Theory and Applications of Categories, Vol. 25, No. 1, 2011, pp. 1–37.

A CATEGORY OF QUANTUM CATEGORIES

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Abstract. Quantum categories were introduced in [5] as generalizations of both bi(co)algebroids and small categories. We clarify details of that work. In particular, we show explicitly how the monadic definition of a quantum category unpacks to a set of axioms close to the definitions of a bialgebroid in the Hopf algebraic literature. We introduce notions of functor and natural transformation for quantum categories and consider various constructions on quantum structures.

1. Introduction

Quantum categories are defined within a monoidal category $\mathcal{V}$. When $\mathcal{V}$ is the opposite category of modules over a commutative ring, a quantum category is the same as a bialgebroid. Bialgebroids can be thought of as "several object" generalisations of bialgebras. They were considered for the first time in M. Takeuchi's paper [17] and appeared later in the work of many authors in different fields, e.g. [10], [13], [20], [3]. A quantum category in a general monoidal category was defined by B. Day and R. Street [5] incorporating both bialgebroids, in the way mentioned above, and ordinary categories, by taking the monoidal category $\mathcal{V}$ to be the category of sets. In this paper we approach quantum categories using the bicategorical version of the formal theory of (co)monads [14]. One benefit of this approach is that it makes clear the connection between quantum categories and ordinary categories, which enables us to reproduce ordinary category theory for quantum categories.

The paper is organised in the following way. In Section 2 we review the formal theory of (co)monads in a bicategory. In Sections 3 and 4, we deal with a bicategory Comod$\mathcal{V}$, which is defined from our base monoidal category $\mathcal{V}$, and which plays an important role in our theory. In sections 5 we give a monadic definition of a quantum category and show explicitly how it translates to a set of axioms close to the definitions of a bialgebroid in the Hopf algebraic literature. In Section 6 we define the notion of functor between quantum categories, obtaining a category qCat$\mathcal{V}$. When $\mathcal{V}$ is the category of sets, then a quantum functor is the same as an ordinary functor, and the qCat$\mathcal{V}$ is the category of categories. In Section 7 making use of the functoriality of the qCat we give examples of constructions on quantum categories. In the Appendix we introduce framed string diagrams designed to ease computations involving the quantum structures.

The author gratefully acknowledges support of an International Macquarie University Scholarship.
Received by the editors 2009-10-02 and, in revised form, 2011-01-19.
Transmitted by R.J. Wood. Published on 2011-01-24.
2000 Mathematics Subject Classification: 18D35.
Key words and phrases: quantum category, monoidal category, comonad.

2. Monoidal comonads

Let $\mathcal{B}$ be a bicategory. We write as if $\mathcal{B}$ were a 2-category, regarding associativity and unitivity isomorphisms as identities.

Recall that a comonad in $\mathcal{B}$ [14], [1] is a pair (B, g), where B is an object of $\mathcal{B}$ and $g = (g, \delta : g \Rightarrow gg, \epsilon : g \Rightarrow 1_g)$ is a comonoid in the homcategory $\mathcal{B}(B, B)$. A map of comonads $(k, \kappa) : (B, g) \to (A, g')$ consists of a morphism $k : B \to A$ and a 2-cell $\kappa : kg \Rightarrow g'k$ satisfying:

$$\Big(kg \xrightarrow{k\delta} kgg \xrightarrow{\kappa g} g'kg \xrightarrow{g'\kappa} g'g'k\Big) = \Big(kg \xrightarrow{\kappa} g'k \xrightarrow{g'\delta} g'g'k\Big),$$

$$\Big(kg \xrightarrow{k\epsilon} k\Big) = \Big(kg \xrightarrow{\kappa} g'k \xrightarrow{\epsilon k} k\Big).$$

A comonad map transformation $\tau : (k, \kappa) \Rightarrow (k', \kappa') : (A, g) \to (B, g')$ is a 2-cell $\tau : k \Rightarrow k'$ satisfying:

$$\Big(kg \xrightarrow{\tau g} k'g \xrightarrow{\kappa'} g'k'\Big) = \Big(kg \xrightarrow{\kappa} g'k \xrightarrow{g\tau} g'k'\Big).$$

Comonads in $\mathcal{B}$, comonad maps and comonad map transformations form a bicategory $\mathrm{Comnd}\mathcal{B}$ under the obvious composition.

$\mathcal{B}$ is said [14] to admit the Eilenberg-Moore construction for comonads if the inclusion $\mathcal{B} \to \mathrm{Comnd}\mathcal{B}$, taking an object B to $(B, 1)$, has a right biadjoint $\mathrm{Comnd}\mathcal{B} \to \mathcal{B}$. The value of this right biadjoint at (B, g) is called an Eilenberg-Moore object of (B, g). It will be denoted by denoted B^g. There is a pseudonatural equivalence

$$\mathcal{B}(X, B^g) \simeq \mathrm{Comnd}\mathcal{B}((X, 1), (B, g))$$

The objects of the right side are called g-coalgebras. Taking $X = B^g$ and evaluating at the identity, we obtain a universal g-coalgebra $(u, \gamma) : (B^g, 1) \to (B, g)$. Every comonad map $k : (B, g) \to (A, g')$ induces a map $\hat{k} : B^g \to A^{g'}$ between Eilenberg-Moore objects so that there is an isomorphism:

$$\begin{tikzcd} B^g \arrow[r, "\hat{k}"] \arrow[d, "u"'] \arrow[dr, phantom, "\cong"] & A^{g'} \arrow[d, "u"] \\ A \arrow[r, "k"'] & B \end{tikzcd} \tag{1}$$

By an equivalence between suitable categories, comonad structures on $k : B \to A$ correspond to diagrams (2) in $\mathcal{B}$.

Let $\mathcal{B}$ be a monoidal bicategory [4]. We specify n-ary tensor product pseudofunctors

$$\mathcal{B}^n \xrightarrow{\otimes_n} \mathcal{B}$$

by choosing bracketing for the tensor product to be from the left. So, the expression $B_1 \otimes \ldots \otimes B_n$ refers to $\otimes_n(B_1 \otimes \ldots \otimes B_n)$.

A monoidale E in $\mathcal{B}$ consists of an object E together with morphisms $p : E \otimes E \to E$ and $j : I \to E$ called the multiplication and the unit respectively, and invertible 2-cells expressing associativity and unitivity, subject to coherence conditions. The n-ary multiplication map

$$E^n \xrightarrow{p_n} E.$$

is defined by consecutive multiplications from the left.

2.1. Example. Let an object B be the right bidual to an object A in $\mathcal{B}$, with the biduality counit $e : A \otimes B \to I$ and the biduality unit $n : I \to B \otimes A$. $B \otimes A$ becomes a monoidale with product $p = 1 \otimes e \otimes 1 : B \otimes A \otimes B \otimes A \to B \otimes A$ and unit $j = n : I \to B \otimes A$.

A monoidal morphism $(f, \phi_2, \phi_0) : E \to D$ between monoidales consists of a morphism $f : E \to D$ and 2-cells $\phi_2 : p(f \otimes f) \Rightarrow fp$, $\phi_0 : j \Rightarrow fj$ satisfying three axioms. The composition of monoidal morphisms $(f, \phi_2, \phi_0) : E \to D$ and $(f', \phi_2, \phi_0) : D \to F$ is defined to be $(f'f, \phi_2, \phi_0) : E \to F$, where

$$\phi_2 = \Big(p(f' \otimes f')(f \otimes f) \xrightarrow{\phi_2(f \otimes f)} f'p(f \otimes f) \xrightarrow{f'\phi_2} f'fp\Big)$$

$$\phi_0 = \Big(j \xrightarrow{\phi_0} f'j \xrightarrow{f'\phi_0} f'fj\Big).$$

A monoidal morphism is called strong when ϕ_2 and ϕ_0 are isomorphisms. Monoidales in $\mathcal{B}$, monoidal morphisms between them and obvious 2-cells form a bicategory Mon$\mathcal{B}$.

There is a biequivalence

$$\mathrm{MonComnd}\mathcal{B} \sim \mathrm{ComndMon}\mathcal{B}, \tag{2}$$

where the left hand side is defined using the monoidal structure on Comnd$\mathcal{B}$ inherited from $\mathcal{B}$.

A monoidal comonad is an object of ComndMon$\mathcal{B}$, or equally, an object of MonComnd$\mathcal{B}$. Explicitly, a monoidal comonad consists of a monoidale E, a comonad g on E and 2-cells $\phi_2 : p(g \otimes g) \Rightarrow gp$, $\phi_0 : j \Rightarrow gj$ such that (g, ϕ_2, ϕ_0) is a monoidal morphism and $(p, \phi_2) : (E \otimes E, g \otimes g) \to (E, g)$ and $(j, \phi_0) : (I, 1) \to (E, g)$ are comonad maps. A morphism of monoidal comonads $(k, \kappa) : (E, g) \to (E', g')$ is a map of underlying comonads such that $\kappa : kg \Rightarrow g'k$ is a map of monoidal morphisms.

Mon$(-)$ can be made into a pseudofunctor from the tricategory of monoidal bicategories and monoidal pseudofunctors to the tricategory of bicategories and pseudofunctors. Since the inclusion $i : \mathcal{B} \to \mathrm{Comnd}\mathcal{B}$ is a strong monoidal pseudofunctor the right biadjoint to it is a monoidal pseudofunctor too. It follows that if i has a right biadjoint, then Mon(i) : Mon$\mathcal{B} \to$ MonComnd$\mathcal{B}$ has a right biadjoint too. Using the the biequivalence (2) we infer that the canonical inclusion Mon$\mathcal{B} \to$ ComndMon$\mathcal{B}$ has a right biadjoint. This proves [12], [11]:

2.2. Proposition. *If $\mathcal{B}$ admits the Eilenberg-Moore construction for comonads, then so does* $\mathrm{Mon}(\mathcal{B})$.

Explicitly an Eilenberg-Moore object of a monoidal comonad (E, g) is obtained in the following way. Let E^g be the Eilenberg-Moore object for the underlying comonad in $\mathcal{B}$ with $(u, \gamma) : (E^g, 1) \to (E, g)$ the universal coalgebra. Then $p(u \otimes u) : E^g \otimes E^g \to E$ becomes a g-coalgebra with coaction

$$p(u \otimes u) \xrightarrow{p(\gamma\otimes\gamma)} p(g \otimes g)(u \otimes u) \xrightarrow{\phi_2 p} gp(u \otimes u),$$

and $j : I \to E$ becomes a g-coalgebra with the coaction

$$j \xrightarrow{\phi_0} gj.$$

The induced morphisms $\hat{p} : E^g \otimes E^g \to E^g$ and $\hat{j} : I \to E^g$ define a monoidale structure on E^g. This monoidale is the Eilenberg-Moore object of (E, g) in $\mathrm{Mon}\mathcal{B}$. Moreover, the map $u : E^g \to E$ is a strong monoidal morphism.

There is an equivalence of categories which establishes a correspondence between monoidal comonad maps $(k, \kappa) : (E, g) \to (E', g')$ and diagrams (2), now in $\mathrm{Mon}\mathcal{B}$.

What we have been discussing so far were standard constructions in a monoidal bicategory. Further we introduce some concepts, which we will later use for our specific purposes.

An opmonoidal morphism $(w, \psi_2, \psi_0) : E \to D$ between monoidales is a monoidal morphism in $\mathcal{B}^{co}$. Thus an opmonoidal morphism consists of a morphism $w : E \to F$ and 2-cells $\psi_2 : wp \Rightarrow p(w \otimes w)$, $\psi_0 : hj \Rightarrow j$ in $\mathcal{B}$ satisfying three axioms.

Monoidal morphisms and opmonoidal morphisms lead us to the setting of a double category [9], [19]. Recall briefly, that a double category has objects and two types of arrows, called horizontal morphisms and vertical morphisms, forming bicategories in the two directions. Also, there is a set of squares, each square having as its sides two horizontal morphisms and two vertical morphisms. Squares can be composed in the two directions.

As suggested, there is a double category with objects the monoidales in $\mathcal{B}$, horizontal arrows the monoidal morphisms and vertical morphisms the opmonoidal morphisms. A square is a 2-cell

$$\begin{tikzcd} E \arrow[r, "f"] \arrow[d, "w"'] \arrow[dr, phantom, "\sigma\Downarrow"] & E' \arrow[d, "w'"] \\ D \arrow[r, "f'"'] & D' \end{tikzcd}$$

with f and f' monoidal morphisms and w and w' opmonoidal morphisms such that:

$$\Big(w'p(f \otimes f) \xrightarrow{\psi_2(f\otimes f)} p(w' \otimes w')(f \otimes f) \xrightarrow{p(\sigma\otimes\sigma)} p(f' \otimes f')(w \otimes w) \xrightarrow{\phi_2(w\otimes w)} f'p(w \otimes w)\Big)$$

$$= \left(w'p(f \otimes f) \xrightarrow{w'\phi_2} w'fp \xrightarrow{\psi_2 p} f'wp \xrightarrow{\sigma} f'p(w \otimes w) \right)$$

and $$\left(w'j \xrightarrow{\phi_0} w'fj \xrightarrow{\sigma j} f'wj \xrightarrow{f'\psi_0} f'j \right) = (w'j \xrightarrow{\psi_0} j \xrightarrow{\phi_0} f'j \Big).$$

Suppose that (E, g) and (D, g') are monoidal comonads. An opmorphism of monoidal comonads $(h, \sigma) : (E, g) \to (D, g')$ is an opmonoidal morphism $h : E \to D$ together with a square $\sigma : hg \Rightarrow g'h$, such that $(h, \sigma) : (E, g) \to (D, g')$ is a map of comonads.

As with the monoidal comonad maps, there is an equivalence of categories which establishes a correspondence between opmorphisms of monoidal comonads $h : (E, g) \to (D, g')$ and diagrams

$$\begin{tikzcd} E^g \arrow[r, "\hat{h}"] \arrow[d, "u"'] \arrow[dr, phantom, "\cong"] & D^{g'} \arrow[d, "u"] \\ E \arrow[r, "h"'] & D \end{tikzcd}$$

of opmonoidal morphisms.

A coaction of an opmonoidal morphism $h : E \to D$ on a morphism $l : E \to D$ of $\mathcal{B}$ is a 2-cell $\lambda : lp \Rightarrow p(h \otimes l)$ satisfying two axioms, relating it to the opmonoidal structure on h.

Suppose that $(h, \sigma) : (E, g) \to (D, g')$ is an opmorphism of monoidal comonads and $(l, \tau) : (E, g) \to (D, g')$ is a comonad map. We will say that a left coaction λ of h on l respects the comonad structure if

$$\left(lp(g \otimes g) \xrightarrow{\lambda(g \otimes g)} p(h \otimes l)(g \otimes g) \xrightarrow{p(\sigma \otimes \tau)} p(g' \otimes g')(h \otimes l) \xrightarrow{\mu(h \otimes l)} g'p(h \otimes l) \right)$$

$$= \left(lp(g \otimes g) \xrightarrow{l\mu} lgp \xrightarrow{\tau p} g'lp \xrightarrow{g'\lambda} g'p(h \otimes l) \right).$$

A left coaction of h on l respects comonad structure if and only if it can be lifted to a coaction of $\hat{l} : E^g \to D^{g'}$ on $\hat{h} : E^g \to D^{g'}$.

There is a similar notion of a right coaction of an opmorphism.

3. The bicategory of comodules

Suppose that $\mathcal{V} = (\mathcal{V}, \otimes, I, c)$ is a braided monoidal category with finite colimits. Assume that each of the functors $X \otimes -$ preserves equalizers of coreflexive pairs.

We will work with monoidal bicategory $\mathcal{C} = \mathrm{Comod}\mathcal{V}$ defined in [2]. Objects of $\mathcal{C}$ are the comonoids

$$C = (C, \delta : C \to C \otimes C, \epsilon : C \to I)$$

in $\mathcal{V}$. The homcategory $\mathcal{C}(C,D)$ is the category of Eilenberg-Moore coalgebras for the comonad $C\otimes-\otimes D : \mathcal{V} \to \mathcal{V}$. A 1-cell from C to D, depicted $C \looparrowright D$, is a comodule from C to D. Recall that this consists of an object M and a coaction map $\delta : M \to C \otimes M \otimes D$ satisfying two axioms. A 2-cell $\alpha : M \Rightarrow N : C \looparrowright D$ is a coaction respecting map $M \to N$. An object of $\mathcal{C}(C, I)$ is a left C-comodule, and an object of $\mathcal{C}(I, C)$ is a right C-comodule. A comodule $M : C \looparrowright D$ becomes a left C-comodule and a right D-comodule via coactions

$$\begin{gathered}\delta_l : M \xrightarrow{\delta} C \otimes M \otimes D \xrightarrow{1\otimes 1\otimes \epsilon} C \otimes M \\ \delta_r : M \xrightarrow{\delta} C \otimes M \otimes D \xrightarrow{\epsilon\otimes 1\otimes 1} M \otimes D.\end{gathered} \tag{3}$$

The maps δ_l and δ_r are called left and right coactions on M. If M is a left C-comodule and N is a right D-comodule, then a tensor product $M \otimes_C N$ over C is defined by a (coreflexive) equalizer:

$$M \otimes_C N \xrightarrow{i} M \otimes N \underset{1\otimes\delta_l}{\overset{\delta_r\otimes 1}{\rightrightarrows}} M \otimes C \otimes N.$$

If M is a comodule $E \looparrowright C$ and N is a comodule $C \looparrowright F$, then using the fact that the functor $E \otimes - \otimes F$ preserves coreflexive equalizers, $M \otimes_C N$ becomes a comodule $E \looparrowright F$. Composition in $\mathcal{C}$ is defined by $N \circ M = M \otimes_C N$. It is associative up to canonical isomorphism.

Any comonoid C is a $C \looparrowright C$ comodule with the coaction

$$C \xrightarrow{\delta_3} C \otimes C \otimes C.$$

The identity comodule on C is C itself.

As it is a convention to name such bicategories after arrows, Comod$\mathcal{V}$ is called the bicategory of comodules. For more on the theory of comodules we refer the reader to [15].

Each comonoid morphism $f : C \to D$ determines an adjoint pair in $\mathcal{C}$:

$$f_* \dashv f^* C \looparrowright D$$

The comodules $f^* : C \looparrowright D$ and $f_* : D \looparrowright C$ are both C as objects of $\mathcal{V}$ with coactions respectively

$$C \xrightarrow{\delta_3} C \otimes C \otimes C \xrightarrow{1\otimes 1\otimes f} C \otimes C \otimes D \quad \text{and}$$

$$C \xrightarrow{\delta_3} C \otimes C \otimes C \xrightarrow{f\otimes 1\otimes 1} D \otimes D \otimes C.$$

The counit

$$\begin{tikzcd} & C \arrow[dr, "f_*", "|"] & \\ D \arrow[ur, "f^*", "|"] \arrow[rr, "D"', "|"] \arrow[rr, phantom, bend left=20, "\beta\Downarrow"] & & D \end{tikzcd}$$

of the adjunction is the map

$$C \otimes_C C \cong C \xrightarrow{f} D.$$

The unit

$$\begin{tikzcd} & D \arrow[dr, "f^*", "|"] & \\ C \arrow[ur, "f_*", "|"] \arrow[rr, "C"', "|"] \arrow[rr, phantom, bend left=20, "\Uparrow\alpha"] & & C \end{tikzcd}$$

is induced by the comultiplication $\delta : C \to C \otimes C$ as shown on the diagram:

$$\begin{tikzcd} C \otimes_D C \arrow[rr, "eq."] & & C \otimes C \arrow[r, shift left, "(1\otimes f\otimes 1)(\delta\otimes 1)"] \arrow[r, shift right, "(1\otimes f\otimes 1)(1\otimes\delta)"'] & C \otimes D \otimes C \\ & C \arrow[ul, "\alpha"] \arrow[ur, "\delta"'] & & \end{tikzcd}$$

In particular, we have comodules $\epsilon^* : I \xrightarrow{|} C$ and $\epsilon_* : C \xrightarrow{|} I$. The compositions

$$C \xrightarrow{M} D \xrightarrow{\epsilon_*} I$$

$$I \xrightarrow{\epsilon_*} C \xrightarrow{M} D$$

reconfirm the fact that M is a left C-comodule and a right D-comodule by (3).

The monoidal structure on $\mathcal{C}$ extends the monoidal structure on $\mathcal{V}$. The tensor product of comonoids $C = (C, \delta, \epsilon)$ and $C' = (C', \delta', \epsilon')$ is $C \otimes C'$ with comultiplication and counit:

$$(1 \otimes c \otimes 1)(\delta \otimes \delta') : C \otimes C' \to C \otimes C' \otimes C \otimes C'$$

$$\epsilon \otimes \epsilon' : C \otimes C' \to I.$$

The monoidal unit of $\mathcal{C}$ is I, which is a comonoid in an obvious way. On 1-cells, the tensor product of comodules $M : C \xrightarrow{|} D$ and $N : C' \xrightarrow{|} D'$ is $M \otimes N$, which is a comodule $C \otimes C' \xrightarrow{|} D \otimes D'$ with coaction:

$$M \otimes N \xrightarrow{\delta\otimes\delta} C \otimes M \otimes D \otimes C' \otimes N \otimes D' \xrightarrow{c_{142536}} C \otimes C' \otimes M \otimes N \otimes D \otimes D'.$$

Here and below a morphism named c subscripted with a permutation is an isomorphism coming from the braiding.

We often encounter comodules going between tensor products of comonoids, like $M : C_1 \otimes C_2 \otimes \ldots C_n \nrightarrow D_1 \otimes D_2 \otimes \ldots D_m$. Such a comodule inherently is a left C_i-comodule, for $1 \leq i \leq n$, and a right D_i-comodule, for $1 \leq i \leq m$. Conversely, given left C_i-comodule and right D_i-comodules structures on M compatible in a certain way, M becomes a $C_1 \otimes C_2 \otimes \ldots C_n \nrightarrow D_1 \otimes D_2 \otimes \ldots D_m$ comodule. This enables us to describe a comodule just by giving left and right coactions. A map $M \to N$ is a comodule map between comodule $M, N : C_1 \otimes C_2 \otimes \ldots C_n \nrightarrow D_1 \otimes D_2 \otimes \ldots D_m$ if and only if it is a left C_i-comodule map for all $1 \leq i \leq n$ and a right D_i-comodule map for all $0 \leq i \leq m$.

$\mathcal{C}$ is a right autonomous monoidal bicategory. The bidual of a comonoid $C = (C, \delta, \epsilon)$ is the comonoid with the opposite comultiplication $C^o = (C, c\delta, \epsilon)$. Unit and counit are comodules $e : C^o \otimes C \nrightarrow I$ and $n : I \nrightarrow C \otimes C^o$, both of which are C as objects of $\mathcal{V}$ and the coactions on them are respectively

$$C \xrightarrow{\delta_3} C \otimes C \otimes C \xrightarrow{1 \otimes c} C \otimes C \otimes C \quad \text{and}$$

$$C \xrightarrow{\delta_3} C \otimes C \otimes C \longrightarrow c \otimes 1 C \otimes C \otimes C.$$

It follows that $C^o \otimes C$ is a monoidale in $\mathcal{C}$. The multiplication is $p = 1 \otimes e \otimes 1$ and the unit is $j = n$. Still more explicitly, the multiplication $C^o \otimes C \otimes C^o \otimes C \nrightarrow C^o \otimes C$ is $p = C \otimes C \otimes C$ with coaction

$$C^{\otimes 3} \xrightarrow{\delta_3 \otimes \delta_3 \otimes \delta_3} C^{\otimes 9} \xrightarrow{c_{146725839}} C^{\otimes 9}$$

and the unit $I \nrightarrow C^o \otimes C$ is $j = C$ with coaction

$$C \xrightarrow{\delta_3} C^{\otimes 3} \xrightarrow{c_{213}} C^{\otimes 3}$$

Let M and N be comodules $I \nrightarrow C^o \otimes C$. Regard these as comodules $C \nrightarrow C$ by the equivalence

$$\mathcal{C}(I, C^o \otimes C) \simeq \mathcal{C}(C, C). \tag{4}$$

The composite

$$I \xrightarrow{M \otimes N} C^o \otimes C \otimes C^o \otimes C \xrightarrow{p} C^o \otimes C$$

is $M \otimes_C N$ with right $C^o \otimes C$-coaction the unique map $\delta_l : M \otimes_C N \to (M \otimes_C N) \otimes C \otimes C$ making

$$\begin{tikzcd}
M \otimes_C N \arrow[r, "i"] \arrow[d, "\delta_l"'] & M \otimes N \arrow[d, "\delta_l"] \\
(M \otimes_C N) \otimes C \otimes C \arrow[r, "i \otimes \otimes C \otimes C"'] & M \otimes N \otimes C \otimes C
\end{tikzcd}$$

commute.

The equivalence (4) is a monoidal equivalence, where the monoidal structure on the left side comes from the pseudomonoid structure on $C^o \otimes C$ and the monoidal structure on the right is defined to be the composition in $\mathcal{C}$.

Next we prove some technical lemmas, which we use in Section 5.

3.1. Lemma. *Suppose that β is a 2-cell:*

$$\begin{tikzcd} & A \otimes C \otimes C^o \otimes B \arrow[dl, "A\otimes e\otimes B"'] \arrow[dr, "M"] \arrow[d, phantom, "\beta\Downarrow"] & \\ A \otimes B \arrow[rr, "N"'] & & D \end{tikzcd}$$

Let $\alpha : M \to N$ be the map in $\mathcal{V}$ determined by the pasting composite

$$\begin{tikzcd} A \otimes B \arrow[r, "A\otimes\epsilon^*\otimes B"] \arrow[d, "A\otimes B"'] \arrow[dr, phantom, "A\otimes\epsilon\otimes B\Downarrow"] & A \otimes C \otimes C^o \otimes B \arrow[d, "A\otimes e\otimes B"'] \arrow[dr, "M"] \arrow[dr, phantom, "\beta\Downarrow"'] & \\ A \otimes B \arrow[r, "A\otimes B"'] & A \otimes B \arrow[r, "N"'] & D \end{tikzcd} \tag{5}$$

It satisfies

$$M \xrightarrow{\delta_l} C \otimes C \otimes M \underset{\epsilon\otimes 1\otimes 1}{\overset{1\otimes\epsilon\otimes 1}{\rightrightarrows}} C \otimes M \xrightarrow{1\otimes\alpha} C \otimes N \tag{6}$$

The 2-cell β is uniquely determined by a left $A \otimes B$- right D-comodule map α which satisfies (6).

Proof. The comodule $N \circ (A \otimes e \otimes B)$ is $C \otimes N$ with coaction

$$C \otimes N \xrightarrow{\delta_3\otimes\delta} C \otimes C \otimes C \otimes A \otimes B \otimes N \otimes D \xrightarrow{c_{4135267}} C \otimes A \otimes B \otimes C \otimes C \otimes N \otimes D.$$

The left C^o and C coactions on $C \otimes N$ both are the cofree coactions, i.e. they are determined by the comultiplications.

The basic property of a cofree comodule is that any comodule map $\beta : M \Rightarrow C \otimes N$ to a cofree comodule is uniquely determined by its corestriction to N, by which is meant the map $\alpha = (\epsilon \otimes N)\beta : M \Rightarrow N$ in $\mathcal{V}$. Specifically, β can be recovered from α as

$$M \xrightarrow{\delta_l} C \otimes M \xrightarrow{1\otimes\alpha} C \otimes N.$$

It follows that in the setting of the lemma β can be reconstructed from α in two ways. The condition (6) asserts that these two reconstructions are the same.

It is easily seen that β is a left A-, B- right D- comodule map if and only if α is. The lemma is proved. ∎

3.2. Lemma. *Let β be a 2-cell:*

$$\begin{tikzcd}
 & I \arrow[dl, "n"'] \arrow[dr, "M"] \arrow[d, phantom, "\beta\Downarrow"] & \\
C^o \otimes C \arrow[rr, "N"'] & & D
\end{tikzcd}$$

Let $\alpha : M \to N$ be a map in $\mathcal{V}$ determined by the pasting composite:

$$\begin{tikzcd}
 & & I \arrow[dll, "1"'] \arrow[d, "j"'] \arrow[drr, "M"] \arrow[dl, phantom, "\delta\Downarrow"] \arrow[dr, phantom, "\beta\Downarrow"] & & \\
I \arrow[rr, "\epsilon^*"'] & & C^o \otimes C \arrow[rr, "N"'] & & D
\end{tikzcd}$$

It satisfies:

$$C \xrightarrow{\alpha} N \xrightarrow{\delta_l} C \otimes C \otimes N \underset{\epsilon\otimes 1\otimes 1}{\overset{1\otimes\epsilon\otimes 1}{\rightrightarrows}} C \otimes N \tag{7}$$

The 2-cell β is uniquely determined by a right D-comodule map α which satisfies (7)*.*

Proof. The 2-cell β is a map $C \to C \otimes_{C^o \otimes C} N$. This is induced by a map $\beta' : C \to C \otimes N$ satisfying

$$(1 \otimes ((\epsilon \otimes 1)\delta_l))\beta' = (c\delta \otimes 1)\beta' \tag{8}$$

$$(1 \otimes ((1 \otimes \epsilon)\delta_l))\beta' = (\delta \otimes 1)\beta' \tag{9}$$

We have $\alpha = (\epsilon \otimes 1)\beta'$. From α we can recover β' in two ways: using (8) it can be shown that β' can be reconstructed from α as the top composite in (7), or using (9) it can be shown that β' can be reconstructed from α as the bottom composite in (7). So, the map α defined from β satisfies (7). Conversely, β can be defined from a map α which satisfies (7).

It is easily checked that β is a right D-comodule map if and only if α is. The lemma is proved. ∎

The maps $\xi_2 = 1 \otimes \epsilon \otimes 1 : C \otimes C \otimes C \to C \otimes C$ and $\xi_0 = \delta : C \to C \otimes C$ define a monoidal morphism structure on $\epsilon^* = \epsilon^* \otimes \epsilon^* : I \longrightarrow C^o \otimes C$. For any $n \geq 0$ we have a 2-cell:

$$\begin{tikzcd}
I \arrow[r, "\epsilon^*"] \arrow[d, "I"'] \arrow[dr, phantom, "\xi_n\Downarrow"] & (C^o \otimes C)^{\otimes n} \arrow[d, "p_n"] \\
I \arrow[r, "\epsilon^*"'] & C^0 \otimes C
\end{tikzcd} \tag{10}$$

3.3. Lemma. *For any n, the function defined on the set of 2-cells*

$$\mathcal{C}((C^o \otimes C)^{\otimes n}, C^o \otimes C)(M, N \circ p_n)$$

with values in

$$\mathcal{C}(I, C^o \otimes C)(M \circ \epsilon^*, N \circ \epsilon^*) \tag{11}$$

taking

$$\begin{tikzcd} & (C^o \otimes C)^{\otimes n} \arrow[dl, "p_n"'] \arrow[dr, "M"] \arrow[d, phantom, "\beta\Downarrow"] & \\ C^o \otimes C \arrow[rr, "A"'] & & C^o \otimes C \end{tikzcd}$$

to the pasting composite

$$\begin{tikzcd} I \arrow[r, "\epsilon^*"] \arrow[d, "I"'] \arrow[dr, phantom, "\xi_n\Downarrow"] & (C^o \otimes C)^{\otimes n} \arrow[d, "p_n"'] \arrow[dr, "M"] \arrow[dr, phantom, "\beta\Downarrow"'] & \\ I \arrow[r, "\epsilon^*"'] & C^o \otimes C \arrow[r, "A"'] & C^o \otimes C \end{tikzcd}$$

is injective.

Proof. The $n = 0$ case follows from Lemma 3.1. For $n = 1$ the function forgets the left $C^o \otimes C$ comodule structure which clearly is injective. For $n \geq 2$, the 2-cell ξ_n can be written as a pasting composite of the 2-cells $\xi_2 \otimes (C^o \otimes C)^{\otimes n-1}, \ldots, \xi_2 \otimes (C^o \otimes C)^{\otimes 2}, \xi_2$. Pasting from the left by each of these is an injective function by Lemma 3.2, hence pasting from the left by ξ_n is injective too. ■

4. Comonads in the bicategory of comodules

We will use the lower case Greek letters ϵ and δ for counits and comultiplications of both comonads in $\mathcal{C}$ and the comonoids. Although these are not the same, below it will become clear that such notation is not confusing.

Let E be a comonoid. There is an equivalence of categories between comonads on E in the bicategory $\mathcal{C}$ of comodule and comonoid maps with codomain E. If $\epsilon : G \to E$ is a comonoid map, then the adjunction

$$\epsilon_* \dashv \epsilon^* : E \longrightarrow G$$

induces a comonad on E. Conversely, if G is a comonad on E with comonad comultiplication $\delta : G \to G \otimes_E G$ and comonad counit $\epsilon : G \to E$, then G itself becomes a comonoid with comultiplication and counit

$$G \xrightarrow{\delta} G \otimes_E G \xrightarrow{i} G \otimes G$$

$$G \xrightarrow{\epsilon} E \xrightarrow{\epsilon} I$$

while $\epsilon : G \to E$ becomes a comonoid map. In fact, the comonoid G is the Eilenberg-Moore object of (E, G) with the universal G-coalgebra

$$\begin{tikzcd}
& G \arrow[dl, "\epsilon_*"'] \arrow[dr, "\epsilon_*"] \arrow[d, phantom, "\delta\Downarrow"] & \\
E \arrow[rr, "g"'] & {} & E
\end{tikzcd}$$

4.1. Proposition. *$\mathcal{C}$ admits the Eilenberg-Moore construction for comonads.*

It follows from Proposition 2.2 that Mon$\mathcal{C}$ also admits the comonad Eilenberg-Moore construction. To wit, given a monoidal structure on a comonad G, the comonoid G becomes a monoidale in $\mathcal{C}$, while $\epsilon_* : G \longrightarrow E$ becomes a strong monoidal morphism.

The correspondence between comonads and comonoid maps lifts to a correspondence between monoidal comonads on the monoidale E and monoidales G in $\mathcal{C}$ together with a comonoid map $G \to E$ such that $\epsilon_* : G \longrightarrow E$ is a strong monoidal morphism.

5. Quantum Categories

Essentially following [5] we define a quantum category in $\mathcal{V}$. In [5] it was shown that a quantum category in Set is the same as a small category and a quantum category in $Vect^{\mathrm{op}}$ is the same as a bialgebroid [17], [10], [20]. Most of the section after the definition is dedicated to proving Statement 5.6, which translates that definition to a set of axioms close to the definitions of bialgebroid in the literature.

5.1. Definition. *A quantum graph (C, A) in $\mathcal{V}$ consists of a comonoid C and a comonad A on $C^o \otimes C$.*

$$\begin{tikzcd}
C^o \otimes C \arrow[rr, "A"] \arrow[dr, "A"'] & \arrow[d, phantom, "\delta\Downarrow"] & C^o \otimes C \arrow[dl, "A"'] \arrow[dl, bend left=60, "C\otimes C"] \\
& C^o \otimes C &
\end{tikzcd}$$

5.2. Definition. *A quantum category (C, A) in $\mathcal{V}$ consists of a comonoid C together with a monoidal comonad A on $C^o \otimes C$.*

A quantum category has an underlying quantum graph and 2-cells

$$\begin{tikzcd}
C^o \otimes C \otimes C^o \otimes C \arrow[r, "A\otimes A"] \arrow[d, "p"'] \arrow[dr, phantom, "\Downarrow\mu_2"] & C^o \otimes C \otimes C^o \otimes C \arrow[d, "p"] \\
C^o \otimes C \arrow[r, "A"'] & C^o \otimes C
\end{tikzcd}
\qquad
\begin{tikzcd}
& I \arrow[dl, "j"'] \arrow[dr, "j"] \arrow[d, phantom, "\Downarrow\mu_0"] & \\
C^o \otimes C \arrow[rr, "A"'] & {} & C^o \otimes C
\end{tikzcd}$$

which make A into a monoidal morphism and both of which are comonad maps.

If $\mathcal{V}$ is the category of sets with the monoidal structure given by the cartesian product, then a quantum category in $\mathcal{V}$ is the same as an ordinary category. We do not consider this fact in details since this was done in [3]. When $\mathcal{V}$ is the opposite category of modules over a commutative ring, then a quantum category in $\mathcal{V}$ is the same as a bialgebroid. This will become clearer with the Statement 5.6. We continue with a general monoidal category $\mathcal{V}$.

By Section 4, a quantum graph amounts to comonoids C, A and a comonoid map $\epsilon : A \to C^o \otimes C$. The latter itself amounts to comonoid maps $s : A \to C^o$ and $t : A \to C$ satisfying:

$$\begin{tikzcd} & A\otimes A \arrow[r, "s\otimes t"] & C\otimes C \arrow[dd, "c"] \\ A \arrow[ur, "\delta"] \arrow[dr, "\delta"'] & & \\ & A\otimes A \arrow[r, "t\otimes s"'] & C\otimes C \end{tikzcd}$$

By s and t we can express ϵ as

$$A \xrightarrow{\delta} A\otimes A \xrightarrow{s\otimes t} C\otimes C.$$

C is called the object of objects of the quantum graph. A is called the object of arrows. The maps s and t are called the source and the target maps respectively.

We regard A as a comodule $C \mathrel{-\!\!\!+\!\!\!\rightarrow} C$ using the right $C^o\otimes C$-coaction on it. In terms of s and t left and right C-coactions on A are

$$A \xrightarrow{\delta_3} A\otimes A\otimes A \xrightarrow{1\otimes s\otimes\epsilon} A\otimes C \xrightarrow{c^{-1}} C\otimes A \quad \text{and}$$

$$A \xrightarrow{\delta_3} A\otimes A\otimes A \xrightarrow{1\otimes\epsilon\otimes t} A\otimes C.$$

The tensor product $H = A\otimes_C A$ of A with itself over C is called the object of composable arrows for the quantum graph. It is defined by the equalizer

$$\begin{tikzcd} H \arrow[r, "i"] & A\otimes A \arrow[r, shift left, "1\otimes(c^{-1}(1\otimes s\otimes\epsilon)\delta_3)"] \arrow[r, shift right, "(1\otimes\epsilon\otimes t)\delta_3)\otimes 1"'] & A\otimes C\otimes A. \end{tikzcd}$$

The composite comodule

$$\begin{tikzcd} C^o\otimes C\otimes C^o\otimes C \arrow[r, "A\otimes A", "|" marking] & C^o\otimes C\otimes C^o\otimes C \arrow[r, "p", "|" marking] & C^o\otimes C \end{tikzcd}$$

is H with left $C^o\otimes C\otimes C^o\otimes C$-coaction the unique map $\delta_l : H \to C\otimes C\otimes C\otimes C\otimes H$ making

$$\begin{tikzcd}
H \arrow[r, "\delta_l"] \arrow[d, "i"'] & C \otimes C \otimes C \otimes C \otimes H \arrow[d, "1\otimes 1\otimes 1\otimes 1\otimes i"] \\
A \otimes A \arrow[r, "\delta_l"'] & C \otimes C \otimes C \otimes C \otimes A \otimes A
\end{tikzcd}$$

commute and right $C^o \otimes C$-coaction the unique map $\delta_r : H \to C \otimes C \otimes H$ making

$$\begin{tikzcd}
H \arrow[r, "\delta_r"] \arrow[d, "i"'] & H \otimes C \otimes C \arrow[d, "i\otimes 1\otimes 1"] \\
A \otimes A \arrow[r, "\delta_r"'] & A \otimes A \otimes C \otimes C
\end{tikzcd}$$

commute. We regard H as a comodule $C \longrightarrow C$ using the right $C^o \otimes C$-coaction on it.

The map $\nu_2 : H \to A$ determined by the pasting composite

$$\begin{tikzcd}
I \arrow[r, "\epsilon^*"] \arrow[d, "I"'] \arrow[dr, phantom, "\Downarrow\xi_2"] & C^o \otimes C \otimes C^o \otimes C \arrow[r, "A\otimes A"] \arrow[d, "p"'] \arrow[dr, phantom, "\Downarrow\mu_2"] & C^o \otimes C \otimes C^o \otimes C \arrow[d, "p"] \\
I \arrow[r, "\epsilon^*"'] & C^o \otimes C \arrow[r, "A"'] & C^o \otimes C
\end{tikzcd} \tag{12}$$

is called the composition map of the quantum category. The map $\nu_0 : C \to H$ determined by the pasting composite

$$\begin{tikzcd}
 & I \arrow[dl, "I"'] \arrow[d, "j"] \arrow[dr, "j"] \arrow[dl, phantom, "\xi_0\Downarrow", bend left] \arrow[dr, phantom, "\mu_0\Downarrow", bend right] & \\
I \arrow[r, "\epsilon^*"'] & C^o \otimes C \arrow[r, "A"'] & C^o \otimes C
\end{tikzcd} \tag{13}$$

is called the unit map of the quantum category.

5.3. Lemma. *The 2-cells μ_2 and μ_0 determine a monoidal morphism structure on the comodule $A : C^o \otimes C \longrightarrow C^o \otimes C$ if and only if (A, ν_2, ν_0) is a monoid in $\mathcal{C}(C, C)$.*

Proof. It follows from Lemma 3.3 that μ_2 and μ_0 determine a monoidal morphism structure on A if and only if the pasting composites (12) and (13) determine a monoidal morphism structure on $\epsilon^* A$. Using the equivalence $\mathcal{C}(I, C^o \otimes C) \simeq \mathcal{C}(C, C)$, the 2-cells (12) and (13) determine a monoidal morphism structure on $\epsilon^* A$ if and only if (A, ν_2, ν_0) is a monoid in $\mathcal{C}(C, C)$. ∎

Consider the pasting composite

$$\begin{tikzcd}
C^o \otimes C \otimes C^o \otimes C \arrow[r, "A\otimes A"] \arrow[rr, bend left, "A\otimes A"] \arrow[rr, phantom, bend left=20, "\delta\Downarrow"] & C^o \otimes C \otimes C^o \otimes C \arrow[r, "A\otimes A"] \arrow[d, "p"'] \arrow[dr, phantom, "\mu_2\Downarrow"] & C^o \otimes C \otimes C^o \otimes C \arrow[d, "p"] \\
 & C^o \otimes C \arrow[r, "A"'] & C^o \otimes C
\end{tikzcd} \tag{14}$$

It is a map $H \to H \otimes_{(C^o \otimes C)} A$. Let γ_r be the composite of this with the canonical injection $H \otimes_{(C^o \otimes C)} A \to H \otimes A$.

5.4. Lemma. *The 2-cell μ_2 is a comonad morphism if and only if the following diagrams commute*

$$\begin{tikzcd} H \arrow[r, "\nu_2"] \arrow[d, "\gamma_r"'] & A \arrow[d, "\delta"] \\ H \otimes A \arrow[r, "\nu_2 \otimes 1"'] & A \otimes A \end{tikzcd} \tag{15}$$

$$\begin{tikzcd} H \arrow[r, "\nu_2"] \arrow[d, "i"'] & A \arrow[d, "\epsilon"] \\ A \otimes A \arrow[r, "\epsilon \otimes \epsilon"'] & I \end{tikzcd} \tag{16}$$

Proof. The map μ_2 is a comonad map if:

$$\begin{tikzcd} C^o \otimes C \otimes C^o \otimes C \arrow[r, "A \otimes A"] \arrow[rr, bend left, "A \otimes A"] \arrow[rr, phantom, bend left=15, "\delta\Downarrow"] \arrow[d, "p"'] & C^o \otimes C \otimes C^o \otimes C \arrow[r, "A \otimes A"] \arrow[d, "p"'] \arrow[dl, phantom, "\mu_2\Downarrow"] & C^o \otimes C \otimes C^o \otimes C \arrow[d, "p"] \arrow[dl, phantom, "\mu_2\Downarrow"] \\ C^o \otimes C \arrow[r, "A"'] & C^o \otimes C \arrow[r, "A"'] & C^o \otimes C \end{tikzcd}$$

$$=$$

$$\begin{tikzcd} C^o \otimes C \otimes C^o \otimes C \arrow[rr, bend left, "A \otimes A"] \arrow[rr, phantom, "\mu_2\Downarrow"] \arrow[d, "p"'] & & C^o \otimes C \otimes C^o \otimes C \arrow[d, "p"] \\ C^o \otimes C \arrow[r, "A"'] \arrow[rr, bend left, "A \otimes A"] \arrow[rr, phantom, bend left=15, "\delta\Downarrow"] & C^o \otimes C \arrow[r, "A"'] & C^o \otimes C \end{tikzcd}$$

and

$$\begin{tikzcd} C^o \otimes C \otimes C^o \otimes C \arrow[r, "C^{\otimes 4}"'] \arrow[r, bend left=50, "A \otimes A"] \arrow[r, phantom, bend left=25, "\epsilon\Downarrow"] \arrow[d, "p"'] \arrow[dr, phantom, "\cong"] & C^o \otimes C \otimes C^o \otimes C \arrow[d, "p"] \\ C^o \otimes C \arrow[r, "C \otimes C"'] & C^o \otimes C \end{tikzcd} = \begin{tikzcd} C^o \otimes C \otimes C^o \otimes C \arrow[r, bend left=50, "A \otimes A"] \arrow[r, phantom, "\mu_2\Downarrow"] \arrow[d, "p"'] & C^o \otimes C \otimes C^o \otimes C \arrow[d, "p"] \\ C^o \otimes C \arrow[r, "C \otimes C"'] \arrow[r, bend left=50, "A"] \arrow[r, phantom, bend left=25, "\epsilon\Downarrow"] & C^o \otimes C \end{tikzcd}$$

By Lemma 3.3 these equalites between pasting diagrams are equivalent to the following equalities obtained by suitably pasting to them the 2-cell (10) for $n = 2$.

$$\begin{tikzcd}
I \arrow[r, "A\otimes A"] \arrow[rr, bend left=30, "A\otimes A"] \arrow[rr, phantom, bend left=15, "\delta\Downarrow"] \arrow[d, "I"'] \arrow[dr, phantom, "\nu_2\Downarrow"] & C^o \otimes C \otimes C^o \otimes C \arrow[r, "A\otimes A"] \arrow[d, "p"'] \arrow[dr, phantom, "\mu_2\Downarrow"] & C^o \otimes C \otimes C^o \otimes C \arrow[d, "p"] \\
I \arrow[r, "A"'] & C^o \otimes C \arrow[r, "A"'] & C^o \otimes C
\end{tikzcd}$$

$$=$$

$$\begin{tikzcd}
I \arrow[rr, bend left, "A\otimes A"] \arrow[rr, phantom, "\nu_2\Downarrow"] \arrow[d, "I"'] & & C^o \otimes C \otimes C^o \otimes C \arrow[d, "p"] \\
I \arrow[r, "A"'] \arrow[rr, bend left, "A\otimes A"] \arrow[rr, phantom, bend left=15, "\delta\Downarrow"] & C^o \otimes C \arrow[r, "A"'] & C^o \otimes C
\end{tikzcd}$$

and

$$\begin{tikzcd}
I \arrow[r, bend left, "A\otimes A"] \arrow[r, phantom, bend left=20, "\epsilon\Downarrow"] \arrow[r, "C^{\otimes 4}"'] \arrow[d, "I"'] \arrow[dr, phantom, "\cong"] & C^o \otimes C \otimes C^o \otimes C \arrow[d, "p"] \\
I \arrow[r, "C\otimes C"'] & C^o \otimes C
\end{tikzcd}
=
\begin{tikzcd}
I \arrow[r, bend left, "A\otimes A"] \arrow[r, phantom, "\nu_2\Downarrow"] \arrow[d, "I"'] & C^o \otimes C \otimes C^o \otimes C \arrow[d, "p"] \\
I \arrow[r, bend left, "A"] \arrow[r, phantom, bend left=15, "\epsilon\Downarrow"] \arrow[r, "C\otimes C"'] & C^o \otimes C
\end{tikzcd}$$

It is easy to translate these equalities into commutative diagrams. The first of them translates to (15). The second translates to the commutativity of

$$\begin{tikzcd}
A \arrow[r, "\delta"] & A \otimes A \arrow[r, "s\otimes t"] & C \otimes C \\
H \arrow[u, "\nu_2"] \arrow[rr, "i"'] & & A \otimes A \arrow[u, "s\otimes t"']
\end{tikzcd} \tag{17}$$

This reduces to the commutativity of (17). Indeed, in the diagram

$$\begin{tikzcd}
A \arrow[r, "\delta_l"] & A \otimes C \otimes C \arrow[r, "\epsilon\otimes 1\otimes 1"] & C \otimes C \\
H \arrow[u, "\nu_2"] \arrow[r, "\delta_l"'] & H \otimes C \otimes C \arrow[u, "\nu_2\otimes 1\otimes 1"'] \arrow[r, "i\otimes 1\otimes 1"'] & A \otimes A \otimes C \otimes C \arrow[u, "\epsilon\otimes\epsilon\otimes 1\otimes 1"']
\end{tikzcd}$$

the left square commutes since ν_2 is a right $C^o \otimes C$-comodule map and the right square commutes given (16). A little calculation shows that the outer part is exactly (17). ∎

The 2-cell μ_0 is a map $\mu_0 : C \to C \otimes_{(C^o \otimes C)} A$. Let γ_r be the composite of this with the canonical injection $C \otimes_{(C^o \otimes C)} A \to C \otimes A$.

5.5. Lemma. *The 2-cell μ_0 is a comonad morphism if and only if the following diagrams commute*

$$\begin{tikzcd} C \arrow[r, "\nu_0"] \arrow[d, "\gamma_r"'] & A \arrow[d, "\delta"] \\ C \otimes A \arrow[r, "\nu_0 \otimes 1"'] & A \otimes A \end{tikzcd} \qquad (18)$$

$$\begin{tikzcd} & C \arrow[dl, "\nu_0"'] \arrow[dr, "\epsilon"] & \\ A \arrow[rr, "\epsilon"'] & & I \end{tikzcd} \qquad (19)$$

Proof. The map μ_0 is a comonad morphism if:

$$\begin{tikzcd} & & I \arrow[dll, "j"'] \arrow[drr, "j"] \arrow[dd, "j"'] & & \\ C^o \otimes C \arrow[drr, "A"'] \arrow[r, phantom, "\mu_0\Downarrow"] & {} & {} \arrow[r, phantom, "\mu_0\Downarrow"] & {} & C^o \otimes C \\ & & C^o \otimes C \arrow[urr, "A"'] & & \end{tikzcd} = \begin{tikzcd} & I \arrow[dl, "j"'] \arrow[dr, "j"] \arrow[d, phantom, "\mu_0\Downarrow"] & \\ C^o \otimes C \arrow[rr, "A"] \arrow[dr, "A"'] & {} \arrow[d, phantom, "\delta\Downarrow"] & C^o \otimes C \\ & C^o \otimes C \arrow[ur, "A"'] & \end{tikzcd}$$

and

$$\begin{tikzcd} & I \arrow[dl, "j"'] \arrow[dr, "j"] \arrow[d, phantom, "\mu_0\Downarrow"] & \\ C^o \otimes C \arrow[rr, "A"] \arrow[rr, bend right=50, "C \otimes C"'] & {} \arrow[d, phantom, "\epsilon\Downarrow"] & C^o \otimes C \\ & {} & \end{tikzcd} = \begin{tikzcd} & I \arrow[dl, "j"'] \arrow[dr, "j"] \arrow[d, phantom, "1_C\Downarrow"] & \\ C^o \otimes C \arrow[rr, bend right=50, "C \otimes C"'] & {} & C^o \otimes C \end{tikzcd}$$

Using Lemma 3.3 these are equivalent to the equalities:

$$\begin{tikzcd} & & I \arrow[dll, "I"'] \arrow[drr, "j"] \arrow[dd, "j"'] & & \\ I \arrow[drr, "A"'] \arrow[r, phantom, "\nu_0\Downarrow"] & {} & {} \arrow[r, phantom, "\mu_0\Downarrow"] & {} & C^o \otimes C \\ & & C^o \otimes C \arrow[urr, "A"'] & & \end{tikzcd} = \begin{tikzcd} & I \arrow[dl, "I"'] \arrow[dr, "j"] \arrow[d, phantom, "\nu_0\Downarrow"] & \\ I \arrow[rr, "A"] \arrow[dr, "A"'] & {} \arrow[d, phantom, "\delta\Downarrow"] & C^o \otimes C \\ & C^o \otimes C \arrow[ur, "A"'] & \end{tikzcd}$$

and

$$
\begin{tikzcd}
 & I \arrow[dl, "I"'] \arrow[dr, "j"] \arrow[d, phantom, "\nu_0\Downarrow"] & \\
I \arrow[rr, "A"] \arrow[rr, bend right=50, "C\otimes C"'] \arrow[rr, phantom, shift right=4, "\epsilon\Downarrow"] & {} & C^o\otimes C
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
 & I \arrow[dl, "I"'] \arrow[dr, "j"] & \\
I \arrow[rr, bend right=50, "C\otimes C"'] \arrow[rr, phantom, "1_C\Downarrow"] & & C^o\otimes C
\end{tikzcd}
$$

The first of these translates to the commutativity of (18). The second translates to

$$
\begin{tikzcd}
A \arrow[r, "\delta"] \arrow[d, "\nu_0"'] & A\otimes A \\
C \arrow[r, "\delta"'] & C\otimes C \arrow[u, "s\otimes t"']
\end{tikzcd}
\tag{20}
$$

Which reduces to the commutativity of (20). Indeed, in the diagram

$$
\begin{tikzcd}
A \arrow[r, "\delta_3"] & A\otimes A\otimes A \arrow[r, "s\otimes t"] & A\otimes C\otimes C \arrow[r, "\epsilon\otimes 1\otimes 1"] & C\otimes C \\
C \arrow[u, "\nu_0"] \arrow[r, "\delta_3"'] & C\otimes C\otimes C \arrow[r, "c\otimes 1"'] & C\otimes C\otimes C \arrow[u, "\nu_0\otimes 1\otimes 1"] \arrow[r, "\epsilon\otimes 1\otimes 1"'] & C\otimes C \arrow[u, "1"']
\end{tikzcd}
$$

the left square commutes since ν_0 is a right $C^o\otimes C$-comodule map, and the right square commutes given (19), while the outer part can be seen to be (20). ∎

Now we are in position to unpack Definition 5.2. Start again with a quantum graph (C, A). There is a unique map $\gamma_l : H \to A\otimes A\otimes H$ making

$$
\begin{tikzcd}
H \arrow[r, "i"] \arrow[d, "\gamma_l"'] & A\otimes A \arrow[r, "\delta\otimes\delta"] & A\otimes A\otimes A\otimes A \arrow[d, "1\otimes c\otimes 1"] \\
A\otimes A\otimes H \arrow[rr, "1\otimes 1\otimes i"'] & & A\otimes A\otimes A\otimes A
\end{tikzcd}
;
\tag{21}
$$

commute. By Lemma 3.1 the 2-cell μ_2 is determined by the map ν_2. The condition of 3.1 says that ν_2 should respects the right coaction by $C^o\otimes C$ and the left coactions by the first and the fourth terms in $C^o\otimes C\otimes C^o\otimes C$ and satisfy (6), which now becomes

$$
\begin{tikzcd}
H \arrow[r, "\gamma_l"] & A\otimes A\otimes H \arrow[r, shift left, "t\otimes\epsilon\otimes 1"] \arrow[r, shift right, "\epsilon\otimes s\otimes 1"'] & C\otimes H \arrow[r, "1\otimes\nu_2"] & C\otimes A.
\end{tikzcd}
\tag{22}
$$

Using (22), it can be shown that there exists a unique map γ_r making

$$
\begin{tikzcd}
H \arrow[r, "\delta_l"] \arrow[d, "\gamma_r"'] & A\otimes A\otimes H \arrow[d, "1\otimes 1\otimes\nu_2"] \\
H\otimes A \arrow[r, "i\otimes 1"'] & A\otimes A\otimes A
\end{tikzcd}
\tag{23}
$$

commute. This is the same as the map defined before by (14). Observe that commutativity of the diagram (15) in Lemma 5.4 implies that ν_2 respects the left coaction by the first and the fourth terms in $C^o \otimes C \otimes C^o \otimes C$.

By Lemma (3.2) the 2-cell μ_0 is determined by the map ν_0. The condition of (3.2) says that ν_0 should respect the right $C^o \otimes C$ coaction and satisfy (7), which now becomes

$$C \xrightarrow{\nu_0} A \underset{(t\otimes 1)\delta}{\overset{(s\otimes 1)\delta}{\rightrightarrows}} C \otimes A. \tag{24}$$

The common value of the two composites in (24) is the same as the map γ_r defined above. Observe that if (A, ν_2, ν_0) is a comonoid in $\mathcal{C}(C, C)$, then (24) follows from (22).

Assembling the established facts we obtain:

5.6. STATEMENT. [B. Day, R. Street] *Giving a quantum category structure on a quantum graph* (C, A) *is equivalent to giving maps* $\nu_2 : H \to A$ *and* $\nu_0 : C \to A$ *satisfying the following axioms:*

Axiom 1: (A, ν_2, ν_0) *is a monoid in* $\mathcal{C}(C, C)$.
Axiom 2: The diagram (22), *in which the map* γ_l *is defined by* (21).
Axiom 3: The diagram (15), *in which the map* γ_r *is defined by* (23) *using Axiom 2.*
Axiom 4: The diagram (16).
Axiom 5: The diagram (18), *in which the map* γ_r *is defined by* (24).
Axiom 6: The diagram (19).

For more clarity see the Appendix where these axioms are presented using string diagrams.

By Section 4, a quantum category structure on a quantum graph (C, A) is the same as a monoidale structure on A, such that $\epsilon : A \to C^o \otimes C$ is strong monoidal. In term of our data this monoidal structure on A can be expressed as follows. The multiplication $A \otimes A \relbar\joinrel\mapstochar\joinrel\rightarrow A$ is H with left and right coactions the maps $\gamma_l : H \to A \otimes A \otimes H$ and $\gamma_r : H \to H \otimes A$. The unit $I \relbar\joinrel\mapstochar\joinrel\rightarrow A$ is C with the right coaction the map $\gamma_r : C \to C \otimes A$. The monoidale A is the Eilenberg-Moore object of the comonad $A : C^o \otimes C \relbar\joinrel\mapstochar\joinrel\rightarrow C^o \otimes C$ in MonComod$(\mathcal{V})$. Applying the representable pseudofunctor Mon$\mathcal{C}(I, -)$: Mon$\mathcal{C} \to$ MonCat to the universal 2-cell

$$\begin{tikzcd} & A \arrow[dl, "\epsilon_*"', "|" marking] \arrow[dr, "\epsilon_*", "|" marking] \arrow[d, phantom, "\Downarrow"] & \\ C^o \otimes C \arrow[rr, "A"', "|" marking] & {} & C^o \otimes C \end{tikzcd}$$

we obtain an Eilenberg-Moore construction in the category of monoidal categories and monoidal functors:

$$\begin{tikzcd}
 & \mathrm{Comod}(I,A) \arrow[dl] \arrow[dr] \arrow[d, phantom, "\Downarrow"] & \\
\mathcal{C}(I,C^o\otimes C) \arrow[rr, "A\circ -"'] & & \mathcal{C}(I,C^o\otimes C)
\end{tikzcd}$$

Using the equivalence (4) we can transport the monoidal comonad $A\circ -$ on the category $\mathcal{C}(I,C^o\otimes C)$ to a monoidal comonad on the category $\mathcal{C}(C,C)$. Thus, a quantum category defines a monoidal comonad on $\mathcal{C}(C,C)$, the Eilenberg-Moore object of which is the category of the right A-comodules.

6. The category of quantum categories

Suppose that $f : C \to C'$ is a comonoid map. Let $\alpha : f^* \circ f_* \to C$ and $\beta : C' \to f_* \circ f^*$ be the unit and the counit of the adjunction $f^* \dashv f_*$ as in Section 3. They are maps respectively in $\mathcal{C}(C,C)$ and $\mathcal{C}(C',C')$. By biduality, from α we get a map $\alpha : e \longrightarrow e\circ (f_*\otimes f^{*o})$ in $\mathcal{C}(C\otimes C^o, I)$ and from β we get a map $\beta : (f_*\otimes f^{*o})\circ n \Rightarrow n$ in $\mathcal{C}(I, C'^o\otimes C')$.

The comodule $f^{*o}\otimes f_* : C^o\otimes C \longrightarrow C'^o\otimes C'$ is an opmonoidal morphism with structure 2-cells

$$\begin{tikzcd}
C^o\otimes C \arrow[r, "f^{*o}\otimes f_*"] \arrow[dr, phantom, "\omega_2\Downarrow"] & C'^o\otimes C' \\
C^o\otimes C\otimes C^o\otimes C \arrow[u, "p"] \arrow[r, "f^{*o}\otimes f_*\otimes f^{*o}\otimes f_*"'] & C'^o\otimes C'\otimes C'^o\otimes C' \arrow[u, "p"']
\end{tikzcd}
\qquad
\begin{tikzcd}
C^o\otimes C \arrow[rr, "f^{*o}\otimes f_*"] & \arrow[d, phantom, "\omega_0\Downarrow"] & C'^o\otimes C' \\
 & I \arrow[ul, "j"] \arrow[ur, "j"'] &
\end{tikzcd}$$

defined as $\omega_2 = f^{*o}\otimes\alpha\otimes f_*$ and $\omega_0 = \beta$.

Suppose that $g : C \to C'$ is another comonoid map. The opmonoidal map $f^{*o}\otimes f_*$ acts from the left on $f^{*o}\otimes g_*$ by

$$\begin{tikzcd}
C^o\otimes C \arrow[r, "f^{*o}\otimes g_*"] \arrow[dr, phantom, "\lambda_l\Downarrow"] & C'^o\otimes C' \\
C^o\otimes C\otimes C^o\otimes C \arrow[u, "p"] \arrow[r, "f^{*o}\otimes f_*\otimes f^{*o}\otimes g_*"'] & C'^o\otimes C'\otimes C'^o\otimes C' \arrow[u, "p"']
\end{tikzcd}$$

defined as $\lambda_l = f^{*o}\otimes\alpha\otimes g_*$. Similarly $g^{*o}\otimes g_*$ acts from the right on $f^{*o}\otimes g_*$ with coaction 2-cell λ_r defined similarly.

6.1. Definition. *A map between quantum graphs $(\sigma, f) : (C,A)\to (C',A')$ consists of a morphism of comonoids $f : C\to C'$ and a 2-cell*

$$\begin{tikzcd}
C^o\otimes C \arrow[r, "A"] \arrow[d, "f^{*o}\otimes f_*"'] \arrow[dr, phantom, "\sigma\Downarrow"] & C^o\otimes C \arrow[d, "f^{*o}\otimes f_*"] \\
C'^o\otimes C' \arrow[r, "A'"'] & C'^o\otimes C
\end{tikzcd} \tag{25}$$

such that $(f^{*o} \otimes f_*, \sigma)$ *is a comonad map.*

6.2. Definition. *A functor* $(f, \sigma) : (C, A) \to (C', A')$ *between quantum categories is a map between the underlying quantum graphs such that the 2-cell* σ *is a square.*

In other words a quantum functor is an opmorphism of monoidal comonads of the form $(f^{*o} \otimes f_*, \varphi) : (C, A) \to (C', A')$. Here are the equalities that σ must satisfy:

$$
\begin{tikzcd}
 & C^o \otimes C \otimes C^o \otimes C \arrow[rr, "A \otimes A"] \arrow[dl, "f^{*o}\otimes f_*\otimes f^{*o}\otimes f_*"'] \arrow[rr, phantom, bend right=40, "\sigma\otimes\sigma \Downarrow"] & & C^o \otimes C \otimes C^o \otimes C \arrow[dl, "f^{*o}\otimes f_*\otimes f^{*o}\otimes f_*"'] \arrow[dr, "p"] & \\
C'^o \otimes C' \otimes C'^o \otimes C' \arrow[rr, "A' \otimes A'"'] \arrow[dr, "p"] & & C'^o \otimes C' \otimes C'^o \otimes C' \arrow[dr, "p"] \arrow[dl, phantom, "\mu_2 \Downarrow"] \arrow[rr, phantom, "\omega_2 \Swarrow"] & & C^o \otimes C \arrow[dl, "f^{*o}\otimes f_*"] \\
 & C'^o \otimes C' \arrow[rr, "A'"'] & & C'^o \otimes C' &
\end{tikzcd}
$$

$$=$$

$$
\begin{tikzcd}
 & C^o \otimes C \otimes C^o \otimes C \arrow[rr, "A \otimes A"] \arrow[dl, "f^{*o}\otimes f_*\otimes f^{*o}\otimes f_*"'] \arrow[dr, "p"] & & C^o \otimes C \otimes C^o \otimes C \arrow[dr, "p"] \arrow[dl, phantom, "\mu_2 \Downarrow"] & \\
C'^o \otimes C' \otimes C'^o \otimes C' \arrow[dr, "p"] \arrow[rr, phantom, "\omega_2 \Swarrow"] & & C^o \otimes C \arrow[rr, "A"'] \arrow[dl, "f^{*o}\otimes f_*"] \arrow[dr, phantom, "\sigma \Downarrow"] & & C^o \otimes C \arrow[dl, "f^{*o}\otimes f_*"] \\
 & C'^o \otimes C' \arrow[rr, "A'"'] & & C'^o \otimes C' &
\end{tikzcd}
$$

and

$$
\begin{tikzcd}
I \arrow[dr, "j"] \arrow[drr, "j"] \arrow[ddr, "j"'] \arrow[dr, phantom, shift left=3, "\eta\Downarrow"] \arrow[ddr, phantom, shift left=2, "\omega_0\Swarrow"] & & \\
& C^o \otimes C \arrow[r, "A"] \arrow[d, "f^{*o}\otimes f_*"'] \arrow[dr, phantom, "\sigma\Downarrow"] & C^o \otimes C \arrow[d, "f^{*o}\otimes f_*"] \\
& C'^o \otimes C' \arrow[r, "A'"'] & C'^o \otimes C'
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
I \arrow[rr, "j"] \arrow[dr, "j"'] \arrow[drr, "j"] \arrow[r, phantom, shift right=6, "\omega_0\Downarrow"] \arrow[d, phantom, shift left=10, "\eta\Downarrow"] & & C^o \otimes C \arrow[d, "f^{*o}\otimes f_*"] \\
& C'^o \otimes C' \arrow[r, "A'"'] & C'^o \otimes C'
\end{tikzcd}
$$

A map of quantum graphs $(C, A) \to (C', A)$ amounts to comonoid maps $f : C \to C'$ and $\varphi : A \to A'$ for which the diagrams

$$
\begin{tikzcd}
A \arrow[r, "s"] \arrow[d, "\varphi"'] & C \arrow[d, "f"] \\
A' \arrow[r, "s"'] & C'
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
A \arrow[r, "t"] \arrow[d, "\varphi"'] & C \arrow[d, "f"] \\
A' \arrow[r, "t"'] & C'
\end{tikzcd}
\tag{26}
$$

commute. The pair (f, φ) is a functor of quantum categories if additionally it satisfies:

$$
\begin{tikzcd}
& A \otimes_C A \arrow[r, "\nu_2"] \arrow[dl, "i"'] & A \arrow[dd, "\varphi"] \\
A \otimes_{C'} A \arrow[dr, "\varphi\otimes_{C'}\varphi"'] & & \\
& A' \otimes_{C'} A' \arrow[r, "\nu_2"'] & A'
\end{tikzcd}
$$

and

$$
\begin{tikzcd}
C \arrow[r, "\nu_0"] \arrow[d, "f"'] & A \arrow[d, "\varphi"] \\
C' \arrow[r, "\nu_0"'] & A'
\end{tikzcd}
$$

The tensor product $A \otimes_{C'} A$ of A with itself over C' is taken by regarding A as a comodule $C' \longrightarrow\!\!\!\!\!+\; C'$ with left and right coactions:

$$A \xrightarrow{\delta} A \otimes A \xrightarrow{1\otimes f} A \otimes A' \xrightarrow{1\otimes s} A \otimes C' \xrightarrow{c} C' \otimes A$$

$$A \xrightarrow{\delta} A \otimes A \xrightarrow{1\otimes f} A \otimes A' \xrightarrow{1\otimes s} A \otimes C'.$$

Observe that in a quantum functor the map f is determined by the map φ via $f = \varphi\nu_0$.

The notion of the quantum functor includes the notion of functor between small categories and the notion of weak morphism of bialgebroids [16].

By Section 2 an opmorphism between monoidal comonads is determined by an opmonoidal morphism between the Eilenberg-Moore objects. Thus given a quantum functor $(f, \varphi) : (C, A) \to (C', A')$ the comodule $\varphi_* : A \longrightarrow\!\!\!\!\!\!\!+\;\; A'$ has an opmonoidal morphism structure which lifts the opmonoidal morphism structure on $f^{*o} \otimes f^o_* : C^o \otimes C \longrightarrow\!\!\!\!\!\!\!+\;\; C^o \otimes C$. By application of the functor $\mathrm{MonComod}(I, -) : \mathrm{MonComod} \to \mathrm{Mon}\mathcal{C}at$ we get an opmonoidal functor between categories of right A-comodules:

$$\mathrm{Comod}(I, A) \xrightarrow{\mathrm{MonComod}(I,\varphi_*)} \mathrm{Comod}(I, A').$$

Define composition of quantum functors $\varphi : (C, A) \to (C', A')$ and $\varphi' : (C', A') \to (C'', A'')$ by

$$A \xrightarrow{\varphi} A' \xrightarrow{\varphi'} A''.$$

The units for this composition are provided by quantum functors of the form $(1, 1)$.

6.3. Theorem. *Quantum categories and functors between them form a category* $\mathrm{qCat}\mathcal{V}$.

6.4. Definition. *A natural transformation* $\tau : (f, \varphi) \Rightarrow (g, \varphi')$

$$\begin{tikzcd} (A, C) \arrow[r, bend left, "{(f,\varphi)}"] \arrow[r, bend right, "{(g,\varphi')}"'] \arrow[r, phantom, "\tau\Downarrow"] & (A', C') \end{tikzcd}$$

between quantum functors is a 2-cell

$$\begin{tikzcd} C^o \otimes C \arrow[r, "A", "+"'] \arrow[d, "f^{*\otimes o}\otimes g_*"', "+"] \arrow[dr, phantom, "\tau\Downarrow"] & C^o \otimes C \arrow[d, "f^{*\otimes o}\otimes g_*", "+"'] \\ C'^o \otimes C' \arrow[r, "A'"', "+"] & C'^o \otimes C' \end{tikzcd}$$

making $f^{*o} \otimes g_*$ into a comonad map so that both the left coaction of $f^{*o} \otimes f_*$ and the right coaction of $g^{*o} \otimes g_*$ on $f^{*o} \otimes g_*$ respect the comonad structure.

Here are the equalities which the 2-cell τ should satisfy:

$$
\begin{tikzcd}
 & C^o\otimes C\otimes C^o\otimes C \arrow[dl, "f^{*o}\otimes f_*\otimes f^{*o}\otimes f_*"'] \arrow[rr, "A\otimes A"] & {} \arrow[d, phantom, "\sigma\otimes\tau\Downarrow"] & C^o\otimes C\otimes C^o\otimes C \arrow[dl, "f^{*o}\otimes f_*\otimes f^{*o}\otimes f_*"'] \arrow[dr, "p"] & \\
C'^o\otimes C'\otimes C'^o\otimes C' \arrow[rr, "A'\otimes A'"'] \arrow[dr, "p"] & & C'^o\otimes C'\otimes C'^o\otimes C' \arrow[dr, "p"] \arrow[d, phantom, "\mu_2\Downarrow"] & {} \arrow[r, phantom, "\lambda_l\swarrow"] & C^o\otimes C \arrow[dl, "f^{*o}\otimes f_*"] \\
 & C'^o\otimes C' \arrow[rr, "A'"'] & {} & C'^o\otimes C' &
\end{tikzcd}
$$

$$=$$

$$
\begin{tikzcd}
 & C^o\otimes C\otimes C^o\otimes C \arrow[dl, "f^{*o}\otimes f_*\otimes f^{*o}\otimes g_*"'] \arrow[dr, "p"] \arrow[rr, "A\otimes A"] & {} \arrow[d, phantom, "\mu_2\Downarrow"] & C^o\otimes C\otimes C^o\otimes C \arrow[dr, "p"] & \\
C'^o\otimes C'\otimes C'^o\otimes C' \arrow[dr, "p"] \arrow[r, phantom, "\lambda_l\swarrow"] & {} & C^o\otimes C \arrow[dl, "f^{*o}\otimes g_*"] \arrow[rr, "A"'] & {} \arrow[d, phantom, "\tau\Downarrow"] & C^o\otimes C \arrow[dl, "f^{*o}\otimes g_*"] \\
 & C'^o\otimes C' \arrow[rr, "A'"'] & {} & C'^o\otimes C' &
\end{tikzcd}
$$

and

$$
\begin{tikzcd}
 & C^o\otimes C\otimes C^o\otimes C \arrow[dl, "f^{*o}\otimes g_*\otimes g^{*o}\otimes g_*"'] \arrow[rr, "A\otimes A"] & {} \arrow[d, phantom, "\tau\otimes\sigma'\Downarrow"] & C^o\otimes C\otimes C^o\otimes C \arrow[dl, "f^{*o}\otimes g_*\otimes g^{*o}\otimes g_*"'] \arrow[dr, "p"] & \\
C'^o\otimes C'\otimes C'^o\otimes C' \arrow[rr, "A'\otimes A'"'] \arrow[dr, "p"] & & C'^o\otimes C'\otimes C'^o\otimes C' \arrow[dr, "p"] \arrow[d, phantom, "\mu_2\Downarrow"] & {} \arrow[r, phantom, "\lambda_r\swarrow"] & C^o\otimes C \arrow[dl, "f^{*o}\otimes g_*"] \\
 & C'^o\otimes C' \arrow[rr, "A'"'] & {} & C'^o\otimes C' &
\end{tikzcd}
$$

$$=$$

$$\begin{tikzcd}
 & C^o \otimes C \otimes C^o \otimes C \arrow[rr, "A\otimes A", "|" marking] \arrow[dl, "f^{*o}\otimes g_*\otimes g^{*o}\otimes g_*"'] \arrow[dr, "p", "|" marking] \arrow[drr, phantom, "\mu_2\Downarrow"] & & C^o \otimes C \otimes C^o \otimes C \arrow[dr, "p", "|" marking] & \\
C'^o \otimes C' \otimes C'^o \otimes C' \arrow[dr, "p", "|" marking] \arrow[rr, phantom, "\lambda_r \Swarrow"] & & C^o \otimes C \arrow[rr, "A"', "|" marking] \arrow[dl, "f^{*o}\otimes g_*", "|" marking] \arrow[r, phantom, "\tau\Downarrow"', shift right=8] & & C^o \otimes C \arrow[dl, "f^{*o}\otimes g_*", "|" marking] \\
 & C'^o \otimes C' \arrow[rr, "A'"', "|" marking] & & C'^o \otimes C' &
\end{tikzcd}$$

A natural transformation $\tau : (f, \varphi) \to (g, \varphi') : (C, A) \to (C', A')$ amounts to a comonoid map $\tau : A \to A'$ such that:

$$\begin{tikzcd}
A \arrow[r, "s"] \arrow[d, "\tau"'] & C \arrow[d, "f"] \\
A' \arrow[r, "s"'] & C'
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
A \arrow[r, "t"] \arrow[d, "\tau"'] & C \arrow[d, "g"] \\
A' \arrow[r, "t"'] & C'
\end{tikzcd}
\tag{27}$$

$$\begin{tikzcd}
 & A' \otimes_C A' \arrow[rr, "i"] & & A' \otimes_{C'} A' \arrow[dr, "\nu_2"] & \\
A \otimes_C A \arrow[ur, "\varphi\otimes_C\tau"] \arrow[rr, "\nu_2"] \arrow[dr, "\tau\otimes_C\varphi"'] & & A \arrow[rr, "\tau"] & & A' \\
 & A' \otimes_C A' \arrow[rr, "i"'] & & A' \otimes'_C A' \arrow[ur, "\nu_2"'] &
\end{tikzcd}$$

7. Constructions on quantum categories

A coreflexive-equalizer-preserving braided strong-monoidal functor $\mathcal{V} \to \mathcal{W}$ defines a functor between the categories of quantum categories $\mathrm{qCat}\mathcal{V} \to \mathrm{qCat}\mathcal{W}$. Thus qCat can be viewed as a 2-functor from the 2-category of braided monoidal categories (satisfying the condition at the beginning of Section 3) and braided strong monoidal functors to the 2-category of categories. This functor preserves finite products since we have isomorphisms

$$\mathrm{qCat}(\mathcal{V} \times \mathcal{W}) \cong \mathrm{qCat}\mathcal{V} \times \mathrm{qCat}\mathcal{W}$$

$$\mathrm{qCat}(1) \cong 1$$

7.1. Example. When $\mathcal{V}$ is a symmetric monoidal category, then the functors $-\otimes- : \mathcal{V}\times\mathcal{V}\to\mathcal{V}$, $I : 1\to\mathcal{V}$ are symmetric monoidal. From them we obtain functors

$$-\otimes- : \mathrm{qCat}\mathcal{V}\times\mathrm{qCat}\mathcal{V}\to\mathrm{qCat}\mathcal{V}$$

$$I : 1\to\mathrm{qCat}\mathcal{V}$$

defining a monoidal structure on $\mathrm{qCat}\mathcal{V}$.

7.2. Example. There is a functor $Set\to\mathcal{V}$, taking a set S to the S-fold coproduct $S\cdot I$ of the monoidal unit, provided these copowers exist. When a certain distributivity law is satisfied, this functor is strong monoidal. Any coreflexive equalizer in Set (which does not involve an empty set) is split, and thus preserved by any functor. We have $\mathrm{qCat}Set = Cat$ [5]. So we get a functor:

$$Cat\to\mathrm{qCat}\mathcal{V}$$

To wit any category determines a quantum category in any (sufficiently good) monoidal category.

7.3. Example. Suppose that $\mathcal{V}$ has small coproducts, and assume that each of the functors $X+-$ preserves coreflexive equalizers. For any finite set S let $S\cdot V$ stand for the S-fold coproduct of an object V of $\mathcal{V}$. There is a coreflexive-equalizer-preserving braided strong-monoidal functor $-\cdot- : Set_f\times\mathcal{V}\to\mathcal{V}$. The preservation of coreflexive equalizers is due to Lemma 0.17 in [6]. We have $\mathrm{qCat}Set_f = Cat_f$. Thus we obtain a functor

$$Cat_f\times\mathrm{qCat}\mathcal{V}\to\mathrm{qCat}\mathcal{V} \tag{28}$$

Let $i : \Delta\to Cat_f$ be the canonical embedding of the simplicial category into the category of finite categories. Precomposing (28) with $i\times 1$ we obtain a functor

$$\Delta\times\mathrm{qCat}\mathcal{V}\to\mathrm{qCat}\mathcal{V}$$

Let A and B be quantum categories in $\mathcal{V}$. Consider the simplicial set

$$\mathrm{qCat}\mathcal{V}(-\cdot A, B). \tag{29}$$

The 1-simplexes of this simplicial set are the quantum functors from A to B. The 2-simplexes are the quantum natural transformation. For quantum categories A, B and C, there is a map of simplicial sets

$$\mathrm{qCat}\mathcal{V}(-\cdot A, B)\times\mathrm{qCat}\mathcal{V}(-\cdot B, C)\to\mathrm{qCat}\mathcal{V}(-\cdot A, C) \tag{30}$$

defined on the n-simplexes by

$$\mathrm{qCat}\mathcal{V}(n\cdot A, B)\times\mathrm{qCat}\mathcal{V}(n\cdot B, C)\xrightarrow{(n\cdot-)\times 1}\mathrm{qCat}\mathcal{V}(n\cdot n\cdot A, n\cdot B)\times\mathrm{qCat}\mathcal{V}(n\cdot B, C)$$

$$\xrightarrow{\mathrm{qCat}\mathcal{V}(\delta_{(n,A)},1)\times 1} \mathrm{qCat}\mathcal{V}(n\cdot A, n\cdot B)\times \mathrm{qCat}\mathcal{V}(n\cdot B, C) \xrightarrow{\mathrm{comp}} \mathrm{qCat}\mathcal{V}(n\cdot A, C).$$

Here δ is the natural transformation

$$\begin{tikzcd} Cat \times \mathrm{qCat}\mathcal{V} \arrow[rr, "\mathrm{diag}\times 1"] \arrow[dr, "-\cdot-"'] & \arrow[d, phantom, "\delta\Downarrow"] & Cat \times Cat \times \mathrm{qCat}\mathcal{V} \arrow[dl, "-\cdot-\cdot-"] \\ & \mathrm{qCat}\mathcal{V} & \end{tikzcd}$$

obtained by applying qCat$-$ to the obvious natural transformation

$$\begin{tikzcd} Set \times \mathcal{V} \arrow[rr, "\mathrm{diag}\times 1"] \arrow[dr, "-\cdot-"'] & \arrow[d, phantom, "\delta\Downarrow"] & Set \times Set \times \mathcal{V} \arrow[dl, "-\cdot-\cdot-"] \\ & \mathcal{V} & \end{tikzcd}$$

Using (30) as compositions we can construct a simplicial set enriched category with objects the quantum categories in $\mathcal{V}$ and a hom simplicial set from A to B given by (29).

7.4. Theorem. *For any (sufficiently good) braided monoidal category $\mathcal{V}$ the above defines a simplicial set enriched category* qCat$\mathcal{V}$*.*

7.5. Example. Consider the category of families Fam$\mathcal{V}$. An object of Fam$\mathcal{V}$ is a pair $(S,\{A_s\})$, where S is a set and $\{A_s\}$ is an S indexed family of objects of $\mathcal{V}$. A morphism $(f,\{\varphi_s\}) : (S,\{A_s\}) \to (S',(A'_s))$ consists of a map $f : S \to S'$ and for each s in S a morphism $\varphi_s : A_s \to A_{f(s)}$ in $\mathcal{V}$. The monoidal structure on $\mathcal{V}$ induces a monoidal structure on Fam$\mathcal{V}$ in the obvious way. Consider the functor

$$\mathrm{Fam}\mathcal{V} \to \mathcal{V}.$$

taking $(S,\{A_s\})$ to the coproduct $\coprod A_s$. This functor is monoidal and under mild conditions on $\mathcal{V}$, it preserves coreflexive equalizers. By applying the qCat we obtain a functor

$$\mathrm{qCatFam}\mathcal{V} \to \mathrm{qCat}\mathcal{V}.$$

The functor

$$\mathrm{Fam}\mathcal{V} \to Set$$

taking $(S,\{A_s\})$ to the set S determines a functor

$$\mathrm{qCatFam}\mathcal{V} \to Cat. \tag{31}$$

In this way each object in qCatFam$\mathcal{V}$ has an underlying category.

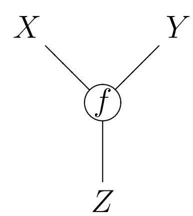
Next we show how Hopf group coalgebras introduced in [18] are quantum categories (groupoids [5]).

Let G denote a group. A Hopf G-coalgebra consists of a family of algebras $\{A_g\}$ indexed by elements g of G together with a family of linear maps $\{A_{gg'} \to A_g \otimes A_{g'}\}$ and an antipode satisfying certain axioms. Such a Hopf group coalgebra without an antipode is an object of qCatFam$Vect^{\mathrm{op}}$ with the object of objects $(1, \{I\})$ and the object of arrows $(G, \{A_g\})$. The underlying category is the group G. Using the functor (31) from a Hopf group coalgebra we obtain a quantum category in $Vect^{\mathrm{op}}$, in other words a bialgebroid. The antipode would make this bialgebroid into a Hopf algebroid.

8. Appendix: Computations for quantum categories in string diagrams

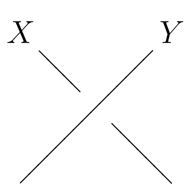
We introduce framed string diagrams to represent morphisms in a braided monoidal bicategory as an enrichment of the string diagrams of [7]. These diagrams are designed to ease presentation of quantum structures.

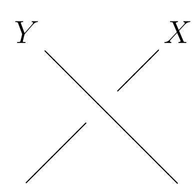
A string diagram of [7] has edges labeled by objects of $\mathcal{V}$ and nodes labeled by morphisms of $\mathcal{V}$ and represents a morphism in $\mathcal{V}$. For example a morphism $f : X \otimes Y \to Z$ in $\mathcal{V}$ is represented by a string diagram:



The identity morphism on an object X is represented by

The braiding isomorphisms c and c^{-1} are represented respectively by



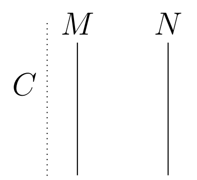


Composition is by concatenation and tensoring is by juxtaposition.

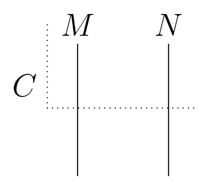
A framed string diagram besides strings and nodes may have framed regions labeled by comonoids in $\mathcal{V}$. A framed region labeled by a comonoid C has two strings passing through it, of which, the left string is labeled by a right C-comodule and the right string is labeled by a left C-comodule. Such a framed region corresponds to taking tensor product over C. Now we give the description.

Suppose that C is a comonoid in $\mathcal{V}$. Suppose that M is a right C-comodule and N is a left C-comodule with coactions:

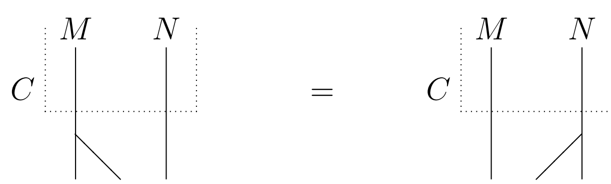
The framed string diagram



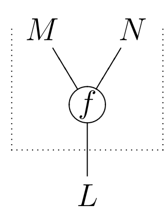
represents the identity morphism on $M \otimes_C N$. The framed string diagram



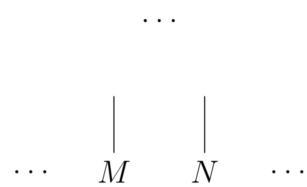
represents a canonical injection $i : M \otimes_C N \to M \otimes N$. Observe that we have:



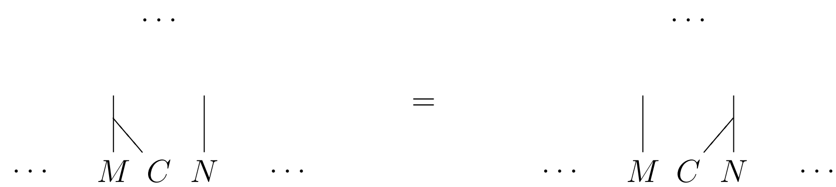
The framed string diagram



represents a morphisms $M \otimes_C N \to L$. The following is a rule for building a new string diagram from a given one. Suppose that a morphism $\ldots \to \ldots M \otimes N \ldots$ is represented by a framed string diagram

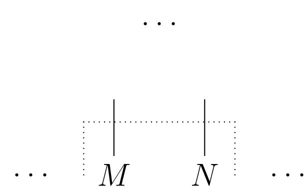


and suppose that it factors through a morphism $\ldots \to \ldots M \otimes_C N \ldots$; that is, the equality

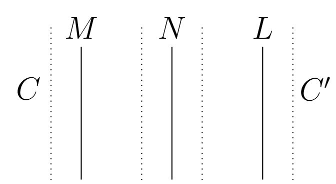


(32)

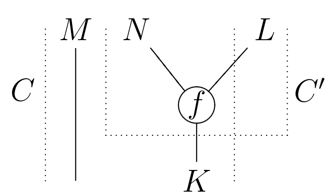
holds, then the latter morphism is represented by

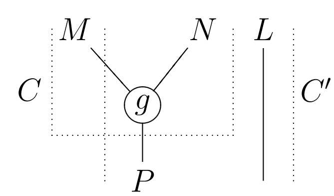


Thus every time we want to introduce a new frame in a string diagram using this rule an extra computation establishing (32) should be performed. We also consider overlapping of framed regions. If M is a right C-comodule, N is a comodule from C to C' and L is a left comodule, then



represents the identity map on $M \otimes_C N \otimes_{C'} L$. If $f : N \otimes_{C'} L \to K$ is a left C-comodule map and $g : M \otimes_C N \to P$ is a right C'-comodule map, then the framed string diagrams





represent $1 \otimes_C f : M \otimes_C N \otimes_{C'} L \to M \otimes_{C'} K$ and $g \otimes_{C'} 1 : M \otimes_C N \otimes_{C'} L \to P \otimes_{C'} L$ respectively.

A quantum graph consists of comonoids C and A, for comultiplications and counits of which we write

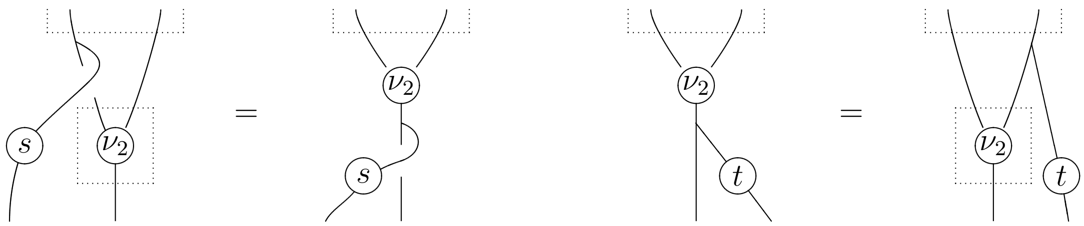
and comonoid maps $s : A \to C$, $t : A \to C$ related by

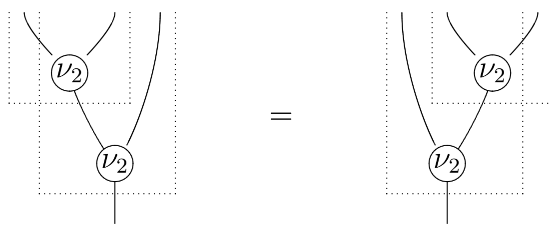
A quantum category consists of a quantum graph together with the composition map $\nu_2 : A \otimes_C A \to A$ and the identity map $\nu_0 : C \to A$:

which satisfy the six axioms in Statement 5.6. Below we quickly go through all of these axioms using framed string diagrams.
A is regarded as a left and right C-comodule by coactions:

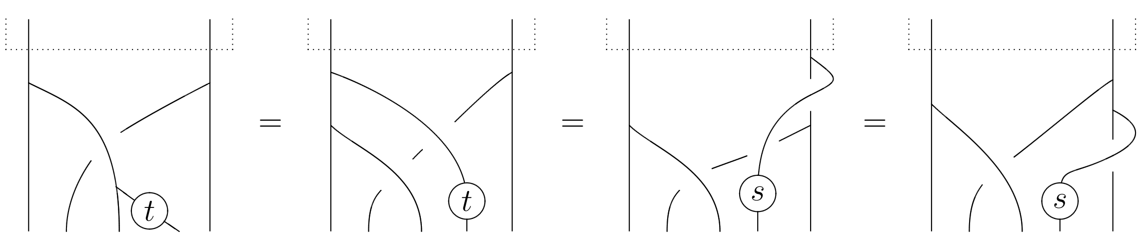
The tensor product $H = A \otimes_C A$ of A with itself over C is a left and a right C-comodule by coaction:

Axiom 1 says that (A, ν_2, ν_0) should be a monoid in Comod$\mathcal{V}(C, C)$. This means that the following conditions should be satisfied.

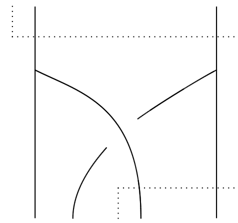




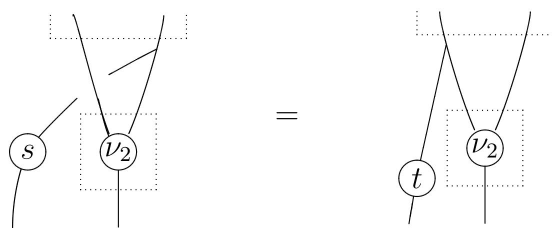
We have:



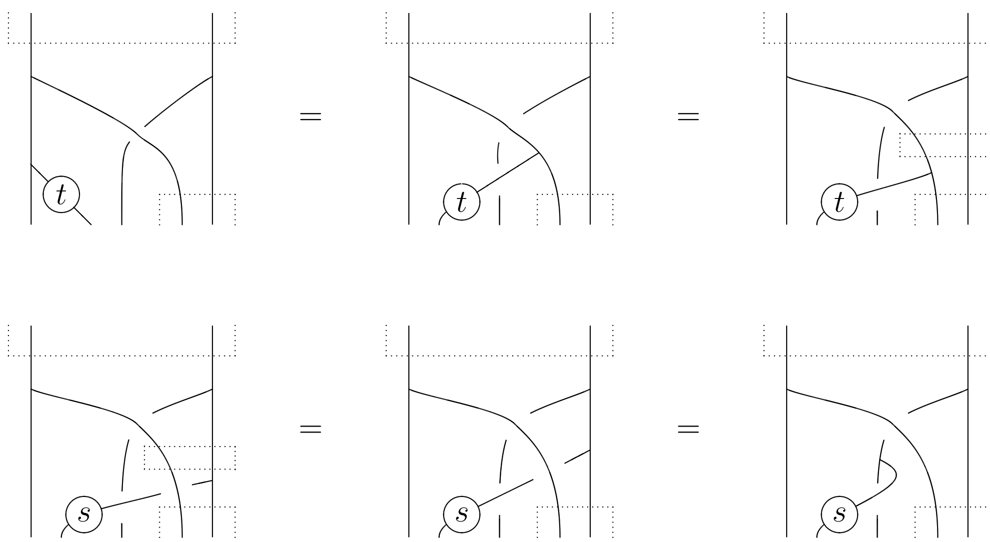
Therefore we can form a framed string diagram



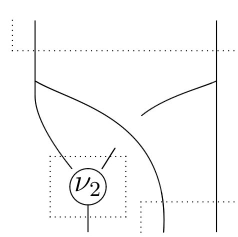
This is the map $\gamma_l : H \to A \otimes A \otimes H$. Axiom 2 is:



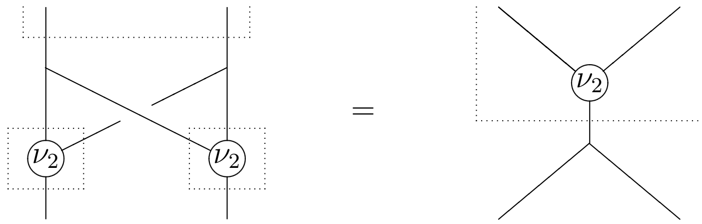
Using Axiom 2 we have:



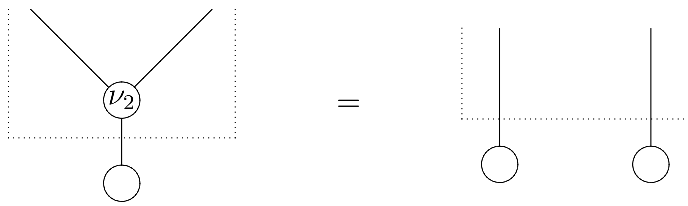
Thus, we can form a framed string diagram



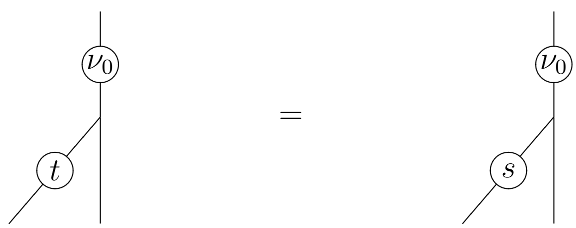
This is the map $\gamma_r : H \to H \otimes A$. Axiom 3:



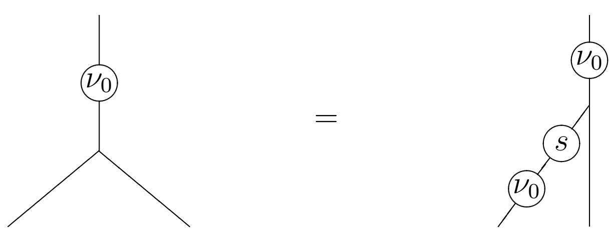
Axiom 4:



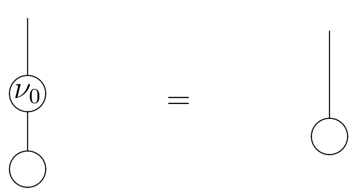
Note that these string diagrams look exactly like the string diagrams for two of the bialgebroid axioms. The map $\gamma_r : C \to C \otimes A$ is either of:



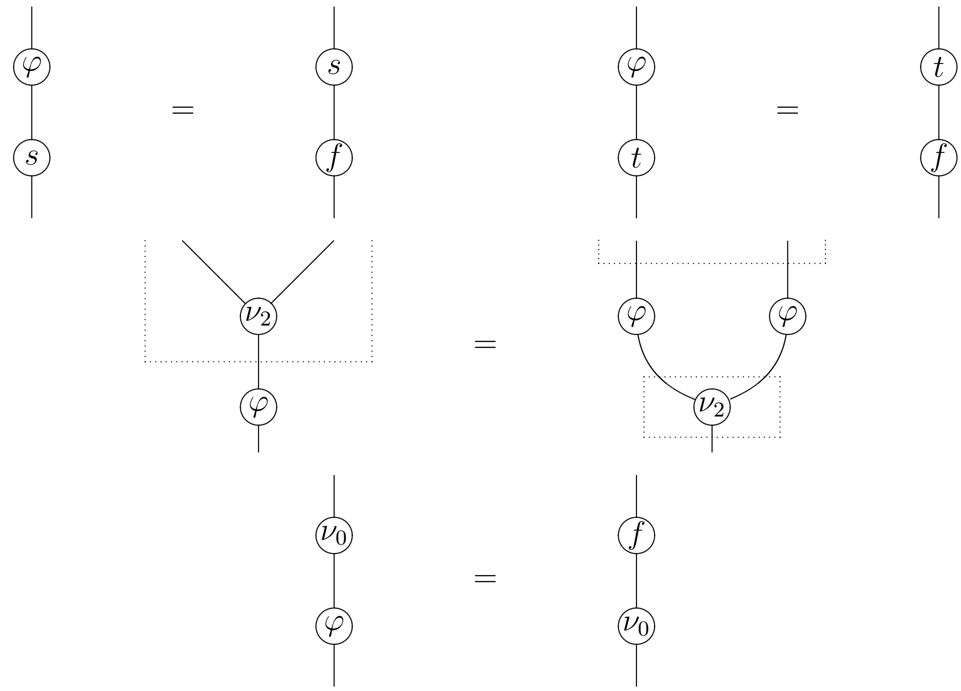
Axiom 5:



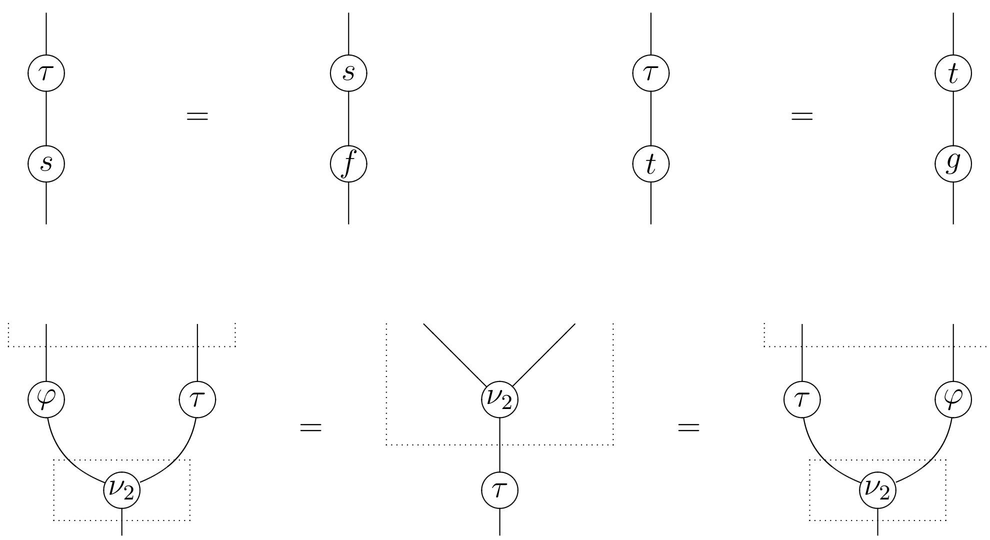
Axiom 6:



Axioms for a quantum functor $(f, \varphi) : (A, C) \to (A', C')$ in framed string diagrams are:



Axioms for a quantum natural transformation $\tau : (f, \varphi) \to (g, \varphi')$ in framed string diagrams are:



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