ENCOURAGING MATHEMATICAL THINKING

# Through Pattern & Structure

## AN INTERVENTION IN THE FIRST YEAR OF SCHOOLING

**JOANNE MULLIGAN**

**MIKE MITCHELMORE,**

**CORAL KEMP,**

**JENNIE MARSTON** and

**KATE HIGHFIELD**

report on a teaching intervention aimed at developing young children's understanding of pattern and structure.

In two earlier articles in this journal, we imagery and structural awareness when they solved triangular number and area tasks (Mulligan, Prescott, & Mitchelmore, 2003; Mulligan, Prescott, Mitchelmore, & Outhred, 2005). In this article, we illustrate Kindergarten students' development of mathematical pattern and structure across number, space and measurement from a recent teaching intervention.

### Pattern and structure in mathematics

Virtually all mathematics is based on pattern and structure. A mathematical *pattern* is any predictable regularity, usually involving numbers or space. In every pattern, the various elements are organised in some regular fashion. The way a pattern is organised is called its *structure,* which may be numerical or spatial. We give an example in Figure 1 from the early development of multiplication (see Mulligan et al., 2005, for a fuller discussion).

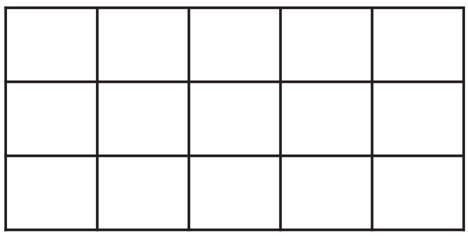

Figure 1. Rectangular grid pattern of 3 x 5 square units.

Figure 1 shows three rows of five (or five columns of three) equal-sized squares with their sides aligned vertically and horizontally. Here we see a "unit of repeat" (individual rows or columns) and ideas of spatial structure (congruence, parallels and perpendiculars). Understanding such grid patterns can connect many mathematical ideas together. For example, counting the squares can lead to "skip counting" (3, 6, 9, 12, 15) and hence to multiplication. The calculation of volume is based on the same pattern. Any multiplication can be represented by a grid pattern, and grid patterns can also be used to illuminate fraction and division situations. Grids are also fundamental to data exploration, geometric concepts and to some applications of technology in mathematics learning.

Children can make many generalisations using the pattern shown in Figure 1. For example, the fact that it can be seen as 3 rows of 5 or 5 columns of 3 shows that $3 \times 5 = 5 \times 3$. Finding such properties is regarded as *pre-algebraic thinking*, because it involves "seeing" a common structure instead of fixed numbers. By exploring the sequence of multiples of 3 (3, 6, 9, 12, 15…), children can see that each number in the sequence is simply three times its position in the sequence (e.g., the 4th number is $3 \times 4$). This process is known as *functional thinking*, and it is part of pre-algebraic thinking (Warren & Cooper, 2006).

## Patterns and algebra in the mathematics curriculum

There has been an increasing attention on connections between pattern and structure and algebraic thinking in mathematics curricula throughout Australia and internationally (Clements & Sarama, 2007). This is an important trend because early algebraic understanding impacts on other mathematics learning. For example, both New South Wales and Queensland now include a Patterns and Algebra strand from the first year of schooling. Working Mathematically, now included in every state syllabus, also reflects an interest in reasoning about mathematical structure. However, pattern and structure are not central to any mathematics syllabus. Current curricula, organised into parallel strands (usually number, space, measurement, data, and patterns and algebra), do not explicitly encourage teachers to see the common processes of pattern and structure, or to make important connections between strands.

## The Pattern and Structure Mathematics Awareness Project (PASMAP)

Over the past five years, we have explored the use of pattern and structure in early mathematical development in a series of studies with 4 to 7 year olds. This classroom research has developed an interview-based assessment of early numeracy called the *Pattern and Structure Assessment (PASA)* and a related *Pattern and Structure Mathematics Awareness Program (PASMAP)*.

We have found that students' identification and use of pattern and structural features reveal common characteristics in their mathematical understanding. In a study of Year 1 and 2 students we observed that low-achievers had a very poor grasp of mathematical pattern and structure and made no progress over the two years. High achievers progressed to an advanced stage of structural development that was reflected in their recordings and explanations. The classroom research that followed in K–2 classrooms has shown that young students can be taught to recognise mathematical pattern and structure. We found that their overall mathematics learning was substantially improved.

In a recent article in this journal, Papic (2007) reported a related intervention study on patterning with pre-schoolers, who were followed through to formal school. The study showed that very young students can be taught the structure of patterns and can symbolise, represent and transfer patterns from one mode to another.

All of these studies have found that recognising similarity and difference in mathematical representations plays a critical role in the development of pattern and structure. These studies also show that young students are capable of developing complex mathematical ideas, rather than being limited to unitary counting, simple arithmetic, shape recognition and informal units of measure. We concluded that a sound understanding of pattern and structure is fundamental to learning multiplicative concepts, the base ten system, unitising and partitioning in early mathematics learning.

## The kindergarten intervention program

In 2006 we implemented an intervention program focused on pattern-eliciting tasks with a group of ten Kindergarten students aged 4 to 6 years. Students were selected by the classroom teachers as representative of the lowest quartile in the cohort in terms of mathematical ability. Students were pre- and post-assessed by our research team using the PASA interview and two subtests of the Woodcock-Johnson standardised mathematics test (Woodcock, McGrew, & Mather, 2001). An experienced classroom teacher, trained in the use of PASMAP, engaged the students in pattern-eliciting tasks that differentiated individual levels of patterning and other mathematical skills.

The program involved withdrawing the students for 15 weekly teaching episodes of one-hour duration during Terms 2 and 3 (May – October). The teaching episodes focused on unitising, counting, partitioning, simple repetition, functional thinking, spatial structuring, and congruence and similarity. Individual profiles of learning were documented through digital recordings, observations and analysis of students' work samples.

Table 1 provides an overview of key components of the program, which were implemented generally in the order shown. Many of the components were regularly revisited because the purpose of the program was to assist students in building on their previous experiences of pattern and structure. Another important feature was encouraging students to make explicit connections between two or more of the components.

Generally, the PASMAP teaching approach can be summarised as follows:

Table 1. Key components of the intervention teaching episodes.

| Component | Focus |
|---|---|
| Counting<br>Rhythmic and perceptual counting | Counting orally by twos and threes, with and without materials<br>Constructing simple patterns using perceptual counting |
| Repetition<br>Simple AB and complex patterns AAB (with and without models) | Constructing, drawing, symbolising and justifying linear and cyclic patterns using a variety of materials |
| Unit of repeat | Chunking, ordering, symbolising and translating |
| Similarity and congruence (2D shapes) | Comparing and drawing similar triangles and squares, distinguishing congruence |
| Symmetry and transformations | Identifying symmetry through matching and congruence |
| Subitising | Identifying number and shape in subitising patterns, three to nine.<br>Spatial structuring of subitising patterns |
| Grids | Identifying number of units in simple grids, 2 x 2, 3 x 3, 4 x 4, 5 x 5 squares and 2 x 3 rectangles<br>Deconstructing and reconstructing from memory the spatial properties of grids |
| Arrays | Identifying number of units in simple arrays, 1 x 2, 1 x 3, 2 x 2, 3 x 3<br>Deconstructing and reconstructing from memory the spatial properties of arrays |
| Table of data: functional thinking | Constructing tables of data, representing ratio as a pattern |

- Students are given pattern-eliciting tasks which require them to copy or produce a model or other representation *(representing)*.
- Students explain their initial, perhaps inaccurate representation *(intuitive justification)*.
- Teachers scaffold and use probing questions, comparing students' representations with those produced by others *(modelling)*.
- Teachers ask students to make their pattern the same as the given pattern, and to explain why it is the same *(focus on similarities and differences)*.
- Students' attention is drawn to crucial attributes of shape, size, spacing and unit of repeat *(focus on units, spatial or numerical structure)*.
- Students reproduce entire patterns or representations with parts increasingly hidden *(successive screening)*.
- Students justify why their representation is accurate (or incorrect) and shows pattern and structure *(justification)*.
- Students reproduce representations including patterns and structural features from memory *(visual memory)*.
- Students attempt to verbalise and record invented symbols to represent patterns and structures *(symbolising)*.
- Students translate structural features from one context to another, or simply reconstruct the pattern using different features *(translation)*.
- Tasks are modified and repeated regularly, extending and linking to prior learning *(repetition and linking)*.

The approach may be exemplified by an initial teaching episode on the rectangular grid pattern. Students make small rectangular patterns using square tiles, drawing around them to complete the pattern. Their attention is drawn to the equal sizes of the tiles they have drawn, and they are asked to count the number of tiles in each row and column. They examine and evaluate their own drawing as well as drawings made by other students; then they attempt to draw the pattern and reflect again. Partially or quickly obscured patterns are presented, and students challenged to find the number of elements. When students are confident with small patterns, they are presented with larger patterns of squares and arrays of objects such as dots. For students whose representations show emerging or partial structure, the activities and challenges are modified accordingly.

## Impact of the PASMAP intervention on students' learning

Figure 2 shows each student's overall improvement on the PASA. Every student showed substantial improvement, not only in gaining correct solutions on the PASA interview but in the growth of their representational skills and the way they could justify their responses. However, these improvements were not necessarily consistent across tasks for individuals. Possibly because of the short time frame of the study there were no consistent gains in general mathematical skills as indicated by the Woodcock-Johnson post-test.

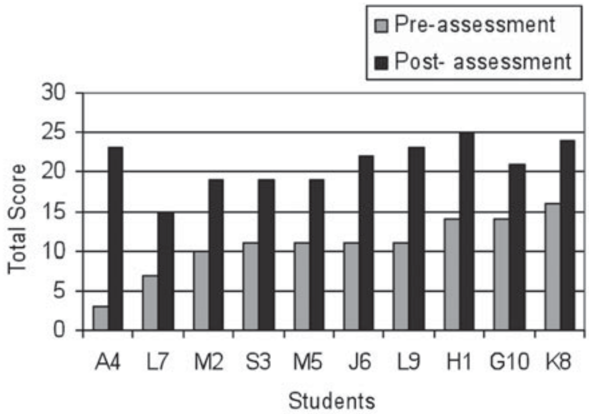


Figure 2. Pre- and post- PASA assessment.

The students showed impressive growth in representing, symbolising and translating simple and complex repetitions, structuring arrays and grids and unitising and partitioning in a variety of ways. The video data and work samples were analysed for improvements in features showing pattern and structure. For example, Figure 3 shows a student's attempted construction of a simple table of data (i.e., an animal represented in first column and the number of "legs" aligned in second column, was demonstrated in the final teaching episode). This revealed multiple count strategies and simple functional thinking.

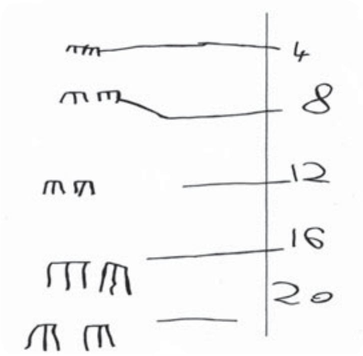


Figure 3. Table of data: Number of animals aligned with number of legs counted by 4s.

Improvements in recognising subitising patterns and counting in multiples of 2, 3 and 4 were also observed as well as some grouping strategies. This improvement could be explained by the varied and repeated PASMAP experiences in grouping and patterning using a unit of repeat. Consistent with the work of Papic (2007), the students represented simple repetitions and growing patterns in a variety of forms (see Figures 4, 5 & 6). We explicitly focused on "chunking" (breaking the pattern into units of repeat) and placing the chunks in the correct sequence (Marston, 2007).

By the end of the program, students could more readily represent the structure of rectangular grids and arrays but the majority of students were initially unable to

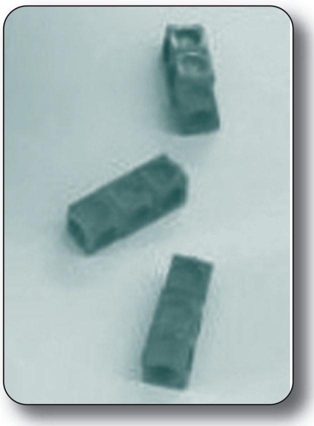

Figure 4. AAB repetition "chunked" showing unit of repeat.

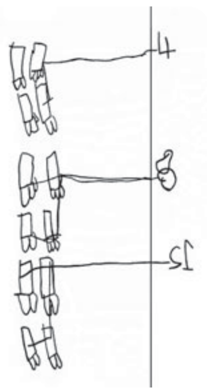

Figure 5. Table of data "4 feet on each dog", showing functional thinking.

Figure 6. AAB repetition shown as blocks and two different types of invented symbols.

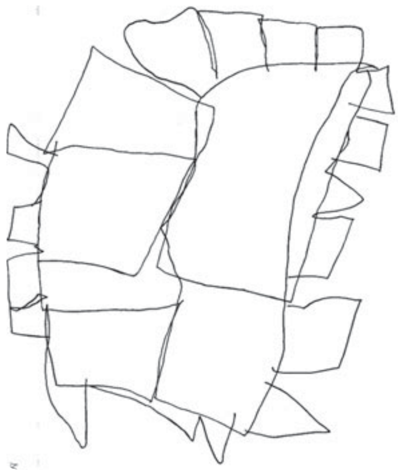

Figure 7. Initial drawing of 3 x 3 grid of squares.

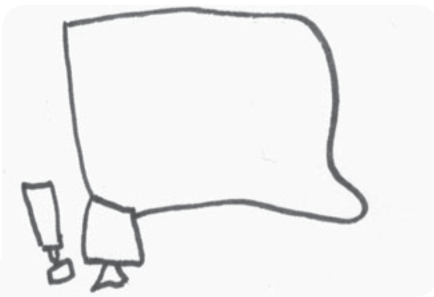

Figure 8. Interim drawing of 3 x 3 grid of squares.

Figure 9. Final drawing of 3 x 2 grid of squares.

represent simple arrays and grids beyond a pattern of four units. Figures 7, 8 and 9 show an individual's progress in representing a $3 \times 3$ grid of squares that was completed as a $3 \times 2$ grid.

## Implications for teaching and learning, curriculum and assessment

We gained the most valuable insights from compiling individual profiles of learning. This process enabled us to not only track individuals' thinking for each component of the program but also to look for connections between the student's use of pattern and structure. Students made rapid and substantial qualitative gains in their structural awareness of mathematics regardless of their learning difficulties or special needs. However, the intervention was limited to a small group of 'low-achieving' students withdrawn for individualised teaching, supported by specialist teachers and well-trialled resources. We cannot assume that the success of this program can be generalised to other pedagogical settings, but it is anticipated that teachers can quite readily integrate some features of the PASMAP approach into their existing programs.

The PASMAP program has undergone recent further development to reflect more explicitly aspects of early algebraic reasoning and data exploration. Plans are being made to make the PASA assessment and the PASMAP pattern-eliciting tasks readily available to classroom teachers and other professionals. The use of PASMAP is intended to align with, rather than replace existing syllabus outcomes. PASMAP is not intended as a prescriptive lockstep teaching sequence; it is a pedagogical approach embedded in a learning framework within which the teacher and students can flexibly move between key components.

Essentially the effectiveness of early mathematics learning will depend on professionals' awareness of the fundamental importance of pattern and structure. We anticipate that the teaching and learning of isolated skills will become more integrated in mathematics teaching and learning through the PASMAP approach. Every mathematics lesson can become an opportunity for students to notice, apply and develop pattern and structure, to represent and symbolise and to reflect and justify. Our research indicates that this process can enable the early abstraction and generalisation of ideas for young children in a way that promotes deep mathematical thinking.

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Joanne Mulligan, Mike Mitchelmore, Coral Kemp, Jennie Marston & Kate Highfield
Macquarie University
<joanne.mulligan@mq.edu.au>

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