EVERY 2-SEGAL SPACE IS UNITAL

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Introduction

2-Segal spaces were introduced by Dyckerhoff and Kapranov [1] for applications in representation theory, homological algebra, and geometry, motivated in particular by Waldhausen's S-construction and Hall algebras. A 2-Segal space is a simplicial space X such that for every triangulation T of every convex plane n-gon (for $n \geqslant 2$), we have $X_n \simeq \lim_{t \in T} X(t)$. Independently, a little later, Gálvez-Carrillo, Kock, and Tonks [2] introduced the notion of *decomposition space* for applications in combinatorics, in connection with Möbius inversion. A decomposition space is a simplicial space $X\colon \mathbb{\Delta}^{\mathrm{op}} \to \mathcal{S}$ for which all pushouts of active maps along inert maps in $\mathbb{\Delta}$ are sent to pullbacks in $\mathcal{S}$. Here, the *inert* maps in $\mathbb{\Delta}$ are generated by the outer coface maps, while the *active* maps are generated by the codegeneracy and inner coface maps. The condition satisfied by X with respect to pushouts of outer coface maps against inner ones is precisely equivalent to the 2-Segal condition. For Dyckerhoff and Kapranov, the condition for pushouts of outer cofaces against codegeneracies is a further axiom which they call *unitality* [1, Definition 2.5.2]. Thus, decomposition spaces are the same thing as unital 2-Segal spaces. While the 2-Segal axiom is expressly the condition required in order to induce a (co)associative (co)multiplication on the linear span of X_1, the unitality condition ensures that this (co)multiplication is (co)unital, which is an important property in many applications.

The present note shows that the unitality condition is actually automatic, by proving:

Theorem. *Every* 2*-Segal space is unital.*

This result is unexpected, as it is not so common in mathematics for (co)associativity to imply (co)unitality.

1 Definitions and theorem

In order to cover all flavours of 2-Segal space that appear in the literature, we give a proof which applies both to 2-Segal objects in an ∞-category with finite limits and to 2-Segal objects in a Quillen model category. From now on, $\mathcal{C}$ will denote either an ∞-category with finite limits or a Quillen model category. In the latter case, "pullback" will mean a (strictly commuting) homotopy pullback.

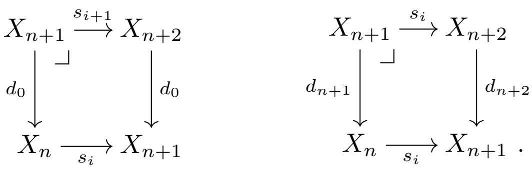
Definition. (cf. [1], [2]) A simplicial object $X\colon \mathbb{\Delta}^{\mathrm{op}} \to \mathcal{C}$ is called 2*-Segal* when the commuting squares that express the simplicial identities between inner and outer face maps of X are pullback squares. More precisely, for all $0 < i < n$ we have pullbacks

$$\begin{array}{ccc} X_{n+1} & \xrightarrow{d_{i+1}} & X_n \\ {\scriptstyle d_0}\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle d_0} \\ X_n & \xrightarrow[d_i]{} & X_{n-1} \end{array} \qquad \text{and} \qquad \begin{array}{ccc} X_{n+1} & \xrightarrow{d_i} & X_n \\ {\scriptstyle d_{n+1}}\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle d_n} \\ X_n & \xrightarrow[d_i]{} & X_{n-1} \end{array} .$$

We say that X is *upper* 2*-Segal* when only squares as to the left are required to be pullbacks, and *lower* 2*-Segal* when this is only required for squares as to the right. [1]

[1]For our purposes, splitting into upper 2-Segal and lower 2-Segal is just for economy; in the theory of higher Segal spaces [5] (k-Segal spaces for $k > 2$), the distinction between upper and lower becomes essential.

Definition. A 2-Segal space X is called *unital* if for all $0 \leqslant i \leqslant n$ the following squares are pullbacks:



We call an upper 2-Segal space *upper unital* when only the pullbacks on the left are required, and call a lower 2-Segal space *lower unital* when only the pullbacks on the right are required.

Theorem. *Every* 2*-Segal space is unital. More precisely, every upper* 2*-Segal space is upper unital, and every lower* 2*-Segal space is lower unital.*

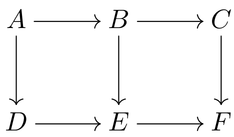
2 The proof

By symmetry, it is enough to prove:

Proposition 2.1. *If* $X\colon \mathbb{\Delta}^{\mathrm{op}} \to \mathcal{C}$ *is upper* 2*-Segal, then it is also upper unital.*

We do so using two lemmas, which are standard both in ∞-category theory and model category theory.

Lemma 2.2 (Prism Lemma)**.** *Given a commuting diagram*



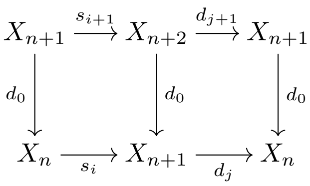
(formally a $\Delta^1 \times \Delta^2$*-diagram in the* ∞*-category case), suppose the right-hand square is a pullback. Then the outer rectangle is a pullback if and only if the left-hand square is a pullback.*

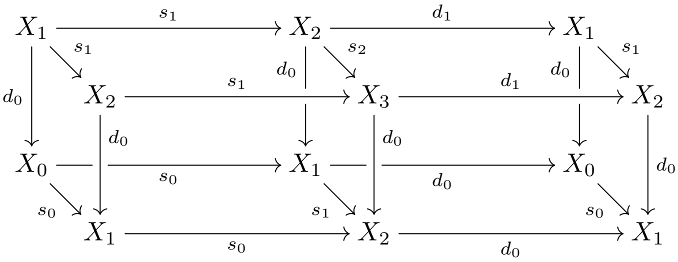
Proof. For the ∞-category version, see (the dual of) [4, Lemma 4.4.2.1]. The model category version is proven in the right-proper case in [3, Proposition 13.3.9]; we give the general case in the appendix. □

Lemma 2.3. *Pullback squares are stable under retract.*

Proof. The ∞-category version follows from (the dual of) [4, Lemma 5.1.6.3]. The model category version is known to experts, but since we do not know of any reference, we give a proof in the appendix. □

Proof of Proposition 2.1. We first establish the pullback condition for $n \geqslant 1$ and $0 \leqslant i \leqslant n$ by following the argument of [2, Proposition 3.5], exploiting that every degeneracy map except $s_0\colon X_0 \to X_1$ is a section of an inner face map. Explicitly, if we choose $j \in \{i, i+1\}$ with $0 < j \leqslant n$, then $s_i\colon X_n \to X_{n+1}$ is a section of the inner face map $d_j\colon X_{n+1} \to X_n$ and $s_{i+1}\colon X_{n+1} \to X_{n+2}$ is a section of d_{j+1}, forming the prism diagram to the left below. Here the outer square is a pullback since its top and bottom edges are the images of identity maps in $\mathbb{\Delta}$, while the right-hand square is a pullback since X is upper 2-Segal and d_j is an inner face map. So by Lemma 2.2, the left-hand square is a pullback as required.

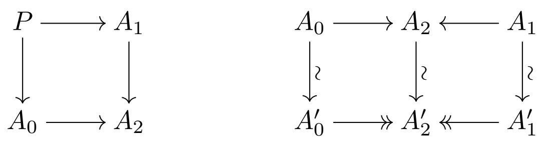




The remaining case, which is not covered by [2, Proposition 3.5], is the square with $n = i = 0$. To see that this is a pullback, we exhibit it as a retract of the square for $n = i = 1$, as displayed above right. Since we already know the $n = i = 1$ square is a pullback, so is the $n = i = 0$ square by Lemma 2.3. □

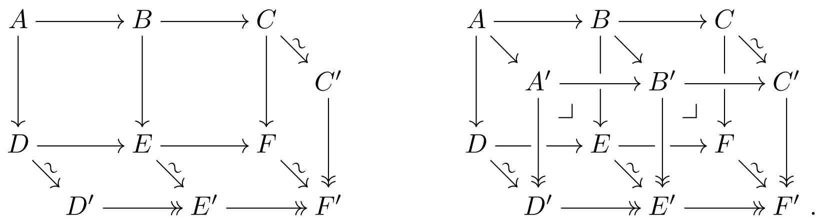
Appendix

We provide proofs of the two lemmas in the context of a model category $\mathcal{C}$. First we recall the notion of (strictly commuting) homotopy pullback. Writing Λ for the cospan category $0 \to 2 \leftarrow 1$, we endow $\mathcal{C}^\Lambda$ with the *injective* model structure, whose weak equivalences and cofibrations are pointwise, and whose fibrant objects are cospans of fibrations between fibrant objects in $\mathcal{C}$. A commuting square in $\mathcal{C}$, as to the left in



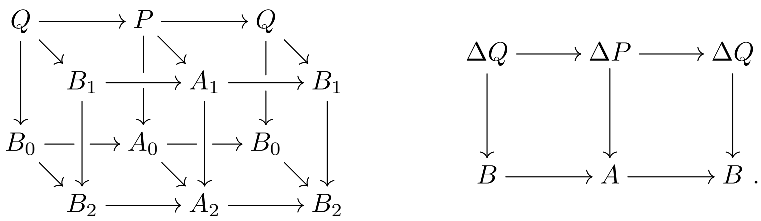
is a *homotopy pullback* if for some (equivalently, any) fibrant replacement in $\mathcal{C}^\Lambda$ for its underlying cospan, as displayed to the right above, the induced map $P \to A'_0 \times_{A'_2} A'_1$ into the strict pullback is a weak equivalence.

Proof of Lemma 2.2 in the model category case. We first replace $D \to E \to F \leftarrow C$ by a diagram of fibrations between fibrant objects, as to the left in:

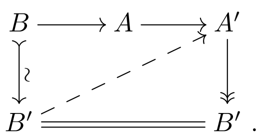


By taking strict pullbacks we complete this to the diagram as to the right. Since the right-hand back face is assumed to be a homotopy pullback, $B \to B'$ is a weak equivalence, and so $D' \twoheadrightarrow E' \twoheadleftarrow B'$ is a fibrant replacement for $D \to E \leftarrow B$ in $\mathcal{C}^\Lambda$. Thus $A \to A'$ is a weak equivalence exactly when the left-hand back face is a homotopy pullback. Since $D' \twoheadrightarrow F' \twoheadleftarrow C'$ is a fibrant replacement for $D \to F \leftarrow C$, we also have that $A \to A'$ is a weak equivalence exactly when the back rectangle is a homotopy pullback, as desired. $\square$

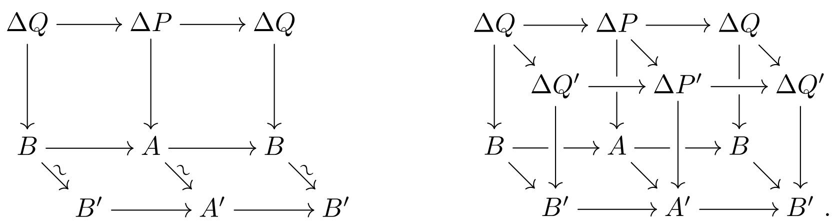
Proof of Lemma 2.3 in the model category case. Suppose given a homotopy pullback square in $\mathcal{C}$, together with a retract of it in the category of commutative squares in $\mathcal{C}$, as to the left in:



We must show that the left (equally, the right) face of this diagram is also a homotopy pullback. Regarding $B_0 \to B_2 \leftarrow B_1$ and $A_0 \to A_2 \leftarrow A_1$ as objects of $\mathcal{C}^\Lambda$, and regarding Q and P as constant objects ΔQ and ΔP, we obtain a retract diagram of arrows in $\mathcal{C}^\Lambda$ as displayed to the right above. We will now fibrantly replace A and B in $\mathcal{C}^\Lambda$ in such a way as to obtain a new retract diagram $B' \to A' \to B'$. To this end, we first fibrantly replace B via a trivial cofibration $B \xrightarrow{\sim} B'$. Now we factor the composite $A \to B \to B'$ as a trivial cofibration $A \xrightarrow{\sim} A'$ followed by a fibration $A' \twoheadrightarrow B'$. Finally, we take a lifting in the square



Altogether, we now have a retract diagram in the category of composable pairs in $\mathcal{C}^\Lambda$, as to the left in:



By forming the strict pullbacks Q' and P' of the cospans B' and A' we may complete this to the retract diagram as to the right above; note in particular that the map $Q \to Q'$ in $\mathcal{C}$ is a retract of the map $P \to P'$. Since $\Delta P \to A$ describes a homotopy pullback, $P \to P'$ is a weak equivalence; so its retract $Q \to Q'$ is also a weak equivalence, which is to say that $\Delta Q \to B$ also describes a homotopy pullback. $\square$

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