta > 0$ and set

$$b_n = a^{n(d-\alpha)}\delta^{2-d+\alpha}.$$

We shall prove the following estimates for the expected values of Z_n and Z_n^2

$$\mathrm{E}_x(Z_n) \geq Cb_n \tag{12}$$

and

$$\mathrm{E}_x(Z_n^2) \leq cb_n^2 \tag{13}$$

valid for all $n \in \mathbb{N}$ such that $a^n \leq \delta^2$. The constants c, C in (12) and (13) do not depend on n. In order to prove (12) we note that if $d > 2$ then for positive constants $C, C', c', c > 0$

$$C' r^{2-d} \exp\left(-\frac{c'r^2}{\delta^2}\right) \leq \int_0^{\delta^2} t^{-d/2} \exp\left(\frac{-r^2}{4t}\right) dt \leq Cr^{2-d} \exp\left(-\frac{cr^2}{\delta^2}\right). \tag{14}$$

Whereas for $d = 2$

$$\begin{aligned} C'(1+|\log(\delta/r)|) \exp\left(-\frac{c'r^2}{\delta^2}\right) &\leq \int_0^{\delta^2} t^{-d/2} \exp\left(\frac{-r^2}{4t}\right) dt \\ &\leq C(1+|\log(\delta/r)|) \exp\left(-\frac{cr^2}{\delta^2}\right). \end{aligned} \tag{15}$$

We first discuss the case $d > 2$. Since $p_t(x,y) = (4\pi t)^{-d/2} e^{-\frac{|x-y|^2}{4t}}$ is the transition density of the considered process X, using Fubini's theorem, we have

$$\begin{aligned} \mathrm{E}_x(Z_n) &= \int \int_0^{\delta^2} \chi_{K_n'}(X_t) dt P_x(dX) \\ &= \int_0^{\delta^2} \int_{\mathbb{R}^d} \chi_{K_n'}(y)(4t\pi)^{-d/2} \exp\left(-\frac{|x-y|^2}{4t}\right) dydt \\ &= \int_{\mathbb{R}^d} \int_0^{\delta^2} \chi_{K_n'}(y)(4t\pi)^{-d/2} \exp\left(-\frac{|x-y|^2}{4t}\right) dtdy, \end{aligned}$$

By (14) and then by (11)

$$\begin{aligned}
\mathrm{E}_x(Z_n) &= \int_{\mathbb{R}^d}\int_0^{\delta^2} \chi_{K_n'}(y)(4t\pi)^{-d/2}\exp\left(-\frac{|x-y|^2}{4t}\right)dtdy \\
&\geq C\int_{|x-y|\leq\delta} \chi_{K_n'}(y)|x-y|^{2-d}dy \geq c\delta^{(2-d)}\int_{K_n'\cap B(x,\delta)} dy \\
&\geq cC_3\delta^{(2-d)}a^{n(d-\alpha)}\delta^\alpha = c'b_n.
\end{aligned}$$

This proves estimate (12).

In the next step of the proof we will verify estimate (13).

$$\begin{aligned}
Z_n^2(X) &= \int_0^{\delta^2}\int_0^{\delta^2} \chi_{K_n'}(X_s)\chi_{K_n'}(X_t)dsdt \\
&= \int_{s\leq t\leq\delta^2} \chi_{K_n'}(X_s)\chi_{K_n'}(X_t)dsdt + \int_{t\leq s\leq\delta^2} \chi_{K_n'}(X_s)\chi_{K_n'}(X_t)dsdt.
\end{aligned}$$

Now for $t > s$ it follows from the independence of X_s and $X_t - X_s$

$$\begin{aligned}
&\int\int_{s\leq t\leq\delta^2} \chi_{K_n'}(X_s)\chi_{K_n'}(X_t)dsdtP_x(dX) \\
=& \int\int_{s\leq t\leq\delta^2} \chi_{K_n'}(X_s)\chi_{K_n'}(X_t - X_s + X_s)dsdtP_x(dX) \\
=& \int_0^{\delta^2}\int_0^t\int\int \frac{\chi_{K_n'}(y)\chi_{K_n'}(z+y)}{(4\pi)^d(s(t-s))^{d/2}}\exp\left(-\frac{|x-y|^2}{4s}\right)\exp\left(-\frac{|z|^2}{4(t-s)}\right)dzdydsdt.
\end{aligned}$$

Next, by (14)

$$\begin{aligned}
&\int_0^{\delta^2}\int_0^t\int\int \frac{\chi_{K_n'}(y)\chi_{K_n'}(z+y)}{(4\pi)^d(s(t-s))^{d/2}}\exp\left(-\frac{|x-y|^2}{4s}\right)\exp\left(-\frac{|z|^2}{4(t-s)}\right)dzdydsdt \\
&\quad\leq C\int\int \frac{\chi_{K_n'}(y)\chi_{K_n'}(z+y)}{|x-y|^{d-2}|z|^{d-2}}\exp\left(-\frac{c|x-y|^2}{\delta^2}\right)\exp\left(-\frac{c|z|^2}{\delta^2}\right)dzdy =: I.
\end{aligned}$$

To estimate the term I we recall from the beginning of proof that without loss of generality we can assume that $K \subset B(x,1)$. For any $m \in \mathbb{N}$ set

$$A_m = \{z \in \mathbb{R}^d\colon\ a^{m+1} < |z| \leq a^m\}.$$

Then by (8)

$$\begin{aligned}
&\int \chi_{K_n'}(z+y)|z|^{2-d}\exp\left(-\frac{c|z|^2}{\delta^2}\right)dz \\
&= \sum_{m\in\mathbb{N}}\int_{A_m}\chi_{K_n'}(z+y)|z|^{2-d}\exp\left(-\frac{c|z|^2}{\delta^2}\right)dz \\
&\le C\sum_{a^m<\delta}a^{m(2-d)}a^{m\alpha}a^{n(d-\alpha)} + C\sum_{a^m\ge\delta}a^{m(2-d)}a^{m\alpha}a^{n(d-\alpha)}\exp\left(-\frac{ca^{2m}}{\delta^2}\right) \qquad (16)\\
&\le Ca^{n(d-\alpha)}\delta^{2-d+\alpha} + Ca^{n(d-\alpha)}\delta^{2-d+\alpha}\sum_{a^m\ge\delta}\left(\frac{a^m}{\delta}\right)^{2-d+\alpha}\exp\left(-\frac{ca^{2m}}{\delta^2}\right) \\
&\le Ca^{n(d-\alpha)}\delta^{2-d+\alpha} = Cb_n.
\end{aligned}$$

By the above estimate

$$\begin{aligned}
\int\int \chi_{K_n'}(y)\chi_{K_n'}(z+y)|x-y|^{2-d}\exp\left(-c\frac{|x-y|^2}{\delta^2}\right)|z|^{2-d}\exp\left(-c\frac{|z|^2}{\delta^2}\right)dzdy \\
\le Ca^{n(d-\alpha)}\delta^{2-d+\alpha}\int\chi_{K_n'}(y)|x-y|^{2-d}\exp\left(-c\frac{|x-y|^2}{\delta^2}\right)dy.
\end{aligned}$$

Now the repetition of the calculation in (16) applied to the last integral above, that is $\int\chi_{K_n'}(y)|x-y|^{2-d}\exp\left(-c\frac{|x-y|^2}{\delta^2}\right)dy$ yields the required estimate

$$I \le Cb_n^2 = C\left(a^{n(d-\alpha)}\delta^{2-d+\alpha}\right)^2.$$

This proves estimate (13).

Next, by the Paley-Zygmund inequality with $\theta = 1/2$ it follows from estimates (12) and (13) that there exists a constant $\sigma > 0$ independent of n and δ such that for an appropriate constant c

$$P_x\Big(Z_n \ge ca^{n(d-\alpha)}\delta^{2-d+\alpha} = cb_n\Big) \ge P_x\left(Z_n \ge \frac{\mathrm{E_x}(Z_n)}{2}\right) \ge \frac{\mathrm{E_x}(Z)^2}{4\,\mathrm{E_x}(Z^2)} \ge \sigma. \qquad (17)$$

Hence for any $\delta > 0$.

$$\begin{aligned}
&P_x\left(\int_0^{\delta^2}\chi_{K_n'}(X_s)ds \ge ca^{n(d-\alpha)}\delta^{2-d+\alpha} \quad \text{for infinitely many } n\right) \\
&= P_x\left(\bigcap_{N=1}^{\infty}\bigcup_{n=N}^{\infty}\left\{\int_0^{\delta^2}\chi_{K_n'}(X_s)ds \ge ca^{n(d-\alpha)}\delta^{2-d+\alpha}\right\}\right) \ge \sigma. \qquad (18)
\end{aligned}$$

Now consider the additive functional A_{V_β} with $V_\beta = d_K^{-\beta}$ for some $\beta \geq d-\alpha$. Then, by (17) and (18), for any sequence δ_j decreasing to 0 we have

$$\begin{aligned} & P_x\left(A_{V_\beta}(\delta^2) = \int_0^{\delta^2} V_\beta(X_s)ds = \infty, \quad \forall \delta > 0\right) \\ = \quad & P_x\left(\bigcap_j \left\{\int_0^{\delta_j^2} V_\beta(X_s)ds = \infty\right\}\right) = \lim_{j\to\infty} P_x\left(\int_0^{\delta_j^2} V_\beta(X_s)ds = \infty\right) \\ \geq \quad & \lim_{j\to\infty} P_x\left(\int_0^{\delta_j^2} \chi_{K'_n}(X_s)ds \geq c a^{n(d-\alpha)}\delta_j^{2-d+\alpha} \quad \text{for infinitely many } n\right) \geq \sigma. \end{aligned}$$

Before we continue our discussion we shall justify the first of the above inequalities in more detailed way. Assume we are given a sample path $X \in \Omega_x$ with

$$a^{n(\alpha-d)} \int_0^{\delta_j^2} \chi_{K'_n}(X_s)ds \geq c\delta_j^{2-d+\alpha}$$

for all $n \in A_X$ and some infinite subset $A_X \subset \mathbb{N}$. Then since $a^{n(\alpha-d)}V_\beta(x) \geq 1$ for all $x \in K'_n$ it follows that

$$\int_0^{\delta_j^2} V_\beta(X_s)ds = \sum_n \int_0^{\delta_j^2} V_\beta(X_s)\chi_{K'_n}(X_s)ds \geq c \sum_{n\in A_X} \delta_j^2 = \infty.$$

Hence the first inequality follows.

Now the event

$$\Omega_{V_\beta} = \left\{X : \int_0^{\delta^2} V_\beta(X_s)ds = \infty: \quad \forall \delta > 0\right\}$$

is measurable with respect to σ-field $\mathbb{F}_{0+}$ defined by (1) so by Blumenthal's zero-one law, see for example [2] or [13, p. 25, Problem 2], it must have probability equal to 0 or 1. Thus $P(\Omega_{V_\beta}) = 1$. This ends the proof of Theorem 3.2 in the case $d > 2$. For $d = 2$ the proof is a simple modification of the above argument. One can essentially repeat the same calculation just replacing estimates (14) by estimates (15) corresponding to the case $d = 2$. We skip the details here. □

We are now in position to state the main result of this section.

Theorem 3.3. *Suppose that the set $D \subset \mathbb{R}^d$ is open simply connected and that its boundary $K = \partial D$ satisfies conditions (8) and (9) for some $d > \alpha > 0$ and $d \geq 2$. Let $\{T_t, t \geq 0\}$ be the Feynman-Kac semigroup generated by $L_{V_\beta} = \Delta - V_\beta$, where the potential V_β is defined by (10). Assume also that $\alpha + \beta \geq d$.*

Then the subspace $L^2(D)$ of $L^2(\mathbb{R}^d)$ is invariant under the action T_t that is

$$T_t(L^2(D)) \subset L^2(D)$$

for all $t \geq 0$.

Proof. Recall that Ω_x is the set of paths starting from $x \in D$. Let $\tau(X)$ denotes the first hitting time of $K = \partial D$ for X. Consider the set

$$\Omega_V = \left\{ X : \int_0^{\delta^2} V(X_s)ds = \infty : \quad \forall \delta > 0 \right\}$$

By Theorem 3.2 and the strong Markov property (2), with $A = \{\tau \le t\} \in \mathbb{F}_{\tau^+}$ and $B = \Omega_V$, see also [13, Point 5b), page 23]

$$\begin{aligned} P_x\left(X \in \Omega_x : \tau \le t, \Theta_\tau X \in \Omega_V\right) &= \int_{\{\tau \le t\}} P_{X_\tau}(\Omega_V) P_x(dX) \\ &= P_x(\{\tau \le t\}). \end{aligned}$$

Hence the event:

$$\text{for all } \delta > 0 \text{ we have } \int_\tau^{\tau+\delta^2} V(X_s)ds = \infty$$

holds a.s on $\{\tau \le t\}$. Clearly for any path X if $X_0 \in D$ and $X_t \in D^c$ then $\tau(X) < t$ so now the theorem follows by applying Feynman-Kac formula (5). □

4. Singularity of V forcing separation.

In this section we prove that Theorem 3.3 is optimal, that is that the condition $\alpha + \beta \ge d$ is also necessary. By $p_t(x, y)$ we denote the Gaussian distribution (corresponding to Δ rather than $\Delta/2$, see the discussion in Section 2). Recall that

$$p_t(x, y) = (4\pi t)^{-d/2} e^{-\frac{|x-y|^2}{4t}}.$$

Then as before by $p_t^V(x, y)$ we denote the kernel of the Feynman-Kac semigroup corresponding to the positive potential V defined by (5). In these terms Theorem 3.3 can be stated in the following way: For any β such that $\alpha + \beta \ge d$

$$p_t^{V_\beta}(x, y) = 0 \quad \forall t > 0, \quad x \in D \quad \text{and} \quad \forall y \in D^c.$$

Note that the kernel $p_t^{V_\beta}$ is symmetric so it automatically follows that if this is the case then $p_t^{V_\beta}(x, y) = 0$ also whenever $x \in D^c$ and $y \in D$.

Next we shall show that if $\alpha + \beta < d$ then

$$p_t^{V_\beta}(x, y) > 0 \quad \forall t > 0, x, y \in \mathbb{R}^d,$$

see Theorem 4.1 and Corollary 4.2 below.

First for any $t > 0$ we set

$$\Gamma_t(x, y) = \int_0^{t/2} p_s(x, y)ds = \int_0^{t/2} (4\pi s)^{-d/2} e^{-\frac{|x-y|^2}{4s}} ds.$$

Theorem 4.1. *Assume that $V \in L^1_{loc}(\mathbb{R}^d)$ is a positive locally integrable potential. Suppose also that for some fixed t*

$$V * \Gamma_t(x) + V * \Gamma_t(y) < \infty.$$

Then $p_t^V(x, y) > 0$.

Proof. Recall that $d\mu_{x,y}$ is the Brownian bridge measure defined on the set $\Omega^t_{x,y}$ of continuous sample paths connecting x and y and normalised in such way that $\int d\mu_{x,y}(\omega) = p_t(x,y)$, see Section 2 above.

Now the transition density formula of the Brownian bridge, see for example [15, p. 359 (6.28)] and Chebyshev's inequality yield

$$\begin{aligned}
\mu_{x,y}\left(\left\{\int_0^t V(Y_s)ds \geq A\right\}\right) &\leq \frac{1}{A}\int\int_0^t V(Y_s)dsd\mu_{x,y}(Y) \\
&= \frac{1}{A}\int_0^t\int_{\mathbb{R}^d} V(z)p_s(x-z)p_{t-s}(z-y)dzds \\
&\leq \frac{C}{At^{d/2}}\left(\int_0^{t/2}\int_{\mathbb{R}^d} V(z)p_s(x-z)dzds + \int_0^{t/2}\int_{\mathbb{R}^d} V(z)p_s(y-z)dzds\right) \\
&\leq \frac{C}{At^{d/2}}\left(V*\Gamma_t(x) + V*\Gamma_t(y)\right).
\end{aligned}$$

Hence for sufficiently large A

$$\mu_{x,y}\left(\left\{\int_0^t V(Y_s)ds \leq A\right\}\right) \geq p_t(x,y) - \frac{C}{At^{d/2}}\left(V*\Gamma_t(x) + V*\Gamma_t(y)\right) \geq \frac{1}{2}p_t(x,y).$$

Thus

$$p^V_t(x,y) = \int \exp\left(-\int_0^t V(Y_s)ds\right)d\mu_{x,y}(Y) \geq \frac{1}{2}e^{-A}p_t(x,y) > 0.$$

This concludes proof of Theorem 4.1. ☐

As a direct consequence of Theorem 4.1 we obtain the following corollary

Corollary 4.2. *Under the assumptions of* Theorem 3.3 *the condition $\alpha+\beta \geq d$ is necessary for separation.*

Proof. The proof is essentially repetition of the proof of estimate (12) from the proof of Theorem 3.2. More precisely one can notice that if we set $t=\delta^2$ then

$$V*\Gamma_t(x) \leq C\sum_{n>0} a^{-n\beta}E(Z_n) \leq C\sum_{n>0} a^{-n\beta}a^{n(d-\alpha)}.$$

Recall that $a<1$ so the above sum is finite if $\alpha+\beta<d$. Now Corollary 4.2 follows from Theorem 4.1. ☐

As an illustration of Theorem 3.3 and Corollary 4.2 we would like to describe the construction of the van Koch snowflake curve.

Example 4.3. *Van Koch snowflake. Consider an equilateral triangle K_0 with sides of unit length. Next define a curve K_1 by replacing the middle of all edges of K_0 by the two sides of the equilateral triangle based on the middle every segment. Next define K_2 by repeating the same procedure on each of the twelfth edges of K_1. Von Koch snowflake, which we denote by K is the self-similar set obtained by iteration of this procedure. Its Minkowski dimension is equal to $\alpha = \log 4/\log 3$ and it satisfies assumptions* (8) *and* (9) *with this α and $d=2$.*

Thus as a straightforward consequence of our results we obtain the following corollary

Corollary 4.4. *Suppose that $D \subset \mathbb{R}^2$ is the region of the plane inside the Von Koch snowflake K and $p_t^{V_\beta}$ is the heat kernel corresponding to the operator $\Delta - d_K^{-\beta}$. Then for any $\beta \geq 2 - \log 4/\log 3$*

$$p_t^{V_\beta}(x,y) = 0 \quad \forall t > 0, x \in D \quad \textit{and} \quad \forall y \in D^c.$$

When $\beta < 2 - \log 4/\log 3$ then

$$p_t^{V_\beta}(x,y) > 0 \quad \forall x, y \in \mathbb{R}^2 \quad \textit{and} \quad \forall t > 0.$$

5. Estimates of the rate of separation for truncated potentials

In this section we consider the potential

$$V_\beta^A(x) = C_V A^\beta \chi_{\{d_K(x) \leq A^{-1}\}} = C_V A^\beta \chi_{K_{A^{-1}}}(x)$$

We fix a supercritical exponent $\alpha + \beta > d$. Denote by $p_t^A(x,y)$ the kernel of the Feynman-Kac semigroup generated by the operator

$$-L_{V_\beta^A} = \Delta - V_\beta^A$$

with the natural domain. It is well known that the functions $p_t^A(x,y)$ are continuous in $x, y, t \in \mathbb{R}^d \times (0,\infty)$.

It is convenient for us to introduce at this point the following definition of the uniform domain type. Our definition is a variant of Definition 3.2 of [11] but is motivated by the definition of NTA (nontangentially accessible) domains introduced by Jerison and Kenig in [14]. See also the discussion in Section 3.1.3 of [11].

Definition 5.1. *Let $D \subset \mathbb{R}^d$ be a connected subset of $\mathbb{R}^d$. We say that D satisfies the inside NTA condition if there are constants c, C such that, for any $x, y \in D$ in the interior of D there exists a continuous curve $\gamma_{x,y} \colon [0,1] \to D$ such that $\gamma_{x,y}(0) = x$ and $\gamma_{x,y}(1) = y$ and the following two properties are satisfied*

1. *The length $L(\gamma_{x,y})$ is at most $C|x-y|$.*
2. *For any $z \in \gamma_{x,y}([0,1])$*

$$d_{\partial D}(z) \geq c \min(L(\gamma_{x,z}), L(\gamma_{y,z})).$$

We say that D is an NTA domain if both D and D^c satisfies the inside NTA condition.

We say that an open domain $D \subset \mathbb{R}^d$ is separated from a given ball $B_0 = B(x,r) \subset \mathbb{R}^d$ if $2B_0 = B(x,2r) \subset D^c = \mathbb{R}^d \setminus D$. The following statement is the main result discussed in this section.

Theorem 5.2. *Suppose that set $D \subset \mathbb{R}^d$ is open simply connected and that its boundary $K = \partial D$ satisfies conditions (8) and (9) for some $d > \alpha > 0$ and that $\alpha + \beta > d \geq 2$. Let D be an NTA domain. There exists a constant $\sigma > 0$ such* *that for any point $x \in D$, any ball B_0 separated from D, and any positive $t > 0$ there exists a constant $C = C(x,t)$ such that*

$$\int_{B_0} p_t^A(x,y)dy \le CA^{-\sigma} \tag{19}$$

for all $A > 0$.

Remark 5.3. By the standard elliptic estimates (or by a slightly more technical variant of our argument) one can obtain a pointwise estimate $p_t^A(x,y) \le CA^{-\sigma}$.

Remark 5.4. In this work we are interested in the existence of a positive σ satisfying (19). We want to point out however that our methods can yield estimates (19) for a larger range of parameter σ at the cost of some further work and more complex and tedious argument. We do not discuss the details of such strengthen result in the present paper.

Remark 5.5. The value of σ in our approach depends on the domain. The constant C depends on the domain and the value of the multiplicative constant C_V.

For the sake of simplicity we will consider only the case $C_V = 1$. Without loss of generality we can assume that $t = 1$. First we prove a series of technical lemmata concerning properties of the Brownian motion.

Lemma 5.6. *There exist constants $0 < \zeta < 1$ and $\sigma_0 > 0$ such that*

$$P_x\left(\int_0^{\delta^2} V_\beta^A(X_t)dt < \zeta\,\mathrm{E}\left(\int_0^{\delta^2} V_\beta^A(X_t)dt\right)\right) \le 1 - \sigma_0$$

for all $\delta > 0$ and $A > 0$ and every starting point $X_0 = x \in K$.

Proof. To prove the statement we use an argument similar to one which we used to verify (17). We have to replace $\chi_{K_n'}$ in the definition of Z_n by $\chi_{\{d_K(x)\le A^{-1}\}}$. Then the statement follows by replicating the calculations leading to (17) which we use in the proof of Theorem 3.2. To avoid repetition we omit the details. $\square$

Let $\mathbf{w}_\lambda = \lambda\dot{(}1,1,\ldots,1) \in \mathbb{R}^d$ where $\lambda \in \mathbb{R}$ will be specified later. Consider the decomposition of $\mathbb{R}^d$ into congruent cubes of side length δ obtained by the $\delta\mathbb{Z}^d$ translations of $\mathbf{w}_\lambda + [0,\delta]^d$ and denote by $Q_1, \ldots, Q_M$ all these cubes, such that $Q_j \cap K \ne \emptyset$ for $j = 1, \ldots, M$. By (8) one immediately gets $\frac{1}{C}\delta^{-\alpha} \le M \le C\delta^{-\alpha}$ where C does not depend on λ. We fix some small $v > 0$, $\delta \approx A^{-v}$ and $H \in \mathbb{N}$ in such a way that $2H + 2 = \delta^{-2}$. For any $h \in \{0, \ldots, H\}$ we set

$$I_h = [2h\delta^2, (2h+1)\delta^2) \quad \text{and} \quad J_h = [(2h+1)\delta^2, (2h+2)\delta^2).$$

Let $1 \le H_0 \le H$ be a fixed number. We will consider multi-indexes of the form $j_1, \ldots, j_{H_0}, k_1, \ldots, k_{H_0}$ such that $1 \le j_s \le M$, $0 \le k_s \le H$, $k_1 < k_2 \ldots < k_{H_0}$ for all $1 \le s \le H_0$.

Let Ω be the set of all Brownian paths X such that $X_0 = x$ and $X_1 \in B_0$. We define the family of the subsets $\Phi_{j_1,\ldots,j_{H_0},k_1,\ldots,k_{H_0}} \subset \Omega$ of Ω by requiring the following conditions:

(1) For every $1 \leq s \leq H_0$ there exists time $t'_s \in I_{k_s}$ such that for the path $X \in \Omega$, $X_{t'_s} \in K$. By t_s we denote the smallest such t'_s (the first hitting time of K for $t \in I_{k_s}$)
(2) For every $1 \leq s \leq H_0$, it holds that $X_{t_s} \in Q_{j_s}$.
(3) For every $t \in I_h$ and for any h not listed in the sequence $k_1, \dots, k_{H_0}$ one has $X_t \notin K$.

The sets $\Phi_{j_1,\dots,j_{H_0},k_1,\dots,k_{H_0}} \subset \Omega$ do not need to be disjoint. We observe that a given sample path belongs to the two different sets $\Phi_{j_1,\dots,j_{H_0},k_1,\dots,k_{H_0}}$ if the first hitting point X_τ for some $\tau \in I_h$ belongs to two different cubes Q of the grid. It is possible only if X_τ belongs to common wall of two neighbouring cubes Q. We have

Claim. There exists $\lambda \in [0, \delta]$ such that the Wiener measure of the set λ of trajectories which do not uniquely determine cubes $Q_{j_1} \dots, Q_{j_{H_0}}$ is zero.

Proof. Indeed, denote by $\tau \in I_{k_j}$ the first hitting time of K. Fix the $d-1$ or lower dimensional common wall $\mathbf{W}$ of two cubes and observe that all of its translations $\mathbf{W}_\lambda = \mathbf{W} + \mathbf{w}_\lambda$ are pairwise disjoint. Hence the events $\{X_\tau \in \mathbf{W}_\lambda\}$ are pairwise disjoint, so only for at most countably many of λ will the event $\{X_\tau \in \mathbf{W}_\lambda\}$ have positive Wiener measure. The claim follows. □

From now on we fix λ given by the claim and consider the grid of cubes corresponding to $\mathbf{w}_\lambda$. We subtract the set E_λ from Ω and $\Phi_{j_1,\dots,j_{H_0},k_1,\dots,k_{H_0}}$ and denote the new sets again by Ω and $\Phi_{j_1,\dots,j_{H_0},k_1,\dots,k_{H_0}}$. Now it is straightforward to see that the sets $\Phi_{j_1,\dots,j_{H_0},k_1,\dots,k_{H_0}}$ are pairwise disjoint. Moreover the set

$$\bigcup_{j_1,\dots,j_{H_0},k_1,\dots,k_{H_0}} \Phi_{j_1,\dots,j_{H_0},k_1,\dots,k_{H_0}}$$

(up to a subset of measure zero) consists of such $X \in \Omega$ that the path X for all $h = 1, \dots, H$ intersects K for exactly H_0 out of H intervals I_h. We will use these facts in the sequel.

Now by q_{j_s} we denote the center of Q_{j_s} and set

$$\gamma_{\delta,H} = \delta^2 (\log H)^{-3} = (\log H)^{-3}/(2H+2).$$

Next, for any $\eta \geq 1$ and $x, y \in \mathbb{R}^d$, such that

$$|x - q_{j_s}| \leq \eta\delta\sqrt{\log H}, \qquad \text{and} \qquad |y - q_{j_s}| \leq \eta\delta\sqrt{\log H}$$

we define a set Ψ_0 by the formula

$$\Psi_0 = \left\{ X \in \Omega^{2\delta^2}_{x,y} \colon \tau_{j_s} \leq \delta^2,\ |X_{\tau_{j_s}} - X_{\tau_{j_s}+\gamma_{\delta,H}}| \leq \frac{\eta\delta}{\log H} \right\},$$

where $\tau_{j_s} = \inf\{t \leq \delta^2 : X_t \in K \cap Q_{j_s}\}$ or $\tau_{j_s} = \infty$ if the set is empty. Then we set

$$p(x, y) = \mu_{x,y}(\Psi_0), \tag{20}$$

where $d\mu_{x,y}$ is the Brownian bridge measure, see the definition in Section 2. Then we define

$$\Psi_{0,r} = \left\{ X \in \Omega_x \colon \tau_{j_s} \leq \delta^2,\ |X_{\tau_{j_s}} - X_{\tau_{j_s}+\gamma_{\delta,H}}| \leq \frac{\eta\delta}{\log H}, |X_{2\delta^2} - y| \leq r \right\}.$$

We put

$$\Psi_1 = \left\{ X \in \Psi_0 \colon \int_{\tau_{j_s}}^{\tau_{j_s}+\gamma_{\delta,H}} V_\beta^A(X_u)du < \zeta E\left(\int_{\tau_{j_s}}^{\tau_{j_s}+\gamma_{\delta,H}} V_\beta^A(X_u)\right)du \right\}$$

where the constant $0 < \zeta < 1$ has been defined in Lemma 5.6 and similarly as above we set

$$\tilde{p}(x,y) = \mu_{x,y}(\Psi_1). \tag{21}$$

We define the set $\Psi_{1,r}$ replacing Ψ_0 by $\Psi_{0,r}$ in the definition of Ψ_1.

In what follows we will need the relation $\tau_{j_s} + \gamma_{\delta,H} \le \tau_{j_{s+1}}$. We have ensured this property by separating subsequent intervals I_s, I_{s+1} by J_s.

The following elementary observation is critical for our argument

Lemma 5.7. *Under the above definitions there exists a constant $\xi < 1$ such that*

$$\tilde{p}(x,y) \le \xi p(x,y).$$

uniformly for all indices x, y, A, j_s defined above and sufficiently large $H \ge H_{min}(\eta)$.

Proof. In the proof of the lemma, for notational convenience, we put $\tau = \tau_{j_s}$. By the definition of Ψ_0, the variable τ is finite for all paths in Ψ_0. Denote by $d\mu(\tau, w)$ the joint distribution of the variables τ, X_τ defined by (3) for $\Psi_0 = A$. Obviously $d\mu$ is supported on $S = [0, \delta^2] \times (K \cap Q_{j_s})$. Next for any subset $G \subset \mathbb{R}^d$ put

$$\nu_1(G) = P_0\left(X_{\gamma_{\delta,H}} \in G\right).$$

Note that at this point ν_1 is just the Gaussian distribution and recall that $\gamma_{\delta,H} = \delta^2(\log H)^{-3}$. Next, we consider the set K_r defined by (7) and we observe that by the strong Markov property (2) applied to $A = \Psi_{0,r}$, $\eta_1(\tau) = \tau + \gamma_{\delta,H}$, $\eta_2(\tau) = 2\delta^2 - \gamma_{\delta,H} - \tau$ and $B = \left\{|w - X_{\tau+\gamma_{\delta,H}}| \le \frac{\eta\delta}{\log H}, |X_{2\delta^2} - y| \le r\right\}$, it follows that

$$\begin{aligned}
p(x,y) &= \mu_{x,y}(\Psi_0) = \lim_{r\to 0} |K_r|^{-1} P_x(\Psi_{0,r}) = \\
&= \lim_{r\to 0} |K_r|^{-1} \int_S P_w\left(|w - X_{\tau+\gamma_{\delta,H}}| \le \frac{\eta\delta}{\log H}, |X_{2\delta^2} - y| \le r\right) d\mu(\tau, w) \\
&= \lim_{r\to 0} |K_r|^{-1} \int_S \int_{|z|\le \frac{\eta\delta}{\log H}} P_{w-z}\left(|X_{2\delta^2-(\tau+\gamma_{\delta,H})} - y| \le r\right) d\nu_1(z) d\mu(\tau, w) \\
&\ge \left(1 - \frac{C\eta^2}{\sqrt{\log H}}\right) \int_S \int_{|z|\le \frac{\eta\delta}{\log H}} p_{2\delta^2-(\tau+\gamma_{\delta,H})}(w-y) d\nu_1(z) d\mu(\tau, w) \\
&\ge \left(1 - \frac{C\eta^2}{\sqrt{\log H}}\right) \int_S \int_{|z|\le \frac{\eta\delta}{\log H}} p_{2\delta^2-(\tau+\gamma_{\delta,H})}(w-y) p_{\gamma_{\delta,H}}(z) dz d\mu(\tau, w) \\
&\ge \left(1 - \frac{C\eta^2}{\sqrt{\log H}}\right)^2 \int_S p_{2\delta^2-(\tau+\gamma_{\delta,H})}(w-y) d\mu(\tau, w).
\end{aligned}$$

In the above estimates we used the first and third of the elementary inequalities

$$\left(1 - \frac{C\eta^2}{\sqrt{\log H}}\right) p_{2\delta^2-(\tau+\gamma_{\delta,H})}(w-y) \le p_{2\delta^2-(\tau+\gamma_{\delta,H})}(w-y+z)$$

$$\left(1-\frac{C\eta^2}{\sqrt{\log H}}\right)p_{2\delta^2-(\tau+\gamma_{\delta,H})}(w-y+z)\le p_{2\delta^2-(\tau+\gamma_{\delta,H})}(w-y)$$

$$\int_{|z|\le\frac{\eta\delta}{\log H}}p_{\gamma_{\delta,H}}(z)dz\ge 1-\frac{C}{\log H}$$

valid for any w and τ from the domain of integration, and z,y,η as above.

Now set

$$\Phi_{V_\beta^A}=\left\{X:\ \int_0^{\gamma_{\delta,H}}V_\beta^A(X_u)du<\zeta E_x\left(\int_0^{\gamma_{\delta,H}}V_\beta^A(X_u)du\right)\right\}$$

and then put

$$\nu_2(G)=P_w\Big(X:\ X\in\Phi_{V_\beta^A},\ X_{\gamma_{\delta,H}}\in w+G\Big).$$

It follows from Lemma 5.6 that $\nu_2(\mathbb{R}^d)\le 1-\sigma_0$ so by (2)

$$\begin{aligned}\tilde p(x,y)&=\mu_{x,y}(\Psi_1)=\lim_{r\to0}|K_r|^{-1}P_x(\Psi_{1,r})=\\&=\lim_{r\to0}|K_r|^{-1}\int_S\int_{|z|\le\frac{\eta\delta}{\log H}}P_{w-z}\left(|X_{2\delta^2-(\tau+\gamma_{\delta,H})}-y|\le r\right)d\nu_2(z)d\mu(\tau,w)\\&\le\left(1+\frac{C\eta^2}{\sqrt{\log H}}\right)(1-\sigma_0)\int_S p_{2\delta^2-(\tau+\gamma_{\delta,H})}(w-y)d\mu(\tau,w).\end{aligned}$$

The lemma follows for $\xi=1-\frac{\sigma_0}{2}$ and sufficiently large $H\ge H_{min}(\eta)$. □

Recall that we denote the centre of the cube Q_{j_s} by q_{j_s}. Slightly abusing the notation we marginally change the meaning of τ_{j_s} be setting by $\tau_{j_s}=\inf\{t\in I_{k_s}:\ X_t\in K\cap Q_{j_s}\}$. Next, we put

$$\Lambda=\bigcup_{1\le H_0\le H}\ \bigcup_{j_1,\dots,j_{H_0},k_1,\dots,k_{H_0}}\Phi_{j_1,\dots,j_{H_0},k_1,\dots,k_{H_0}}$$

where we sum over the set of all indices $1\le H_0\le H$ and $j_1\dots,j_{H_0},k_1,\dots,k_{H_0}$ such that the system of inequalities

$$|X_{2k_s\delta^2}-q_{j_s}|\le\eta\delta\sqrt{\log H}\,,\qquad |X_{(2k_s+2)\delta^2}-q_{j_s}|\le\eta\delta\sqrt{\log H}$$

and

$$|X_{\tau_{j_s}}-X_{\tau_{j_s}+\gamma_{\delta,H}}|\le\frac{\eta\delta}{\log H}$$

is not satisfied for at least one of $1\le s\le H_0$. Then obviously

$$\begin{aligned}\Lambda\subset\tilde\Lambda&=\left\{X:\ \max_{h\in\{0,1,\dots,H\}}|X_{2h\delta^2}-X_{(2h+2)\delta^2}|>2\eta\delta\sqrt{\log H}\right\}\\&\cup\left\{X:\ \sup_{t_1,t_2\in[0,1],|t_1-t_2|\le\gamma_{\delta,H}}|X_{t_1}-X_{t_2}|>\frac{\eta\delta}{\log H}\right\}.\end{aligned}\tag{22}$$

The set $\tilde\Lambda$ is of small probability. In fact we have

Lemma 5.8. *For any $\rho > 0$ there exists a constant η such that the set $\tilde{\Lambda}$ defined above satisfies the estimate*

$$P(\tilde{\Lambda}) \leq H^{-\rho}$$

for all sufficiently large H.

Proof. This lemma follows exactly in the same way as the proof of Hölder regularity of the Brownian motion. Directly one can easily check that

$$P_x\left(|X_{2h\delta^2} - X_{(2h+2)\delta^2}| > \eta\delta\sqrt{\log H}/2\right) \leq \exp(-c\eta \log H)$$

and then sum-up the estimates. The estimates for the probability of the second set in (22) follows from the following two observations:

- Firstly
$$\left\{X\colon \sup_{t_1,t_2\in[0,1],|t_1-t_2|\leq\gamma_{\delta,H}} |X_{t_1} - X_{t_2}| > \frac{\eta\delta}{\log H}\right\}$$
$$\subset \bigcup_{0\leq j\leq\gamma_{\delta,H}^{-1}} \left\{X : \sup_{j\gamma_{\delta,H}\leq t_1<(j+1)\gamma_{\delta,H}} |X_{j\gamma_{\delta,H}} - X_{t_1}| > \frac{\eta\delta}{2\log H}\right\}.$$
- Secondly, each event of the RHS of above inclusion has the probability estimated from above. Indeed, if $\{Y_t\}_{t\geq 0}$ is a one dimensional Brownian motion starting at 0 then by the reflection principle, see for example [13, p. 26, formula 4)]
$$P_x\left(\left\{Y : \sup_{s<t} Y_s > a\right\}\right) = 2P_x\left(\{Y_t > a\}\right)$$
for all $t, a > 0$. By Markov property it follows that
$$P_x\left(\left\{X : \sup_{j\gamma_{\delta,H}\leq t_1<(j+1)\gamma_{\delta,H}} |X_{j\gamma_{\delta,H}} - X_{t_1}| > \frac{\eta\delta}{2\log H}\right\}\right)$$
$$\leq 2P_x\left(\left\{X : |X_{j\gamma_{\delta,H}} - X_{(j+1)\gamma_{\delta,H}}| > \frac{\eta\delta}{2\sqrt{n}\log H}\right\}\right).$$

Then again we sum-up the estimates and the lemma follows. □

In what follow we will also need the following standard fact about NTA domains

Lemma 5.9. A) *Assume $D \subset \mathbb{R}^d$ is an NTA domain, and let B_0, be a closed ball contained in the interior of D and separated from $K = \partial D$. Then, there exists $\gamma > 0$, such that the harmonic function on $D \setminus B_0$ vanishing on K and equal to 1 on ∂B_0 satisfies*

$$h(x) \leq C(d_K(x))^\gamma. \tag{23}$$

The same statement is valid for D^c.

B) *For dimension $d = 2$ estimate (23) holds both for the domain D and its complement D^c with exponent $\gamma = 1/2$.*

Proof. We briefly sketch the proof. We start with Part A. Since D satisfies NTA condition, the Dirichlet problem for D is solvable and there exists a function, harmonic in $D \setminus B_0$ such that $h(x) = 0$ for $x \in K$ and $h(y) = 1$ for $y \in \partial B_0$. Define $K_j = \{x \in D : d_K(x) = a^j\}$ for sufficiently small fixed $a < 1$. Fix $x \in K_j$. Let $B_1 \subset D^c$ be a ball with center y_0 and radius r such that $|x - y_0| \leq 2d_K(x)$ and, for some c depending only on the domain D

$$2a^j = 2d_K(x) \geq r \geq cd_K(x) = ca^j.$$

Such a ball exists by the NTA conditions for D. Now observe that $P_x(X_{r^2} \in B_1) \geq p_0$ and $P_x(\text{there exists } t \leq r^2, \text{ such that } X_t \in K_{j-1}) \leq C_1 \exp(-\frac{1}{C_1 a})$ where p_0, C_1 depends only on the NTA constants of the domain, not on a. Consequently, for some small enough $a > 0$, we have

$$\begin{aligned} &P_x(\text{diffusion starting from } K_j \text{ hits } K, \text{ not hitting } K_{j-1} \text{ before }) \\ &\quad \geq P_x(X_{r^2} \in B_1) - P_x(\text{there exists } t \leq r^2, \text{ such that } X_t \in K_{j-1}) \geq \frac{p_0}{2} \end{aligned}$$

for all $j \in \mathbb{N}$. Hence, any harmonic function bounded by 1 on K_{j-1} and vanishing on K must be bounded by $1 - \frac{p_0}{2}$ on K_j. By a simple induction argument, we obtain $h(x) \leq (1 - \frac{p_0}{2})^{j-j_0}$ for $x \in K_j$, where j_0 is the minimal index such that K_{j_0} does not intersect B_0. Now Part A of Lemma 5.9 follows by the maximum principle.

Part B of the theorem is a consequence of Beurling projection theorem, see [1, Theorem 3-6, page 43]. A probabilistic proof of Beurling projection theorem is described in [16].

□

Remark Note that in the whole paper we use the NTA conditions only to obtain estimate (23).

From now on we fix large ρ and the corresponding $\eta = \eta_0$ given by Lemma 5.8. We assume $H \geq H_{min}(\eta_0)$, where $H_{min}(\eta_0)$ is chosen in the same way as in Lemma 5.7.

In the next lemma we estimate the probability of the set of paths for which the number H_0 is small.

Lemma 5.10. *There exist constant C such that for any $\kappa'' > \kappa' > 0$ the following estimate holds*

$$P_x\left(X : \#\{h : X \text{ hits the boundary for } t \in I_h\} < H^{\kappa'}\right) < CH^{\kappa''}\delta^{\theta}.$$

where $\theta = 2\gamma$ and γ is the exponent from Lemma 5.9.

Proof. Let $\Phi_{j_1,\dots,j_{H_0},k_1,\dots,k_{H_0},0} = \Phi_{j_1,\dots,j_{H_0},k_1,\dots,k_{H_0}} \setminus \Lambda$. Note that

$$\Phi_{j_1,\dots,j_{H_0},k_1,\dots,k_{H_0},0} \subset \widetilde{\Phi}_{j_1,\dots,j_{H_0},k_1,\dots,k_{H_0}}$$

where

$$\begin{aligned} &\widetilde{\Phi}_{j_1,\dots,j_{H_0},k_1,\dots,k_{H_0}} = \\ &\quad = \left\{X :\ X_t = \tilde{X}_t \text{ for all } t \leq (2k_{H_0} + 2)\delta^2 \text{ and some } \tilde{X} \in \Phi_{j_1,\dots,j_{H_0},k_1,\dots,k_{H_0},0}\right\}. \end{aligned}$$

Next, observe that for a fixed $H_0 \geq 1$, the sets $\widetilde{\Phi}_{j_1,\dots,j_{H_0},k_1,\dots,k_{H_0}}$ are mutually disjoint. Let $\theta = 2\gamma$, (γ defined by (23)). We will prove the estimate

$$P_x\left(\Phi_{j_1,\dots,j_{H_0},k_1,\dots,k_{H_0},0}\right) \leq C_\theta \delta^\theta P_x\left(\widetilde{\Phi}_{j_1,\dots,j_{H_0},k_1,\dots,k_{H_0}}\right).$$

Next, we define the measure ν_3 by the formula

$$\nu_3(G) = P_x\left(X \in \widetilde{\Phi}_{j_1,\dots,j_{H_0},k_1,\dots,k_{H_0}} : X_{(2k_{H_0}+2)\delta^2} \in G\right).$$

By the Markov property of Brownian motion we have

$$\begin{aligned} & P_x\left(\Phi_{j_1,\dots,j_{H_0},k_1,\dots,k_{H_0},0}\right) \\ = & \int P_x\left(X_{1-(2k_{H_0}+2)\delta^2} \in B_0 \text{ and } X_s \cap K = \emptyset \text{ for } 0 \leq s \leq 1-(2k_{H_0}+2)\delta^2\right) d\nu_3(x) \end{aligned}$$

where by the definition of $\Phi_{j_1,\dots,j_{H_0},k_1,\dots,k_{H_0}}$, for $x \in \operatorname{supp}\{d\nu_3\}$ we have

$$|x - q_{k_{H_0}}| \leq \delta^2 \sqrt{\log H}.$$

By Lemma 5.9

$$\begin{aligned} P_x\left(X_{1-(2k_{H_0}+2)\delta^2} \in B_0 \text{ and } X_s \cap K = \emptyset \text{ for } 0 \leq s \leq 1-(2k_{H_0}+2)\delta^2\right) \\ \leq P_x\Big(X : X \text{ hits first time into } B_0, \text{ not into } K\Big) \\ \leq C\left(\delta^2\sqrt{\log H}\right)^\gamma \end{aligned}$$

and consequently, since $\nu_3(\mathbb{R}^d) = P_x\left(\widetilde{\Phi}_{j_1,\dots,j_{H_0},k_1,\dots,k_{H_0}}\right)$ it follows that

$$P_x\left(\Phi_{j_1,\dots,j_{H_0},k_1,\dots,k_{H_0},0}\right) \leq C\left(\delta^2\sqrt{\log H}\right)^\gamma P_x\left(\widetilde{\Phi}_{j_1,\dots,j_{H_0},k_1,\dots,k_{H_0}}\right).$$

Hence, since $\widetilde{\Phi}_{j_1,\dots,j_{H_0},k_1,\dots,k_{H_0}}$ are mutually disjoint, we obtain

$$\sum_{1 \leq H_0 \leq H^{\kappa'}} \sum_{j_1,\dots,j_{H_0},k_1,\dots,k_{H_0}} P_x\left(\Phi_{j_1,\dots,j_{H_0},k_1,\dots,k_{H_0},0}\right) \leq C H^{\kappa''}\delta^\theta.$$

Now the lemma follows from the above estimates and Lemma 5.8.

□

Now we are ready to prove Theorem 5.2.

Proof of Theorem 5.2. Our aim is to apply Feynman-Kac formula (5). Observe that without loss of generality, in (5) we can consider the expectation over the set of trajectories which hit K at a time $\tau \in I_h$ for some h. Indeed, if the path does not cross K at any $t \in I_h$ for any h, then it must cross K at a time $\tau \in J_h$ for some h, and we repeat the argument replacing I_h by J_h.

Now consider the set

$$\Phi_{small} = \bigcup_{H_0 \leq H^{\kappa'}} \bigcup_{j_1,\dots,j_{H_0},k_1,\dots,k_{H_0}} \Phi_{j_1,\dots,j_{H_0},k_1,\dots,k_{H_0},0}$$

Recall, that we set $\delta \approx A^{-v}$, $H \approx \delta^{-2}$ so by Lemma 5.10

$$\int_{\Phi_{small}} e_{V_\beta^A}(t)\chi_{B_0}(X_t)P_x(dX) \le P_x(\Phi_{small}) \le CH^{\kappa''}\delta^\theta \le CA^{-2v(\gamma-\kappa'')}. \tag{24}$$

Next, consider the following sets

$$\Phi^{bad}_{j_1,\dots,j_{H_0},k_1,\dots,k_{H_0},0} = \Phi_{j_1,\dots,j_{H_0},k_1,\dots,k_{H_0},0}$$
$$\bigcap \left\{ \int_{\tau_{j_s}}^{\tau_{j_s}+\gamma_{\delta,H}} V_\beta^A(X_s)ds \le \zeta E\left(\int_{\tau_{j_s}}^{\tau_{j_s}+\gamma_{\delta,H}} V_\beta^A(X_s)\right) ds, \text{ on every } I_{k_s} \right\};$$

$$\Phi_{essential} = \bigcup_{H_0 > H^{\kappa'}} \bigcup_{j_1,\dots,j_{H_0},k_1,\dots,k_{H_0}} \Phi_{j_1,\dots,j_{H_0},k_1,\dots,k_{H_0},0};$$

$$\Phi_{bad} = \bigcup_{H_0 > H^{\kappa'}} \bigcup_{j_1,\dots,j_{H_0},k_1,\dots,k_{H_0}} \Phi^{bad}_{j_1,\dots,j_{H_0},k_1,\dots,k_{H_0},0};$$

$$\Phi_{good} = \Phi_{essential} \setminus \Phi_{bad}.$$

Note that $\gamma_{\delta,H} = \delta^2(\log H)^{-3} \approx A^{-2v}(\log H)^{-3}$. Let $2vd < \beta+\alpha-d$. If $X \in \Phi_{good}$ then, for at least one j_s we have

$$\int_{\tau_{j_s}}^{\tau_{j_s}+\gamma_{\delta,H}} V_\beta^A(X_s)ds \ge \zeta E_x\left(\int_{\tau_{j_s}}^{\tau_{j_s}+\gamma_{\delta,H}} V_\beta^A(X_s)ds\right) \ge CA^{\beta+\alpha-d}\frac{\delta^{2d}}{(\log H)^{3d}}$$
$$\ge C\frac{A^{\beta+\alpha-d-2dv}}{(\log H)^{3d}} \ge CA^\iota$$

for any $\iota < \beta+\alpha-d-2dv$ and for some constant $C>0$ depending only on the domain and γ, but not on A. Hence, for any fixed $\rho>0$ and sufficiently large A we have

$$\int_{\Phi_{good}} e_{V_\beta^A}(t)\chi_{B_0}(X_t)P_x(dX) \le \exp(-CA^\iota) \le C'A^{-\rho}. \tag{25}$$

Next we shall prove that for any $\rho>0$ we have $P(\Phi_{bad}) \le CA^{-\rho}$. To that end we will show that

$$P_x\left(\Phi^{bad}_{j_1,\dots,j_{H_0},k_1,\dots,k_{H_0}}\right) \le \xi^{H_0}P_x\Big(\Phi_{j_1,\dots,j_{H_0},k_1,\dots,k_{H_0}}\Big), \tag{26}$$

where ξ is the constant from Lemma 5.7.

We fix some notation. For $\epsilon \in \{0,1\}$ and for $s \in 1,\dots,H_0$ we denote $p^\epsilon_{k_s}(x,y)$ simply setting $p^0_{k_s}(x,y) = p(x,y)$ and $p^1_{k_s}(x,y) = \tilde p(x,y)$ where p and $\tilde p$ are defined by (20) and (21). Secondly, for $h \in \{1,\dots,H\}\setminus\{k_1,\dots,k_{H_0}\}$ we denote $p^0_h(x,y)$ as the Brownian bridge measure $d\mu_{x,y}$ of

$$p^0_h(x,y) = \mu_{x,y}\Big(X \in \Omega^{2\delta^2}_{x,y} \colon\ X_t \notin K \quad \forall t \in [0,\delta^2]\Big)$$

We put $\epsilon_h = 1$ if $h \in \{k_1,\dots,k_{H_0}\}$ $\epsilon_h = 0$ for $h \in \{1,\dots,H\}\setminus\{k_1,\dots,k_{H_0}\}$. Now, by Lemma 5.7 $p^1_h(x,y) \le \xi p^0_k(x,y)$ for $h \in \{k_1,\dots,k_{H_0}\}$ so by the Markov property:

$$P_x\Big(\Phi^{bad}_{j_1,\dots,j_{H_0},k_1,\dots,k_{H_0}}\Big) =$$
$$\int_{B_0}\int p_0^{\epsilon_0}(x,x_1)p_1^{\epsilon_1}(x_1,x_2)\dots p_{H-1}^{\epsilon_{H-1}}(x_{H-1},y)dx_1\dots dx_{H-1}dy \le$$
$$\le \xi^{H_0}\int_{B_0}\int p_0^{0}(x,x_1)p_1^{0}(x_1,x_2)\dots p_{H-1}^{0}(x_{H-1},y)dx_1\dots dx_{H-1}dy$$
$$= \xi^{H_0}P_x\Big(\Phi_{j_1,\dots,j_{H_0},k_1,\dots,k_{H_0}}\Big).$$

The above inequality proves (26). Now, since the sets $\Phi_{j_1,\dots,j_{H_0},k_1,\dots,k_{H_0}}$ are disjoint, for $\delta \approx A^{-v}$ and any $C > 0$ we have

$$P_x\Big(\Phi_{bad}\Big) \le \sum_{H_0\ge H^{\kappa'}}\ \sum_{j_1,\dots,j_{H_0},k_1,\dots,k_{H_0}} P_x\Big(\Phi^{bad}_{j_1,\dots,j_{H_0},k_1,\dots,k_{H_0}}\Big) \le \tag{27}$$
$$\le C\xi^{H^{\kappa'}}\sum_{H_0\ge H^{\kappa'}}\ \sum_{j_1,\dots,j_{H_0},k_1,\dots,k_{H_0}} P_x\Big(\Phi_{j_1,\dots,j_{H_0},k_1,\dots,k_{H_0}}\Big) \le C_{v,C}A^{-\rho}$$

for sufficiently large H. Combining the estimates (24), (25) and (27) we get that if $0 < \beta+\alpha-d-2dv$ then for any $\rho < \gamma\frac{\beta+\alpha-d}{d}$ and κ'', κ' small enough

$$\int_{\Phi_{small}\cup\Phi_{good}\cup\Phi_{bad}} e_{V^A_\beta}(t)\chi_{B_0}(X_t)P_x(dX)$$
$$\le CA^{-2v(\gamma-\kappa'')} + CA^{-\rho} + P_x\Big(\Phi_{bad}\Big) \le CA^{-v\gamma}.$$

This ends the proof.

□

Acknowledgements: Adam Sikora was partially supported by an Australian Research Council (ARC) Discovery Grants DP130101302 and DP160100941. Jacek Zienkiewicz was partially supported by NCN grant UMO-2014/15/B/ST1/00060. We would like to thank the anonymous referee for carefully reading our manuscript and making several valuable comments and suggestions.

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