Optimal strategies for dealing with store discounts

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Introduction

While recently shopping in Adelaide for a pair of shoes, I came across a store that advertised 'second pair half price' with the fine print stating that the 'second pair' was the pair of lesser value. Intrigued by such a fine deal, I enquired about the situation if I were to purchase more than two pairs. The reply was that if I purchased three pairs then the cheapest pair would be half price, four pairs would mean the cheapest two pairs would be half price and so on. In mathematical terms, the deal for a customer purchasing n pairs of shoes was:

Total price = full price of the most expensive $n/2$ pairs plus half the price of the least expensive $n/2$ pairs if n is even
= full price of the most expensive $(n + 1)/2$ pairs plus half the price of the least expensive $(n - 1)/2$ pairs if n is odd.

I also discovered that if more than two pairs were purchased then they could be broken up into multiple transactions of any combination. This naturally led to my wondering what would be the optimal way to mix these transactions for multiple purchases so as to minimise the amount paid. (The kindly assistant informed me that in her view it did not make any difference how the purchases were broken up since it would still cost the same!)

In the following calculations, in the case of purchasing n pairs of shoes, it is assumed that the dearest pair cost x_1, the next dearest pair x_2 and so on. That is:

$$x_1 \geq x_2 \geq x_3 \geq x_4 \geq \ldots\ x_{n-1} \geq x_n$$

Before determining a general strategy, it is instructive to consider the first few cases to determine if a pattern appears. In all cases the option of making n single purchases for n pairs of shoes can be ignored, as it will never be superior since no discount at all will apply

Two pairs (n = 2)

This is the simplest case. The two pairs are purchased in a single transaction with a total price of $x_1 + 0.5x_2$.

Three pairs (n = 3)

According to the store rules, making a single transaction would cost $x_1 + x_2 + 0.5x_3$ since the cheapest pair is half price. There are three ways in which the purchases can be broken into two transactions. These, along with the corresponding total cost, are shown in Table 1.

Table 1. The total cost of various transactions for three pairs of shoes.

Combination	Transaction 1	Transaction 2	Total cost
1	x_1, x_2	x_3	x_1 + X$_3$ + $0.5x_2$
2	x_1, x_3	x_2	$x_1 + x_2 + 0.5x_3$
3	x_2, x_3	x_1	$x_1 + x_2 + 0.5x_3$

From Table 1 it can be seen that combinations 2 and 3 cost the same as purchasing all three pairs in one transaction. Since $x_2 \geq x_3$, it can be easily shown that the cheapest option is transaction 1 with a saving of $0.50(x_2 - x_3)$. In practical terms, this means combining the two most expensive pairs in one transaction and the least expensive in a separate transaction.

Four pairs (n = 4)

According to store rules, purchasing all four pairs in a single transaction would cost $x_1 + x_2 + 0.5x_3 + 0.5x_4$. This time there are seven ways of splitting the transactions as shown in Table 2.

Table 2. The total cost of various transactions for four pairs of shoes.

Combination	Transaction 1	Transaction 2	Total cost
1	x_1, x_2, x_3	x_4	$x_1 + x_2 + x_4 + 0.5x_3$
2	x_1, x_2, x_4	x_3	$x_1 + x_2 + x_3 + 0.5x_4$
3	x_1, x_3, x_4	x_2	$x_1 + x_2 + x_3 + 0.5x_4$
4	x_2, x_3, x_4	x_1	$x_1 + x_2 + x_3 + 0.5x_4$
5	x_1, x_2	x_3, x_4	$x_1 + x_3 + 0.5x_2 + 0.5x_4$
6	x_1, x_3	x_2, x_4	$x_1 + x_2 + 0.5x_3 + 0.5x_4$
7	x_1, x_3	x_1, x_4	$x_1 + x_2 + 0.5x_3 + 0.5x_4$

Since $x_4 \le x_3$, combination 1 is cheaper than either of combinations 2, 3 or 4 (which all cost the same); but combinations 6 and 7 (which both cost the same and is the same cost of purchasing them all in a single transaction) are clearly both cheaper than combination 1. Since $x_3 \le x_2$, the cheapest mixture of all is combination 5. This means that the greatest savings can be made by having one transaction of the two most expensive pairs and the other transaction of the two cheapest pairs. It can easily be shown that this represents a saving of $0.50(x_2 - x_3)$ over simply purchasing all four pairs in one transaction. Note that this expression is the same as for three pairs.

n pairs

By now a pattern is beginning to emerge as to the optimal strategy and its proof is left as an exercise. In words, however, it can be expressed as below:

If n is even, make $n/2$ transactions of the type
$(x_1, x_2), (x_3, x_4), (x_5, x_6), (x_7, x_8), \ldots (x_{n-1}, x_n)$
The total costs are:

$$\text{Cost of optimal strategy} = \sum_{i=1}^{\frac{n}{2}} x_{2i-1} + 0.5\sum_{i=1}^{\frac{n}{2}} x_{2i} \qquad (1)$$

If the n pairs are bought in a single transaction, then the total cost will be:

$$\text{Cost of a single transaction} = \sum_{i=1}^{\frac{n}{2}} x_i + 0.5 \sum_{i=\frac{n}{2}+1}^{n} x_i \qquad (2)$$

Example

A customer decides to purchase six pairs of shoes with ticket prices \$40, \$28, \$70, \$14, \$85 and \$56. Find:

(a) The cost of purchasing these on a single transaction
(b) The cost of purchasing these with an optimal split of transactions
(c) The percentage savings made by using (b) over (a)

Solution

Ordering the prices (in \$) yields:

$x_1 = 85,\ x_2 = 70,\ x_3 = 56,\ x_4 = 40,\ x_5 = 28,\ x_6 = 14$

(a) From (2), the cost (in \$) of making a single transaction
$= 85 + 70 + 56 + 20 + 14 + 7$
$= 252$

(b) From (1), the cost (in $) of making three transactions
= 85 + 35 + 56 + 20 + 28 + 7
= 231

(c) From (a) and (b) the actual savings
= $252 – $231
= $21.
The percentage savings is therefore 100 × ($21/$252) = 8.3%

If n is odd, the optimal strategy is to make $(n+1)/2$ transactions of the type (x_1, x_2), (x_3, x_4), (x_5, x_6)… (x_{n-2}, x_{n-1}), x_n. That is, it is basically the same pattern as when n is even, except that the cheapest pair is made as a separate transaction.

The total costs are shown in (3 and (4).

$$\text{Cost of optimal strategy} = \sum_{i=1}^{\frac{(n+1)}{2}} x_{2i-1} + 0.5 \sum_{i=1}^{\frac{(n-1)}{2}} x_{2i} \qquad (3)$$

If the n pairs are bought in a single transaction, then the total cost will be:

$$\text{Cost of a single transaction} = \sum_{i=1}^{\frac{(n+1)}{2}} x_i + 0.5 \sum_{i=\frac{(n+3)}{2}}^{n} x_i \qquad (4)$$

Remarks

Despite the assurances of store employees, whenever a deal is on offer it always pays to apply a little logic (and mathematics) to see just how to use it most effectively. The use of information to gain an advantage has been demonstrated in a number of similar problems, such as the three box problem in which a game show host gives seemingly useless information to try to dissuade a contestant from selecting a particular box that may or may not contain a valuable prize. Croucher and Byun (1998) shows just how this information can be used to increase the probability of winning. Once again, the mathematics involved was not too difficult but absolutely essential.

Reference

Croucher, J. & Byun, K. (1998). The game show dilemma. *Economics, 34* (2), 7–9

A ham sandwich is better than nothing: Some thoughts about transitivity

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Introduction

There is an old joke that says that given the choice between eternal happiness and a ham sandwich, one should choose the ham sandwich. The proof is quite simple: (1) nothing is better than eternal happiness (2) a ham sandwich is better than nothing; and therefore, it straightforwardly follows from (1) and (2) that (3) a ham sandwich is better than eternal happiness. So, given the choice, one should choose a ham sandwich. QED.

This article takes a slightly more serious view of transitive relationships, and their interpretation in the real world.

Transitivity and intransitivity in mathematics

In most mathematical schema, relationships tend to be transitive. That is, if for a particular relationship R, if we have (a R b), and (b R c), then we can deduce (a R c). This holds true for many mathematical relationships R; for example, if $a > b$, and $b > c$, then $a > c$.

Exercise 1

Let students try to find other examples of common mathematical relationships that are transitive. Possible solutions are given at the end of this article.

Mathematical relations are often (but not invariably) transitive because such relationships can be thought of as linear — they can be mapped to the real numbers, for example. Consider the proposition that if $a > b$ and $b > c$, then $a >$ c. If one thinks of an x-axis, and $a > b$, we think of a as 'further along' the x-axis than b; and if $b > c$, we think of b as 'further along' the x-axis than c. Thus a is clearly 'further along' the x-axis than c. More formally, $a > b \Rightarrow a = b + N_1$, and $b > c \Rightarrow b = c + N_2$, where N_1 and N_2 are positive real numbers. Therefore it follows directly that $a = c + N_1 + N_2$, so $a > c$. The three numbers a, b, and c can be thought of as points on a straight line, with N_1 and

N_2 representing the distances between them.

However, some mathematical relationships are not transitive.

Exercise 2

Let students try to find an example of a common mathematical relationship that is not transitive.

What about the real world?

Take Peter, Paul, and Mary: if Peter likes Paul, and Paul likes Mary, then we cannot deduce that Peter likes Mary. The relationship 'likes' cannot be mapped to the real numbers. This is fairly obvious stuff; few of us would expect the 'likes' relationship to be necessarily transitive. However, some relationships we would intuitively expect to be transitive.

For example, if Richard is taller than Sam, and Sam is taller than Terry, then we can logically deduce that Richard is taller than Terry. It cannot be any other way.

Exercise 3

Let students try to find examples of common real-world relationships: which are transitive, and which are not? Which are the more common?

Games

If A beats B, and B beats C, we would expect A to beat C. However, any game utilising such a rule would be boring indeed; intransitivity is needed to make the game interesting!

One of the simplest examples of intransitivity in a game is the simple children's game of rock-paper-scissors, where rock beats scissors, and scissors beats paper, but rock does not beat paper — on the contrary, paper beats rock.

There are many other games that make use of intransitive relationships for their intrinsic interest. For example, in some war games, a cruiser may 'beat' a destroyer; an aircraft carrier may 'beat' a cruiser; and, as expected, an aircraft carrier may 'beat' a destroyer; but a submarine may 'beat' an aircraft carrier, and yet itself be beaten by a destroyer. So in this case, to beat, or be better than, is intransitive. In one particular game marketed in the 1960s, the ships were matched to integers; so that a destroyer was a 2, a cruiser was a 7, and an aircraft carrier was a 10, for example; but the submarine lay outside this integer hierarchy altogether. It beat a 10 (and most other numbers), but could itself be beaten by a 2. The submarine was somehow outside of the real-number line.

Exercise 4

Let students try to find examples of games which make use of intransitivity.

Dice

Ainley (1978) has described a set of four dice that illustrate that the concept 'outscores' (similar to 'beats') may also be intransitive. The dice have sides marked as follows:

Die A	7, 7, 7, 7, 1, 1
Die B	6, 6, 5, 5, 4, 4
Die C	9, 9, 3, 3, 3, 3
Die D	8, 8, 8, 2, 2, 2

It is quite easy to see that the probability of each die outscoring its neighbors is as follows:

Pr(A outscores B) = 2/3
Pr(B outscores C) = 2/3
Pr(C outscores D) = 2/3
Pr(D outscores A) = 2/3

Thus, no one die is 'the best'. Rather, given that player 1 has selected a die, player 2 can always select a die that is more likely to win — which is of 'higher quality'.

Exercise 5
In fact, the same point can be made with only three dice. Let students try to find examples of three dice in which die A outscores die B 2/3 of the time, die B outscores die C 2/3 of the time, and die C outscores die A 2/3 of the time.

Elections

Although the design of such a series of dice is intriguing, perhaps of more interest are the relationships that lead to real problems concerning democratic elections, thought to have been first pointed out several centuries ago by the French mathematician Condorcet; see, for example, Paielli and Ossipoff (2000).

Suppose in a particular election there are three candidates, Alex, Bernie and Chris, and 10 000 constituents who have exercised their right to vote in the election. Suppose further that 3600 voters put Alex first, Bernie second, and Chris third; 3500 voters put Bernie first, Chris second, and Alex third; and 2900 put Chris first, Alex second, and Bernie third.

Alex has the most first preferences, 3600, so he should win the election; but hold on: 64% of voters prefer Chris to Alex! So should Chris be elected instead? Well, maybe — but wait — a massive 71% of voters prefer Bernie to Chris! So should Bernie be elected? No, because 65% of electors prefer Alex to Bernie!

So elector preference is intransitive. Alex is preferred to Bernie, and Bernie is preferred to Chris, but Alex is not preferred to Chris! Indeed, in this particular situation, regulations pertaining in different countries and constituencies would result in different people being elected; in at least some of these Alex would be the victor, and in others Bernie.

One other matter is worthy of note here: that is, that although it has been demonstrated above that voter preference can be (and often is) intransitive, it is still intuitively assumed that the preference of any single voter cannot be.

Final remarks

Research on intransitivity of preferences dates back at least as far as Tversky's 1969 paper in *Psychological Review*, and more recently, Dawes (1998) provided an excellent review in the *Handbook of Social Psychology*. There has been the occasional excursion into economic theory, though most examples tend to focus on theoretical specific issues, such as Humphrey (1999), who centres on regret theory, and Bergstrom (1992), on competitive equilibrium. Unfortunately, attempts to relate the research to a more widespread audience have been few.

Transitivity and intransitivity are fascinating concepts that relate both to mathematics and to the real world we live in. A couple of lessons devoted to this topic are almost certain to interest and engage students of almost any age, as they seek to discover which relationships are transitive, and which are not, and further to try to discover any general rules that might distinguish between the two.

Oh, and I do not know about you, but I would take eternal happiness over a ham sandwich any day.

Solutions to exercises

Exercise 1

Possible examples of transitive relations would include (but not be limited to) equal to (if $a = b$, and $b = c$, then $a = c$), less than (if $a < b$, and $b < c$, then $a < c$), less than or equal to, greater than, greater than or equal to, and many others relationships such as subsets (if $a \subseteq b$, and $b \subseteq c$, then $a \subseteq c$), supersets, and integer division (if a divides b, and b divides c, then a divides c).

Exercise 2

The most common example of a non-transitive relation would be ‘not equal to’. If a is not equal to b, and b is not equal to c, it does not follow that a is not equal to c. Less common would be the concept of ‘relatively prime’. If a is relatively prime to b (that is, a and b do not contain any common factors), and b is relatively prime to c, then it does not follow that a is relatively prime to c.

Exercise 3
'Loves', 'likes' and 'hates' are all intransitive. 'Is a brother to' and 'is a sister to' are both transitive, but 'is a parent of' and 'is a child of' are not. 'Lives on the same street as' is, but 'lives around the corner from' and 'lives within a mile of' are not.

Exercise 4
Probably many games, in one form or another.

Exercise 5
Each die beating the next 2/3 of the time is not possible!
One example (of many) that comes very close would be:

die A: 6 6 6 6 1 1; die B: 5 5 4 4 3 3; die C: 7 7 2 2 2 2

but the neatest solution is probably

die A: 8 5 5 3 3 3; die B 7 7 7 2 2 2; die C: 6 6 6 4 4 1

where each die beats the next 7/12 of the time.

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